# EDGE LENGTH INDUCES END TOPOLOGIES

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Abstract. The topologies Etop and Mtop are two possible extensions of the well known Freudenthal compactification of locally finite graphs to arbitrary graphs. It is known that both of these topologies can be obtained by using metric completion of a metric on graphs given by certain edge length functions. In this paper we extend the family of functions that are able to achieve this.

## 1. Introduction

For locally finite graphs it is well-known that the Freudenthal compactification. gives rise to a suitable notion of a boundary. If the graph contains vertices of infinite degree it turns out to be more difficile and there are distinct notions of its boundary, e.g. its vertex or its edge ends. These can also be obtained by a suitable topology. In this paper we use two of them, called Etop and Mtop. For locally finite graphs these two topologies coincide.

Given a function assigning lengths to edges one can define a metric space from a graph. Georgakopoulos showed in [4] that certain such functions can be used such that the completion of this metric space, the completion is called $\ell$-Top, can be used to obtain Etop or Mtop. The functions used in [4] had the property that the sum over length of all the edges is finite. We construct a larger natural family of functions such that $\ell$-Top will still generate Etop or Mtop. That each of the assumptions on this family is necessary can be shown by easy exposition, which can be found in the an extended version of this paper. We prove the following two theorems:

**Theorem 1.** *Every 2-connected graph $G$ and $\ell : E \to \mathbb{R}^+$ satisfy $|G|_\ell \approx ||G||$ if every ray has finite $\ell$-length and the $\ell$-lengths of the spokes of any infinite fan or edge bundle tend to 0.*

**Theorem 2.** *Every 2-connected graph $G$ and $\ell : E \to \mathbb{R}^+$ satisfy $|G|_\ell \approx |G|_M$ if $|G|$ is metrizable, every ray has finite $\ell$-length, for every vertex $v$ there is a $\delta_v \in \mathbb{R}^+$ such that the $\ell$-lengths of all edges incident to $v$ are at least $\delta_v$, and the $\ell$-lengths of the open spokes of any infinite fan tend to 0.*

We use the terminology and notation of [2] and [4] unless explicitly stated otherwise. All important definitions used in this paper can be found in Section 2. All the necessary definitions and notations are given in Section 2. In Section 3 we give some insight into the motivation of the premise of Theorem 1 and Theorem 2. We give some lemma that are used for the proof of both theorems in Section 4. The proof of Theorem 1 is in Section 5, and the proof of Theorem 2 is in Section 6.

## 2. Notation

In this section we give the most important definitions and notations used in this paper. We start with recalling the definitions of a ray, double ray, ends and tails from [2]. An infinite graph $G = (V, E)$ with

$$V = \{x_0, x_1, \ldots\} \text{ and } E = \{x_0x_1, x_1x_2 \ldots\}$$

is call a *ray.* An infinite graph $G = (V, E)$ with

$$V = \{\ldots x_{-2}, x_{-1}, x_0, x_1, \ldots\} \text{ and } E = \{\ldots x_{-2}x_{-1}, x_{-1}x_0, x_0x_1, x_1x_2 \ldots\}$$

is called a *double ray.* The subrays of a ray or double ray are its *tails.* An *end* of a graph $G$ is an equivalence class of rays in $G$. For a ray $R$ with vertex set $r_1, r_2, \ldots$ and edge set $r_1r_2, r_2r_3, \ldots$ we will sometimes just write $R = r_1r_2\ldots$, analog for double rays.

In this paper we consider two different equivalence relations on rays. For Mtop two rays are equivalent, if and only if for every finite $S \subseteq V$, both rays have a tail in the same component of $G - S$, those equivalence classes of rays are the *vertex-ends* of $G$. For Etop two rays are equivalent, if and only if for every finite $F \subseteq E$, both rays have a tail in the same component of $G - F$, those equivalence classes of rays are the *edge-ends* of $G$. The set of ends of a graph $G$ is denoted by $\Omega(G)$. Note that depending on the context, $\Omega(G)$ might either mean vertex or edge-ends. We say a vertex $v$ *dominates a vertex-end* $\omega$ if there is no finite vertex separator that separates $v$ from the tail of any ray in $\omega$. In this case we call $\omega$ a dominated vertex-end. Analog we say that a vertex *dominates an edge-end* if there is no finite edge separator that separates this vertex from any tail of any ray in this edge-end. We call ends, vertex- and edge-, *dominated* if they are dominiated by any vertex. An end that is not dominated is called an *undominated end*, vertex- or edge-, respectively.

Next we define the topology $\ell$-Top, [4]. Given a connected graph $G = (V, E)$ and $\ell : E \to \mathbb{R}^+$, define on $V^2$

$$d_\ell(u, v) = \inf \left\{ \sum_{e \in P} \ell(e) \mid P \text{ is a } u - v \text{ path in } G \right\},$$

and identify any $v, u$ with $d_\ell(u, v) = 0$ to obtain a metric space $V/\!\sim$. The equivalence relation used for identifying points in $\ell$-Top will be called $\sim$. Let $(G, \ell)$ be the metric space obtained from $V/\!\sim$ by taking the disjoint union of isometric copies of $[0, \ell(e)]$ for every edge $e \in E(G)$ and identifying the end points of the interval with points in $V/\!\sim$ corresponding to the end vertices of the edge, including edges between vertices which got identified in $V/\!\sim$. We define the distance between a vertex $v$ and an inner point of an edge, say $x \in e = x_1x_2$, to be the $\min_{y \in \{x_1, x_2\}}(d(y, v) + d(y, x))$. We define the distance between two inner points of edges, say $x_1 \in e_1 = v_1w_1$ and $x_2 \in e_2 = v_2w_2$, to be the minimum of $d(x_1, v_2) + d(v_2, x_2)$ and $d(x_1, w_2) + d(w_2, x_2)$. Let $|G|_\ell$ be the metric completion of $(G, \ell)$, i.e. the unique[1] complete metric space containing $(G, \ell)$ as a dense subspace. For a ray $R = r_1r_2\ldots$ in $G$ we define $\ell(R) = \sum_{e = r_ir_{i+1}} \ell(e)$. Note that by definition the premises of each of our theorems ensures that this sum will always be finite.

We write $|G|$ for topological spaces on a graph $G$, considered as a 1-complex, and its ends. Depending on the topology we are considering, those ends might be edge-ends or vertex-ends of $G$.

We will need to pick or be given some real numbers throughout this paper. Whenever we write $\varepsilon > 0$ we mean that $\varepsilon \in \mathbb{R}^+$. In this paper we will use the term

[1] unique up to isometries fixing $(G, \ell)$ pointwise

tending to 0 at different occasions. We call a set of objects, mostly a set of real numbers, *tending* to 0, if for every $\varepsilon > 0$ there are only finitely many elements of this set larger than $\varepsilon$. We will mostly use tending to 0 for the $\ell$-lengths of sets of edges or the $\ell$-lengths of paths in an infinite set of paths.

We recall the definition of ETOP. see [4]. We endow the space consisting of $G = (V, E)$, considered as a 1-complex, and its edge-ends with the topology ETOP. Firstly, every edge $e \in E$ inherits the open sets corresponding to open sets of $[0, 1]$. Moreover, for every finite edge-set $F \subseteq E$, we declare all sets of the form

$$C \cup \Omega(C) \cup E'(C)$$

to be open, where $C$ is any component of $G - F$ and $\Omega(C)$ denotes the set of all edge-ends of $G$ having a ray in $C$ and $E'(C)$ is any union of half-edges $(z, y]$, one for every edge $e = xy$ in $F$ with $y$ lying in $C$. Let ETOP $'(G)$ denote the topological space of $G \cup \Omega(G)$ endowed with the topology generated by the above open sets. Moreover, let *ETOP*$(G)$ denote the space obtained from ETOP $'(G)$ by identifying any two points that have the same open neighborhoods. Instead of ETOP $(G)$ we will often just write $G$ endowed with ETOP or if $G$ is fixed just ETOP or $||G||$.

Let us recall the definition of MTOP. see [4]. To define MTOP $(G)$ we consider each edge of $G$ to be a copy of the real interval $[0, 1]$, bearing the corresponding metric and topology. The basic open neighborhoods of a vertex $v$ are the open stars of radius $\varepsilon$ centered at $v$ for any $\varepsilon < 1$. Those are call *open $\varepsilon$-stars* around $v$. For a vertex-end $\omega \in \Omega(G)$ we declare all sets of the form

$$\hat{C}_\varepsilon(S, \omega) := C(S, \omega) \cup \Omega(S, \omega) \cup \mathring{E}_\varepsilon(S, \omega)$$

to be open, where $S$ is an arbitrary finite subset of $V(G)$, the unique component of $G - S$ containing a ray in $\omega$ is called $C(S, \omega)$, and $\mathring{E}_\varepsilon(S, \omega)$ is the set of all inner points of $S - C(S, \omega)$ edges at distance less than $\varepsilon$ from their endpoint in $C(S, \omega)$. We call $\mathring{E}_\varepsilon(S, \omega)$ also the *$\varepsilon$-collar* of this open set. Let $|G|_{\mathrm{M}}$ denote $|G|$ endowed with MTOP. To avoid possible confusion, as we will be working with both $|G|_{\mathrm{M}}$ and $||G||$ in close proximity, we will always only consider vertex-ends when we are looking at $|G|_{\mathrm{M}}$ which might also be called the MTOP case and we will always only consider edge-ends whenever we are working with $||G||$ or $|G|$ endowed with ETOP. which will sometimes be called the ETOP case.

Sometimes we want to work with the graph $G$ in the space $|G|_{\mathrm{M}}$. In slight abuse of notation we will call $G$ be also be a graph in $|G|_{\mathrm{M}}$. In addition we consider subgraphs of $G$ to also be a subgraph of $|G|_{\mathrm{M}}$. As there is a natural bijection embedding $G$ into $|G|_{\mathrm{M}}$ so this does not pose a problem.

For $||G||$ we do basically the same. But we also have consider the identification that occurs when we from $||G||$. This means that if in $||G||$ there is an edge between two vertices then either there is a edge between those vertices in $G$ or there is no finite edge separator separating those vertices in $G$. The later means, that there are an infinite number of edge disjoint paths between those two vertices. So keep in mind that when we look at path or ray in $||G||$ this does not necessarily correspond to a path of ray in $G$.

Let $X, X'$ be topological spaces that contain a graph $G$, considered as a 1-complex, as a topological space. We will write $X \approx X'$ if the identity on $G$ extends to a homeomorphism between $X$ and $X'$. In this paper we prove that $|G|_\ell \approx ||G||$ and $|G|_\ell \approx |G|_{\mathrm{M}}$ if $G$ is 2-connected and $\ell$ complies with some conditions.

We call a set of paths *independent* if they only meet at their end vertices but are disjoint otherwise. For two vertices $x, y$ we call a path starting in $x$ and ending in $y$ an *$x - y$ path.* For an end $x$, we call a ray starting in the vertex $x$ and belonging to the end $y$ an *$y - x$ ray.* If $y$ is also an end, then we call a double ray with tails in both ends $y$ and $x$ an $y-x$ *double ray*. For a vertex $x$ and a vertex set $A$ we call a path an $x-A$ path if it starts in $x$, ends in $A$ and does contain no vertices in $A$ besides its terminal vertex. Let $v$ be a vertex of $G$ and $A$ a set of vertices in $G$. An *infinite $v-A$ fan* is a subgraph $H$ of $G$ such that $H$ consists of an infinite set of $v-A$ paths such that any two of those paths only have $v$ in common and such that $H$ also contains a ray $R$ that meets every vertex in $A$ but is disjoint to $v$. It follows that in $H$ the vertex $v$ has infinite degree and we will call $v$ the vertex of infinite of this fan. We call this ray $R$ the ray of this infinite fan. We will call this, in slight abuse of notation, just fan, as we will only ever need infinite fans. The *spokes* of a fan are the paths from its vertex of infinite degree to its ray. The *open spokes* of a fan are its spokes minus their edge at the vertex of infinite degree. A *bundle* is a union of infinitely many independent paths between two fixed vertices; the paths are its *spokes*. An *edge bundle* is a union of infinitely many edge disjoint paths between two fixed vertices; we also call the paths its *spokes*.

We say for two vertices to *have no finite edge separator between them* if there is no finite edge separator separating them. For a ray $R = r_1, r_2, \ldots$ and a vertex $v$ or another ray $R' = r'_1, r'_2, \ldots$ we say *there is no finite edge separator between $R$ and $v$ or $R'$* if there is no finite edge separator that separates a tail of $R$ from $v$ or from a tail of $R'$, i.e. for every $N \in \mathbb{N}$ there is a path from some $r_i$ with $i \geq N$ to $v$ or $r'_i$. We define the analog for not having a finite vertex separator.

Let $X$ be a topological space and let $G$, seen as a 1-complex, be embedded into $X$. We call the the embedded image of a vertex or an edge of $G$ in $X$ also a vertex or an edge, respectively. In addition we will also say $X$ contains a vertex or an edge of $G$ if $X$ contains the image of that vertex or edge under the embedding.

## 3. Motivation

In this section we give some motivation for the premises of Theorem 1 and Theorem 2. We show that the premises of the theorems are necessary in the sense that there are graphs together with length functions $\ell$ that make the theorems false if one of the conditions in the premise of Theorem 1 or Theorem 2 is not fulfilled. The first examples will work for both $||G||$ and $|G|_{\mathrm{M}}$.

3.1. **Example 1.** In this example we show why 2-connectivity is necessary. To define $G$ first let $R = r_0 r_1 \ldots$ be a ray. Then for every vertex $r_i$ in $R$ we add a new vertex $r'_i$ and join $r_i$ and $r'_i$ by an edge. We define $\ell$ to be the following: For edges $e = r_i r_{i+1}$ we define $\ell(e) = \frac{1}{2^i}$, for edges $e = r_i r'_i$ we define $\ell(e) = 1$. This graph and length function comply with every premise in Theorem 1 and Theorem 2 but the 2-connectivity. As $G$ is locally finite, $||G||$ and $|G|_{\mathrm{M}}$ are the same, see [3]. Let $O$ be any basic open set in either of those topologies around the only end $\omega$ of $G$. There is some $N \in \mathbb{N}$ such that this open set will contain all of the vertices $r'_i$ for $i \geq N$ as there is no finite edge or vertex separator separating those vertices from the ray $R$. But in $\ell$-Top, any basic open set around the metric completion point added for the ray $R$, say an $\varepsilon$-ball with $0 < \varepsilon < 1$ will miss all the vertices $r'_i$. This graph is shown together with such an open set $O$ in Figure 1 on the left.

3.2. **Example 2.** In this example we show why the $\ell$-length of the spokes of infinite fans need to tend to 0. Next we show that just adding 2-connectivity to the premises of the theorems is insufficient to make the theorems true. We start with the graph and length function defined in Subsection 3.1. Now we add a vertex $v$ to $G$ and join $v$ to every vertex $r'_i$. Here we have to be a bit careful with how to extend $\ell$ depending on if we want to look at Etop or Mtop. Say we want to look at the Etop case first, then we can extend all for the edges $e = vr'_i$ to be $\ell(e) = \frac{1}{2^i}$. For this graph and length function, every premise of Theorem 1 and Theorem 1 is

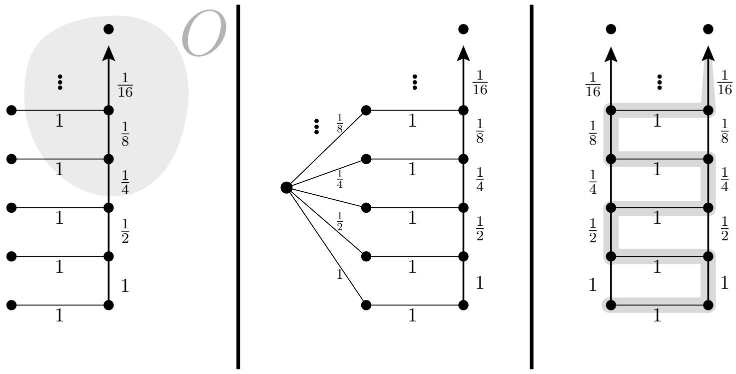


FIGURE 1. Example 1 and the two versions of Example 2

fulfilled but that the spokes of any infinite fan must tend to 0. Again we can see that for this length function $||G|| \approx |G|_\ell$ is false. One easy way to see that is, that in $||G||$, the end $\omega$ and the vertex $v$ will get identified, as there is no finite edge separator separating them. But in $|G|_\ell$, every open $\varepsilon$-ball around $v$ with $0 < \varepsilon < 1$ will miss the metric completion point at the ray, and vice versa. So those points will not get identified in $|G|_\ell$. This graph is shown in Figure 1 as the middle figure.

For the MTOP case, we can extend $\ell$ by just giving every edge $e = vr'_i$ the $\ell$-lengths 1. Now this graph and length function complies with every premise of Theorem 2 but that the open spokes of any infinite fan must tend to 0. To see that MTOP will not get induced by this length function $\ell$ we can again look at open sets around the end in MTOP and open sets around the metric completion point in $\ell$-TOP the same way we have done in Subsection 3.1.

Another possible modification to get 2-connectivity to Example 1 is to extend $G$ not by adding a vertex of infinite degree but by adding another ray. Say adding edges $r'_i r'_{i+1}$ for all $i \in \mathbb{N}$, this yields an infinite ladder. We extend $\ell$ for the edges $r'_i r'_{i+1}$ to be $\frac{1}{2^i}$. As $G$ is again locally finite, ETOP and MTOP coincide, so we can check them together. It seems like this graph complies with every premise in Theorem 1 and Theorem 2. But this time we have created a ray of infinite $\ell$-length. Starting in $r_1$ we move to $r'_1$ taking an edge of $\ell$-length 1. Then we move up on to $r'_2$ and back to $r_2$. By doing so, we used another edge of $\ell$-length 1. Repeating this process yields a ray of infinite $\ell$-length. In this graph, there will be two metric completion points, but $G$ only contains one end. The Figure 1 shows this graph in right together the ray of infinite $\ell$-length in gray and the two metric completion points.

3.3. **Example 3.** In this example we will look at the remaining conditions in the premise of Theorem 1 and Theorem 2. We start with the premise of Theorem 1 that for every two vertices $v, w$, such that there are infinitely many vertex disjoint $v - w$ paths, the $\ell$-lengths of those paths must tend to 0. To define $G$ we start with two vertices, call them $v$ and $w$. Then we add infinitely many more vertices $v_1, v_2, \ldots$ and join all of them to $v$ and $w$. We define $\ell(e) = 1$ for all edges $e$ in $G$. This graph complies with every premise of Theorem 1 and Theorem 2 but that the spokes of every bundle must tend to 0. In $||G||$ the vertices $v$ and $w$ get identified, but in $\ell$-TOP they have distance 2, so they will not get identified.

For showing that in the MTOP case, the $\ell$-length of every edge must be bounded from below we can reuse $G$. For that let $G$ be the same as in the first part of the Example 3. We change $\ell$ like follows: We define $\ell(e) = \frac{1}{2^i}$ if $e$ contains the vertex

$v_i$. Now we got the opposite of the above case. In MTOP $v$ and $w$ are not identified, but in $\ell$-TOP they are.

3.4. **Example 4.** In subsection we sketch examples that shows that Theorem 1 and Theorem 2 are not characterizations. We show that there are graphs together with length functions such that for those graphs ETOP or MTOP coincide with $\ell$-TOP, but some of the conditions in the premise of Theorem 1 and Theorem 2 are not met.

If the graph $G$ is only a single ray then any length function that gives this ray a finite length induces the topologies MTOP and ETOP on $G$ as those coincide for locally finite graphs. This example is shown in Figure 2 on the left hand side.

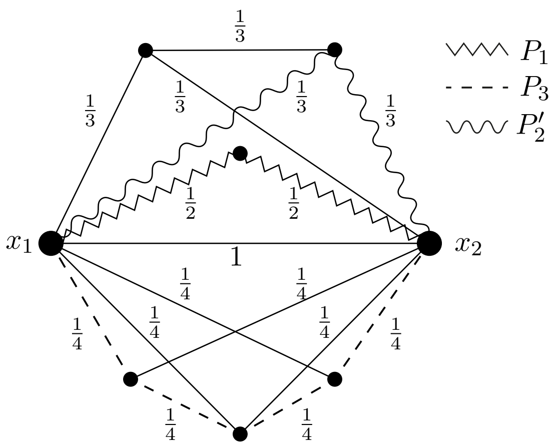


FIGURE 2. Example in which topologies coincide

Next we give an example graph together with a length function $\ell$ such that not every edge bundle tends to 0, but the topology of ETOP still coincides with the topology $\ell$-TOP. We only sketch the proof that ETOP and $\ell$-TOP coincide.

We define the graph $G$ together with the length function $\ell$ recursively. We start with two vertices $x_1$ and $x_2$ and the edge $x_1x_2$ and set $\ell(x_1x_2) = 1$. Now we add an $x_1 - x_2$ paths $P_i$ to $G$ such that $P_i$ only meets $x_1$ and $x_2$ and is disjoint to everything else defined so far and such that $P_i$ contains $i$ vertices outside of $x_1$ and $x_2$ and such that $\ell(P_i) = 1$ for all $i$. For that we set $\ell(e_i) = \frac{1}{i+1}$ for every edge $e_i \in P_i$. For each vertex $p_i$ in $P_i$ we add the edge $x_1p_i$ and $x_2p_i$ to $G$ unless $G$ already contains it. We also set the $\ell$-length of those edges to be $\frac{1}{i+1}$. This graph is shown in Figure 2 after adding the paths $P_1, P_2$ and $P_3$.

It is obvious that there is no finite edge separator separating $x_1$ and $x_2$. And by definition the paths $P_1, P_2, \ldots$ form a bundle even tough each path $P_i$ has $\ell$-length 1. Still it is straight forward to check that $d(x_1, x_2) = 0$.[2] We leave it up to the interessted reader to check that the topologies coincide.

3.5. **Example 5.** Now we show an example of a graph together with a length function such that for this graph MTOP and $\ell$-TOP coincide but there are open spokes such that their $\ell$-length does not tend to 0. Let $G$ be a graph that consist of a ray $R = r_1r_2\ldots$ and a vertex $v$ that is not on $R$. We set $\ell(r_ir_{i+1}) = \frac{1}{2^i}$. For every $i \in \mathbb{N}^+$ we now add a cycle $C_i$ to $G$. The cycle $C_i$ contains the vertex $v$ and $i + 1$ additional vertices, that are disjoint to $R$ and vertices contained in any $C_j$ with $j <i$. All edges incident with $v$ get $\ell$-length 1. Each edge in $C_i$ that does not

[2]Using the paths like the indicated path $P_2'$ in Figure 2.

have an $\ell$-length so far gets the $\ell$-length $\frac{1}{i+1}$. For each vertex in $C_i - v$ we add an edge $e_i$ from this vertex to $r_i$, those edges get $\ell$-length $\frac{1}{i+1}$. Figure 3 shows a part of $G$ containing the first three cycles.

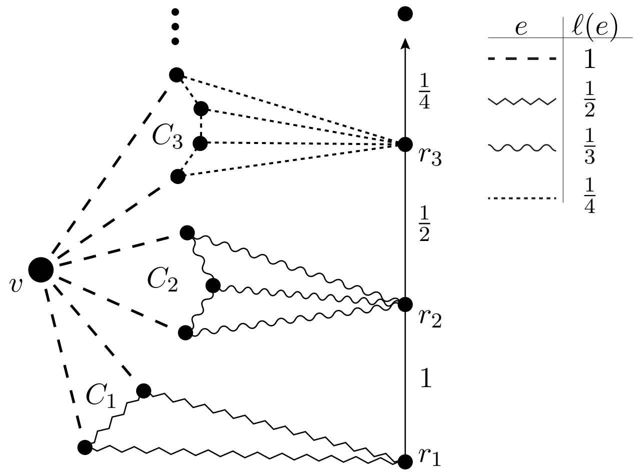


FIGURE 3. The graph $G$ with addition of $C_1, C_2$ and $C_3$

Each vertex in $G$ besides $v$ has finite degree by definition. We now show that every ray in $G$ has finite $\ell$-length. For that we look at 2-separators of $G$. For every $i \in \mathbb{N}$ the set of vertices $S_i := \{r_i, v\}$ is a 2-separator of $G$. The deletion of any $S_1$ leaves two components of $G$. The deletion of any other $S_i, i \geq 2$ leaves three components and only one of those is infinite. Let $R'$ be a ray in $G$. Let $G_1$ be the component of $G - S_i$ that does not contain a tail of $R$. For $i \geq 2$ let $G_i$ and $G'_i$ be the two components of $G - S_i$ that do not contain a tail of $R$. Suppose that $R'$ contains vertices of more than two different $G_i$ or more than two different $G'_i$, say $R'$ contains vertices of more than two different $G_i$, let say for $i = j, k$ and $i = l$. Also say that $R'$ meets $G_k$ after it met $G_j$ and before it meets $G_l$. If $v$ was the last vertex of $R'$ before it met $G_k$, then it has to use $r_k$ to leave $G_k$. But then it must use $r_l$ to move into $G_l$, but this means the ray cannot leave $G_l$, which is a contraction as $G_l$ is finite. So we may assume that $R'$ meets $G_k$ after using $r_k$, but then is has to leave $G_k$ by using $v$. This yields the same contraction as before. So $R'$ has a tail such that each vertex of that tail is in $R$. Then this tail of $R'$ has finite $\ell$-length, and as initial segments do not add infinite $\ell$ to a ray, $R'$ has finite $\ell$-length. Now we need to check that the graph $G$ is metrizable. Any graph is metrizable if it has a normal spanning tree by [1]. Any graph that does not contain a $K^{\aleph_0}$ as a minor has a normal spanning tree by [2] and this is obviously true for $G$.

Now we show that for this graph $G$ together with this length function $\ell$ there is an infinite set of open spokes such that the $\ell$-lengths of those spokes do not tend to zero. We define $P'_i$ to be the path starting in $v$ and moving along the cycle $C_i$. We move along $C_i$ until the we have met every vertex of $C_i$ once. By definition of $G$, we can now use an edge to $R$. This defines an infinite set of independent $v - R$ paths. By definition the path $P'_i$ contains exactly $i + 3$ vertices and $i + 2$ edges. Each edge in $P'_i$ that not incident with $v$ has $\ell$-length $\frac{1}{i+1}$, as we have $i + 1$ such edges this sums to an $\ell$-length of 1 for all $i \in \mathbb{N}$. As $P'_i - v$ is exactly an open spoke, the $\ell$-lengths of these open spokes do not tend to zero.

The interested reader may check that MTOP and $\ell$-TOP coincide. This can be done in the same way that we prove Theorem 2, see Section 4 and Section 6 for details.

## 4. General lemmas

In this section we will prove general lemmas that are mostly usable for the proofs of Theorem 1 and Theorem 2. Before we start with those lemmas, we will state the very general proof structure for the theorems of this paper. We will prove Theorem 1 and Theorem 2 by following the same general steps that were used in [4, Theorem 3.1]. First we will show that $(G, \ell)$ and the two topological spaces, $|G|_{\mathrm{M}}$ and $||G||$, without their respective (edge-)ends have the same points, so there is a bijection between $(G, \ell)$ and $|G|_{\mathrm{M}}$ or $||G||$, that is indifferent under the identity on $G$. Note that in the in the case of $||G||$, we will not remove dominated ends, as those will be represented by a vertex. We then let those spaces without those ends inherit the metric $d$ from the metric $d_\ell$ of $(G, \ell)$. We will extends this metric $d$ to all of $|G|_{\mathrm{M}}$ and $||G||$ in a way that makes $d$ induce the existing topology on $|G|_{\mathrm{M}}$ and $||G||$ such that both $|G|_{\mathrm{M}}$ and $||G||$ are complete and $(G, \ell)$ is dense in $||G||$ and $|G|_{\mathrm{M}}$, respectively. As metric completions are, as stated above, unique, it follows that the theorems are true.

We now continue with general lemmas that can be used for the Etop and the Mtop case. For that we need the definition of the above mentioned metrics. In this section we will just give the definition of the metrics used for $||G||$ and $|G|_{\mathrm{M}}$ but skip the proof that those actually define metrics. We also skip the proof that there is bijection between $(G, \ell)$ and $||G||$ or $|G|_{\mathrm{M}}$ without, in the case of Etop. undominated ends. All of this will be done in Section 5 for Etop and Section 6 for Mtop. As the definition of the new metric $d$ will be almost the same for the case of Etop and Mtop. we will just handle both of them at the same time. But keep in mind, that because the length function $\ell$ is different by assumption in the Etop and the Mtop case, we are actually defining different metrics.

Let $\ell : E(G) \to \mathbb{R}^+$ be a length function. For the Etop case write $\widetilde{\Omega}$ for the set of edge-ends of $||G||$ that are not represented by a vertex, which are exactly the undominated edge-ends of $G$. We call the metric given by $\ell$-Top $d_\ell$. Let $d_\ell$ induce a metric on $||G|| \setminus \widetilde{\Omega}$ or $|G|_{\mathrm{M}} \setminus \Omega$, call this new metric $d$. We now extend this metric to all of $||G||$ or $|G|_{\mathrm{M}}$, respectively. For $x \in \widetilde{\Omega} \cup V$ and $y \in \widetilde{\Omega}$ in the Etop case and $x \in \Omega \cup V$ and $y \in \Omega$ in the Mtop case define

$$d(x, y) = \inf \{\ell(R) \mid R \text{ is an } x - y \text{ ray or double ray}\},$$

where $\ell(R)$ is the sum over the $\ell$-lengths of all edges contained in $R$. The premises of Theorem 1 and Theorem 2 ensure that this sum will always be finite. This gives us two slightly different metrics; one on $||G||$ and one on $|G|_{\mathrm{M}}$. The metric space on the point set of $||G||$ induced by $d$ will be called $||G||_d$ and metric space on the point set of $|G|_{\mathrm{M}}$ induced by $d$ will be called $|G|_d$.

The first lemma we prove is a version of the Jumping Arc Lemma [2] for our topological space $||G||$. The core of the Jumping Arc Lemma states, that if there is a finite cut $F \subseteq E(G)$ with sides $V_1, V_2$ in $G$, then there is no arc in $|G|$ that meets both sides $V_1$ and $V_2$ without containing an inner point of an edge $e \in F$. For the set inner points of an edge $e$ we write $\mathring{e}$. For the set of inner points of edges within an edge set $F$ we write $\mathring{F}$. For this paper it is sufficient to not consider general arcs but only rays.

**Lemma 1.** *Let $G$ be a graph and let $F$ be a finite cut of $G$ with sides $V_1, V_2$. Then there is no ray in $||G||$ that has a tail in both $V_1$ and $V_2$.*

*Proof.* Since $F$ is finite and separates $V_1$ from $V_2$ in $G$, there is no equivalence class of points in $||G||$ that meets $V_1$ and $V_2$. If there would be such a class meeting $V_1$ in a point $x$ and $V_2$ in a point $y$, then $F$ would have separated those points, which would have placed $x$ and $y$ in different equivalence classes.

Let $R = r_1 r_2 \dots$ be any ray in $||G||$. Let $V_i'$ be the set of equivalence classes whose representatives are in $V_i$. As $F$ is finite there is a vertex $r_i$ on $R$, say $r_i \in V_1$ such that there is no $r_j$ in $R$ such that $r_j \in V_2$ and $j > i$. This means that $R$ cannot have a tail in $V_2$. □

Note that we might call edge-ends only ends for the remainder of this section, but like states above, by end in the Etop case, we always mean edge-end. The next lemma states that different rays in the same end converge to the same point in $||G||$ and $|G|_{\mathrm{M}}$, respectively.

**Lemma 2.** *Let $G$ be a graph and $\omega$ an undominated edge-end of $G$. Then for any two rays $R$ and $R'$ in $\omega$ there are infinitely many vertex disjoint $R - R'$ paths in $G$.*

*Proof.* As $R$ and $R'$ are in the same edge-end, we can find an infinite set of edge disjoint paths $\mathcal{P}$ between them. Suppose that the paths in $\mathcal{P}$ are not vertex disjoint. But as $\omega$ is an undominated edge-end, for every vertex $v$ of $G$ there is a finite edge separator $F$ separating $v$ from any tail of any ray in $\omega$. Each path in $\mathcal{P}$ only meets finitely many other paths in $\mathcal{P}$ as all the paths in $\mathcal{P}$ are edge disjoint by assumption. To find an infinite set of disjoint $R - R'$ paths now is easy. We start with $\mathcal{P}' = \emptyset$ and in each step we pick any path still in $\mathcal{P}$ and add it to $\mathcal{P}'$ and then delete every path in $\mathcal{P}$ from $\mathcal{P}$ that meets this just chosen path. As there are only finitely many paths that do, we can repeat this infinitely many times. □

**Lemma 3.** *Let $\omega \in \widetilde{\Omega}$, for $||G||$, or $\omega \in \Omega$, for $|G|_{\mathrm{M}}$. For any two rays $R, R'$ in $\omega$ there is an infinite set $\mathcal{P}$ of disjoint $R - R'$ paths. And for every such $\mathcal{P}$ the $\ell$-lengths of all paths in $\mathcal{P}$ tends to 0.*

*Proof.* Let $R$ and $R'$ be two rays in an end $\omega \in \widetilde{\Omega}$ for $||G||$ and $\omega \in \Omega$ for $|G|_{\mathrm{M}}$. Let $R = v_1 v_2 v_3 \dots$ and $R' = v_1' v_2' v_3' \dots$ be two rays, we set $v_i < v_j$ if $i < j$, and the same for the $v_i'$ and $v_j'$. We assume that $R \cap R' = \emptyset$: if $R$ and $R'$ intersect only finitely often, we can delete the initial segments to get tails $\overline{R}$ and $\overline{R}'$ of $R$ and $R'$ such that $\overline{R} \cap \overline{R}' = \emptyset$ and instead work with those. If $R$ and $R'$ meet an infinite number of times the conclusion follows directly. As $R$ and $R'$ belong to the same end, there is no finite separator that separates them; in the case of Etop edge separator, in the case of Mtop vertex separator. So in the case of Mtop. we get the existence of an infinite set of disjoints $R - R'$ paths from the fact that there is no finite vertex separator separating $R$ from $R'$, as $R$ and are $R'$ are in the same end. Picking an $R - R'$ path, deleting this path and the initial segments on $R$ and $R'$ up to this path will only delete finitely many vertices. We pick tails of $R$ and $R'$ disjoint to everything defined so far. Now we can find another $R - R'$ path between those tails that is disjoint to all paths found in an earlier stage. Repeating this infinitely many times yields infinitely many vertex disjoint $R - R'$ paths. By Lemma 2 there are infinitely many vertex disjoint $R - R'$ paths in $G$ in the Etop case. So we can find infinitely $R - R'$ paths both in $||G||$ and $|G|_{\mathrm{M}}$, respectively.

Now assume that there is an infinite set of $R - R'$ paths $\mathcal{P}$ whose $\ell$-lengths do not tend to 0. Chose $\delta > 0$ such that there is an infinte set of paths $\mathcal{P}' \subseteq \mathcal{P}$ such that all paths in $\mathcal{P}'$ have $\ell$-lengths at least $\delta$.

We can assume $\mathcal{P}'$ consists of paths $P_1, P_2, \dots$ such that if $P_i$ starts on $R$ in $v_k$ and $P_j$ starts on $R$ in $v_l$ if $i < j$ and $v_k < v_l$. In addition we also assume that if $v_u'$ is the end vertex of $P_i$ on $R'$ and $v_z'$ is the end vertex of $P_j$ on $R'$ and $i < j$, then $v_u' < v_z'$. We define the ray of infinite $\ell$-length $R_{inf}$ as follows: Starting in $v_1$ we move up along $R$ till we find the first vertex of $P_1$, we use $P_1$ to get to $R'$ and move along $R'$ till we get to $P_2$ and move along $P_2$ to $R$. Repeating this process infinitely many times defines a ray. This ray $R_{inf}$ has infinite $\ell$-length as it contains infinitely many paths of $\mathcal{P}$ each of $\ell$-length at least $\delta > 0$. This is a contraction to our initial assumption on $\ell$. $\square$

In the prove of Lemma 3 only used that boths rays were in the same end to find an infinite set of disjoint paths between them. If such condition is given, we can use the same arguments to show that the $\ell$-lenghts of those paths must tend to 0.

**Lemma 4.** *Let $R = r_1 r_2 \dots$ and let $R' = r'_1 r'_2 \dots$ and $\mathcal{P}$ be an infinite set of vertex disjoint paths from $R$ to $R'$. Then the $\ell$-lengths of the paths in $\mathcal{P}$ tends to 0.* $\square$

We will prove some lemmas that are true for both $||G||_d$ and $|G|_d$. We will make the necessary distinctions but one should keep in mind, that the metric $d$ will, depending on the context, be different. The following lemma states, that if we take a basic open set in the metric space on $||G||_d$ or $|G|_d$, those open sets are connected as graphs for as long as $\ell$ complies with the appropriate theorem, so Theorem 1 for ETOP and Theorem 2 for MTOP.

**Lemma 5.** *Let $O$ be an open $\varepsilon$-ball around any point $x$ in $||G||_d$ or $|G|_d$, with $\ell$ complying with the appropriate theorem. Then $O$ is connected in the sense, that for any two vertices $z, y$ in $O$ there is path in $O$ from $z$ to $y$.*

*Proof.* We will prove the theorem simultaneously for both lemma. Let $O$ be an open $\varepsilon$-ball around any point $x$ in either metric space $||G||_d$ or $|G|_d$. Now let $y, z$ be two vertices in $O$. We may assume that $y \neq z$. We want to find a path $P_{yz}$ from $y$ to $z$ that is contained in $O$. If $x$ is a vertex, or in the case of ETOP a class representing vertices and maybe an edge-end, then there must be a path $P_1$ from $x$ to $z$ and a path $P_2$ from $x$ to $y$ in $O$, by definition of $d$. If there were no path $P_1$ from $x$ to $z$ in $O$, every path in $G$ from $x$ to $z$ must have $\ell$-length at least $\varepsilon$. So the infimum over the $\ell$-length of all those paths is at least $\varepsilon$, but $d(x, v) < \varepsilon$, which would be contraction. So there is a path from $z$ to $y$ in $O$, as there is one contained in $P_1 \cup P_2$.

If $x$ is an inner point of an edge $e = x_1 x_2$, then at least one of its vertices is contained in $O$, say $x_1$. As any path from $y, z$ to $x$ either contains a paths to $x_1$ or can be extend to $x_1$ using the partial edge $x_1 - x$, we use the same argument as if $x$ were a vertex by using $x_1$ instead of $x$.

The last case is that $x$ is an end, that is not represented by a vertex. We define: $d_{xy} := d(x, y)$ and $d_{xz} := d(x, z)$. As $z, y$ are in $O$ it follows that $d_{xy}, d_{xy} < \varepsilon$. Let $d_{zy} = \min_{\{y,z\}}\{\varepsilon - (d_{xz}), \varepsilon - (d_{xy})\}$. We now pick rays $R_y, R_z$ starting in $y, z$, respectively, belonging to $x$ such that, $\ell(R_y) < d_{xy} + \frac{d_{zy}}{3}$ and $\ell(R_z) < d_{xz} + \frac{d_{zy}}{3}$ which is possible by the definition of $\ell$. In addition, by definition, the rays $R_y$ and $R_z$ are contained in $O$. If $R_y$ and $R_z$ are not disjoint in $O$, then $O$ obviously does contain a path from $z$ to $y$ that is contained in $O$. So we assume that $R_z$ and $R_y$ are disjoint in $O$. By the Lemma 4, the $\ell$-length of any paths in any infinite set of disjoint paths between $R_z$ and $R_y$ tends to 0. As $x$ is an end not represented by a vertex, there is an infinite set of vertex disjoint paths between those two rays by Lemma 2. Now we pick a path $P$ from $R_y$ to $R_z$ such that $d(u, x) < \frac{\varepsilon}{2}$ for the vertex $u := R_y \cap P$ and that $\ell(P) < \frac{\varepsilon}{2}$. As every point on $P$ has distance less than $\varepsilon$ from $x$, the entire path $P$ is contained in $O$.

We can now easily define a path from $y$ to $z$ contained in $O$. We start in $y$ and follow $R_y$ till we meet $P$, we than follow $P$ to $R_z$ and follow that to $z$. $\square$

Next we prove a lemma that states that for a ray $R$ and a set $B$ of vertices 1such that $d(R, b) \geq \delta$ for some $\delta \in \mathbb{R}^+$ and for all $b \in B$ there is a finite vertex separator that separates $R$ from every vertex in $B$.

**Lemma 6.** *Let $G$ be a 2-connected graph and $\ell : E(G) \to \mathbb{R}^+$ be a length function such that the following is true: Each ray of $G$ has finite $\ell$-length and the open spokes of each infinite fan tend to 0. Let $B$ be an infinite set of vertices and $R = r_1r_2\dots$ be a ray. If $d(r_i, b) \geq \delta$ for some $\delta \in \mathbb{R}^+$ for all $r_i$ in $R$ and all $b$ in $B$ then there are only finitely many vertex disjoint $B - R$ paths in $G$.*

*Proof.* Let $G$ and $\ell$ fulfill the premise of Lemma 6. Suppose there is a set $B$ of vertices and a ray $R = r_1r_2\dots$ such that following is true: There are infinitely many vertex disjoint $B - R$ paths in $G$. And $d(r_i, b) \geq \delta$ for some $\delta \in \mathbb{R}^+$ for all $r_i$ in $R$ and for all $b$ in $B$.

As $G$ is 2-connected, for every vertex $b \in B$ there are two $b - R$ paths in $G$ that only meet in $b$. Together with a subpath of $R$, these form a cycle $C_b$. Suppose that we can find an infinite set of such cycles vertex disjoint. Then we create a ray $\widetilde{R}$ of infinite $\ell$-length: For each $C_b$ we start in $b$ and move on both paths to $R$, let $v_b^+$ and $v_b^-$ be the first vertices of $C_b$ that are also on $R$, such that if $v_b^+ = r_j$ and $v_b^- = r_k$ then $k < j$. To define $\widetilde{R}$ we start in any vertex on $R$ and move up on $R$ till we hit the first vertex $v_b^-$ of such a cycle $C_b$. We then move on $C_b$ to the vertex $v_b^+$ and then keep moving up along $R$ and repeat. By the definition of $R$, each such cycle $C_b$ has $\ell$-length at least $\delta$ outside its part on $R$. So $\widetilde{R}$ has infinite $\ell$-length, which is a contraction to your assumptions.

So that there is no infinite set of disjoint cycles $C_b$. This means that after we have found a finite number of such disjoint cycles $C_b$, there is no additional disjoint cycle. Let $p \in \mathbb{N}$ be the largest natural number such that $r_p \in R$ gets met by one of those finitely many cycles. We define $\widetilde{S}$ to be the union of $\{r_1, \dots, r_p\}$ and all the vertices that are contained in those finitely many cycles. We now take a tail $T$ of $R$ such that no vertex in $\widetilde{S}$ is contained in $T$, this is possible because $\widetilde{S}$ is finite.

By our assumptions we know that there is an infinite set of vertex disjoint $B - R$ paths, so there is also an infinite set of vertex disjoint $B - T$ paths in $G$. Call this set of paths $\mathcal{Q}$. Let $\widehat{B}$ be subset of vertices in $B$ which gets met by paths in $\mathcal{Q}$.

For two sets of vertices $B'$ and $T'$ of $G$ we say a finite set $S'$ of vertices of $G$ *almost disconnects* $B'$ from $T'$, if for every vertex $b \in B'$ there is a vertex $x$ in $V(G) - (S' \cup \{b\})$ such that in $G - (S' \cup \{x\})$, the vertex $b$ is disconnected from $T'$. In other words, for every $b \in B'$ there is an $x \neq b$ such that $S' \cup \{x\}$ separates $b$ from $T'$.

By our above argument, we know $S'$ almost disconnects $B$ from $T$. Otherwise we could find infinitely many disjoint cycle $C_b$. Note that for all but finitely many vertices in $B$, the set of vertices $S'$ does not separate $\widehat{B}$ from $T$ as there are infinitely vertex disjoint $\widehat{B} - T$ paths in $G$, one start in each vertex of $\widehat{B}$. We assume that $S'$ is disjoint to $T$. If not, we could just pick a tail of $T$ instead of $T$ such that this is true. Say $S' = \{s_1, \dots, s_n\}$ and set $S_0 = \emptyset$ and let $S_i = \{s_1, \dots, s_i\}$.

Let $B_i$ be the set of vertices in $\widehat{B}$ that $S_i$ almost disconnects from $T$. Obviously $B_0 = \emptyset$ holds as $G$ is 2-connected and $B_n = \widehat{B}$. Now we note that for $B_0$ there is finite vertex separator separating $B_0$ from $T$. So there must be some $i \in \{1, \dots .n-1\}$ such that there is a finite vertex separator that separates $B_i$ from $T$ but there is no finite vertex separator that separates $B_{i+1}$ from $T$.

Let $G' = G - S_i$ and $B' = B_{i+1} \setminus B_i$. By assumption on $i$ we know that $B'$ is infinite. Otherwise we could extend the finite separator that separates $B_i$ from $T$ by all the vertices in $B'$, and have a finite separator that separates $B_{i+1}$ from $T$. By the definition of $B_{i+1}$ there is no 1-separator that separates any $b \in B'$ from $T$ in $G'$. So we know that for each vertex $b$ in $B'$ there are two $b - T$ paths in $G'$ that only meet in $b$.

We pick any vertex in $B'$, call that $b_1$. By assumption on $B_{i+1}$ there is only a 2-separator containing $s_{i+1}$ and a vertex other than $b_1$, say $x_{b_1}$ that separates $b_1$ from $T$. Let $P$ be any path from $b_1$ to $T$ that does not contain $s_{i+1}$. By definition, any 2-separator containing $s_{i+1}$ that separates $b_1$ from $T$ must contain a vertex of $P$. So there are only finitely many ways to choose the vertex $x_{b_1}$. Under all those finitely many candidates for $x_{b_1}$ we want to choose the one whose distance to $T$ on $P$ is smallest with this property, counting edges not in terms of $d$ or $\ell$. Suppose that there is no unique vertex under all the candidates for $x_{b_1}$ that is closest to $T$. This means there are two paths $P_1$ and $P_2$ from $b_1$ to $T$ that avoid $s_{i+1}$, such that for $P_1$ the closest candidate for $x_{b_1}$ is a different one than for $P_2$. Let $c_1$ and $c_2$ be those two candidates, $c_1$ for $P_1$ and $c_2$ for $P_2$. So we know that the path $P_1$ meets $c_2$ before it meets $c_1$. It must meet $c_1$ and $c_2$, otherwise $\{c_1\} \cup \{s_{i+1}\}$ and $\{c_2\} \cup \{s_{i+1}\}$ would not separates $b_1$ from $T$ in $G'$. Now we show that $\{c_1\} \cup \{s_{i+1}\}$ does not separate $b_1$ from $T$ in $G'$. We now take the path $P_1$ from $b_1$ to $c_2$, and then use the path $P_2$ from $c_2$ to $T$. By definition, this path does not use the vertex $c_1$ nor the vertex $s_{i+1}$, which is contraction. So let $x_{b_1}$ be the vertex closest to $T$ such that $S'_{j+1}$ separates $b_1$ from $T$.

Let $S'_1$ be $\{s_{i+1}, x_{b_1}\}$. Now define $A_1$ to be all the vertices of $B'$ that are contained in the same component of $G' - S'_1$ as $b_1$. Set $B'_1$ to be $B' \setminus A_1$. Again by assumption on $i$, we know that $B'_1$ is infinite, otherwise we would find a finite separator separating $B_{i+1}$ from $T$. Now we repeat this process, say we have found $b_1, \ldots, b_j$, let $S'_j$ be the 2-separator consisting of $s_{i+1}$ and $x_{b_j}$ and let $A_j$ be all the vertices that are in the same component of $G'$ after deletion of $S'_j$ and let $B'_j = B'_{j-1} \setminus A_j$ with $B'_0 = B'$. By the above argument is $B'_j$ infinite. We now choose any vertex $b_{j+1} \in B_j$. Again we choose a vertex $x_{b_{j+1}}$ closest to $T$ such that the union of $x_{b_{j+1}}$ and $s_{i+1}$ separates $b_{j+1}$ from $T$. We repeat this process infinite times. For each $b_j$ there are two independent paths from $b_j$, one to $s_{i+1}$ and one to $x_{b_j}$. Let $P^s_j$ be the path to $s_{i+1}$ and $P^x_j$ the path to $x_{b_j}$. By definition, there is a path $Q_j$ in $\mathcal{Q}$ from $b_j$ to $T$ that contains $x_{b_j}$. We define the path $\widetilde{P^x_j}$ to be the path $P^x_j$ from $b_j$ to $x_{b_j}$ and the path $Q_j$ from $x_{b_j}$ to $T$.

We now note, that for different $b_j$ and $b_k$, say $j < k$, the paths $P^s_j$ and $P^s_k$ only meet in $s_{i+1}$. Otherwise they were to meet outside of $s_{i+1}$, say in a vertex $v$, we would get a contraction to the choice of $b_k$ as follows: We move from $b_j$ to $v$ without using $s_{i+1}$,$x_{b_k}$ or $x_{b_j}$. Then we move from $v$ to $x_{b_k}$ again without using $s_{i+1}$,$x_{b_k}$ or $x_{b_j}$ But this would mean that $b_k \in A_j$ was it is in the same component as $b_j$ of $G' - S'_j$ as we have just seen.

We note that the paths $\widetilde{P^x_j}$ and $\widetilde{P^x_k}$ are vertex disjoint for $j \neq k$ as $\widetilde{P^x_k}$ cannot meet $P^x_j$ by definition of the paths in $\mathcal{Q}$. Because the subpathsf $\widetilde{P^x_k}$ and $\widetilde{P^x_j}$ are that are paths in $\mathcal{Q}$, and those do not meet.

Now we have a contraction like follows. For all $j \in \mathbb{N}$ let $P_j$ be the path from $s_{i+1}$ to $b_j$ by the path $P^s_j$ and then from $b_j$ to $T$ by $P^x_j$. This gives us infinitely many independent paths from $s_{i+1}$ to $T$, as each contains a path from a vertex $b_j \in B$ to $T$, all of those paths have $\ell$-length at least $\delta$, so the open spokes of this infinite fan do not tend to 0. $\square$

We finish this section by quoting a useful lemma, the so called Star-Comb Lemma from [2]. For that let us recall some definitions. Let $R$ be a ray and let $\mathcal{P}$ be a set of infinitely many vertex disjoint paths that have exactly their first vertex on $R$. The graph that is the union of $R$ and all the paths in $\mathcal{P}$ is called a *comb*, the paths in $\mathcal{P}$ are called the *teeth* of the comb. The ray $R$ is the *spine* of the comb. Let $G$ be a subdivided star or a comb, we call the set of vertices in $G$ with vertex degree 1 the *leaves* of this subdivided star or comb.

**Lemma 7** (Star-Comb Lemma)**.** *Let $U$ be an infinite set of vertices in a connected graph $G$. Then $G$ contains either a comb with all leaves in $U$ or a subdivision of an infinite star with all leaves in $U$.*

## 5. Inducing Etop

In this section we prove Theorem 1. We split this section into three subsections. In Subsection 5.1 we prove some lemmas that can only be used in the Etop case as well as proving that the metric $d$ defined in Section 4 is indeed a metric. In Subsection 5.2. we prove the main lemma for the proof of Theorem 1. In the final subsection Subsection 5.3 we put all those lemmas together and prove Theorem 1. The challenge in this paper will be to find or use a finite edge set $F$ used to generate an open set $O$, choosing $z$ for all the half edges $(z, x]$ with $x \in O$ in $O$ will be easy. Because of this we will most of the time just talk about the finite edge set $F$, instead of referring to inner points of edges when searching or defining open sets in $||G||$. We will only make a distinction between the finite edge set $F$ and the inner points within it.

### 5.1. **Basics Lemmas for** Etop**.**

We start by proving that the metric $d$ is indeed a metric. For that we need the lemma that $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$ have the same points, i.e. that there is a bijection between $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$ that extends the identity on $G$.

**Lemma 8.** *Let $G$ be a 2-connected graph, $\ell$: $E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 1. If $x, y, z$ are vertices in $G$ such that $d(x, y) = 0$ and $d(y, z) = 0$, then $d(x, z) = 0$.*

*Proof.* The union of the an $x - y$ path with an $y - z$ path always contains an $x - z$ path. When taking the infimum for the definition of $d$, all those $x - z$ paths were taken into account. It follows that $d(x, z)$ is equal to 0. □

The following lemma states that rays in the same edge-end tend to a common point. Should this edge-end be dominated by a vertex $v$, this point will be $[v]$.

**Lemma 9.** *Let $G$ be a 2-connected graph, $\ell$: $E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 1. Let $\omega$ be an edge-end of $G$, and let $R = r_1r_2\dots$ and $R' = r'_1r'_2\dots$ be two rays in $G$ that are $\omega$. Then $\inf_{i\in\mathbb{N}} d(r_i, r'_i) = 0$. If $v$ is a vertex such that there is no finite edge separator separating $v$ from $R$ in $G$, then $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$.*

*Proof.* Let $G$ be a 2-connected graph, $\ell$: $E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 1. Also let $\omega$ be an edge-end of $G$, let $R = r_1r_2\dots$, and $R' = r'_1r'_2\dots$ be two rays in $\omega$. Let $v$ be a vertex such that there is no finite edge separator separating $v$ from $R$. We show that $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$. Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a countable infinite set of edge disjoint $v - R$ paths in $G$ such that each vertex in $R$ gets met by at most one path in $\mathcal{P}$. If there is an infinite subset $\mathcal{P}' \subseteq \mathcal{P}$ such that all the paths in $\mathcal{P}'$ are independent, then by our assumptions on $\ell$ the $\inf_{i\in\mathbb{N}} d(v, r_i)$ is equal to 0.

We consider all the paths in $\mathcal{P}$ to be directed towards $R$. For the path $P_i$ in $\mathcal{P}$ let $p_i$ be the last vertex of $P_i$ that meets infinitely many other paths in $\mathcal{P}$. We can now assume that the subpath of $P_i$ from $p_i$ to $R$ does not meet any other paths in $\mathcal{P}$ as each vertex of this subpath only meets finitely many path in $\mathcal{P}$, and we can delete all of those for each of the finitely many vertices of that path, so $\mathcal{P}$ would still be infinite after that, which is all we need. By our assumption on $\ell$ we know that $d(v, p_i) = 0$ for all paths $P_i$, as those form an edge bundle. If there are only finitely many such vertices $p_i$, then at least for one of those vertices, say $p_j$, there are infinitely many independent $p_j - R$ paths. So the lengths of all those has to tend to 0 this forms an infinite fan, and with $d(v, p_j) = 0$ follows that $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$. So we may assume that there are infinitely many such $p_i$. For each path $P_i \in \mathcal{P}$ that contains a vertex $p_i$ we define the subpath of $P_i$ from $p_i$ to $R$ as $P_i^R$, and the path from $v$ to $p_i$ as $P_i^v$. Now let $H$ be a graph that is the union of all the paths $P_i^v$. As all those paths contain $v$, the graph $H$ is connected. We apply the Star-Comb Lemma to $H$ with the vertex set of all the vertices $p_i$. If the Star-Comb Lemma returns a subdivided star with center say $c$, then $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$ follows from assumptions on $\ell$. Going from $c$ to some leaf of this subdivided star and then to $R$ by the disjoint path gives us an infinite $c - R$ fan, so the bundle must tend to 0. By definition of $c$, there is no finite edge separator separating $c$ from $v$, so $d(v, c) = 0$ by our assumption on $\ell$. This means that $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$. If the Star-Comb Lemma returns a comb with spine say $R'$ then the paths to the leaves extend to disjoint paths to $R$. The Lemma 4 states that the $\ell$-length of all those paths must tend to 0, as they are disjoint paths between two rays. But each of those paths contains a vertex $p_i$, and as there is no finite edge separator separating $p_i$ from $v$, $d(p_i, v) = 0$, this means that $\inf_{i\in\mathbb{N}} d(v, r_i) = 0$.

Now we prove the first part of this lemma. As there is no finite edge separator separating $R$ from $R'$, we can find an infinite set of edge disjoint paths $\mathcal{P}$. Assume we are able to find an infinite set of vertex disjoint $R - R'$ paths $\mathcal{P}$. By Lemma 4 this means that the $\ell$-lengths of the paths in $\mathcal{P}$ tend to 0. Let $\varepsilon > 0$ be given. We show that there is an $N \in \mathbb{N}$ such that $d(r_i, r'_i) < \varepsilon$ for all $i \geq N$. Let $R_1$ and $R'_1$ be tails of $R$ and $R'$, respective, such that $\ell(R_1) < \frac{\varepsilon}{3}$ and $\ell(R'_1) < \frac{\varepsilon}{3}$, as $R$ and $R'$ have finite $\ell$-length, this is possible. Now we chose a path $P \in \mathcal{P}$ with $\ell(P) < \frac{\varepsilon}{3}$ that meets $R$ and $R'$ in the tails $R_1$ and $R'_1$, respectively. For each vertex $r_i$ in $R_1$ the path $P$ ensures that $d(r_i, r'_j) \leq \varepsilon$ for all vertices $r'_j \in R'_1$. It follows that $\inf_{i\in\mathbb{N}} d(r_i, r'_i) = 0$. So we can assume that we can find a vertex $x$ that meets infinitely many paths in $\mathcal{P}$. But now by the above argument, $\inf_{i\in\mathbb{N}} d(x, r_i) = 0 = \inf_{i\in\mathbb{N}} d(x, r'_i)$, so $\inf_{i\in\mathbb{N}} d(r_i, r'_i) = 0$. □

**Lemma 10.** *Let $G$ be a 2-connected graph and $\ell$: $E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 1. Then the identity on $G$ extends to a bijection between $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$.*

*Proof.* As both $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$ contain all the edges and no edge gets identified, the identity on $G$ induces a bijection between the edges of $(G, \ell)$ and the edges of $||G|| \setminus \widetilde{\Omega}$. Neither $(G, \ell)$ nor $||G|| \setminus \widetilde{\Omega}$ contains any points not contained in $G$, considered as a one complex. It remains to show that the identity on $G$ for vertices extends to a bijection between the vertices in $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$. So we have to show that in $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$ the same vertices get identified. We start with the direction, that vertices get identified in $(G, \ell)$ get also identified in $||G|| \setminus \widetilde{\Omega}$. If two vertices $x, y$ get identified in $(G, \ell)$, they have distance 0 by definition. Suppose there is a finite edge separator $F$ separating $x$ and $y$. But as $\ell(e) > 0$ for all $e \in F$ this means that $d(x, y) \neq 0$, a contraction to the assumption that $x$ and $y$ get identified in $(G, \ell)$.

Let $x, y$ be the vertices that get identified in $||G|| \setminus \widetilde{\Omega}$. We have to show that $d(x, y) = 0$. This is true by our assumptions on $\ell$. □

The next lemma we prove shows that $d$ is indeed a metric

**Lemma 11.** *Let $G$ be a 2-connected graph, $\ell$: $E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 1, and let $d$ be the metric on $||G|| \setminus \widetilde{\Omega}$ inherited by the metric $d_\ell$ from $(G, \ell)$. Given an $x \in \widetilde{\Omega} \cup V$ and $y \in \widetilde{\Omega}$ we define*

$$d_\ell(x,y) = \inf\left\{\ell(R) | R \textit{ is an } x-y \textit{ ray or double ray}\right\},$$

*where* $l(R) := \sum_{e\in E(R)} \ell(e)$ *for any ray or double ray* $R$*. Then* $d$ *is a metric on* $||G||$.

*Proof.* First we check $d(x,y) = 0$ if and only if $x = y$. Let $x, y$ be two points in $||G||$: By the definition of $||G||$, if $x \neq y$ then there is a finite set $F$ of edges that separates $x$ from $y$ in $G$, and due to the fact that $\ell(e) > 0$ for all $e \in E(G)$ follows that $d(x,y) \geq \min\{\ell(e) \mid e \in F\}$, for $d(x,y) = 0$ if $x = y$.

Secondly we have $d(x,y) = d(y,x)$ directly by the definition of $d$.

Last to check is the triangle inequality. We only consider the case $x, y, z \in \widetilde{\Omega}$, the other cases are easier versions of the same proof or follow directly from the fact that in the infimum over all the $x - z$ paths, every path contained in the union of an $x-y$ and an $y-z$ path was a candidate. We show that $d(x,z) \leq d(x,y) + d(y,z)$ by showing that $d(x,z) \leq d(x,y) + d(y,z) + \varepsilon$ for every $\varepsilon > 0$, which is sufficient because in the definition of $d$ we take the infimum.

Let $\varepsilon > 0$ be given. Without loss of generality we assume $d(x,z) \neq 0$. We will find double rays from $x$ to $y$ and from $y$ to $z$ that exceed $d(x,y)$ and $d(y,z)$ only by $\frac{\varepsilon}{3}$. We will find a path with total $\ell$-length less than $\frac{\varepsilon}{3}$ that connects these two double rays. Together this will create the desired double rays. There are $x - y$ and $y - z$ double ray that exceed the $\ell$-length of $d(x,y)$ and $d(y,z)$ by at most $\frac{\varepsilon}{3}$ by the definition of $d$. If there were no such double rays, then $d(x,y)$ or $d(y,z)$ would be larger, by at least $\frac{\varepsilon}{3}$. Call those double rays $R_{xy}$ and $R_{yz}$. Now we need to find a path with $\ell$-length at most $\frac{\varepsilon}{3}$ that connect $R_{xy}$ and $R_{yz}$. For the double rays $R_{xy}$ and $R_{yz}$ call a tail of those the $y$-tail, if this tail belongs to $y$. Since $y$ is an undominated end there is an infinite set of disjoint paths connecting those $y$-tails by Lemma 2. Now we need to find a path with $\ell$-length at most $\frac{\varepsilon}{3}$ that connect $R_{xy}$ and $R_{yz}$. By Lemma 3 we know that the infimum over the $\ell$-lengths of all $R_{xy} - R_{yx}$ paths is 0, so such a path must exist, call this path $P_{xy}$. It follows that there is a double ray $R_{xz}$ from $x$ to $z$ with $\ell$-length less than $d(x,y) + d(y,z) + \varepsilon$. So $d$ is indeed a metric on $||G||$. □

5.2. **Main Lemmas for** Etop**.** In this subsection we will prove lemmas that are only used for the proof of Theorem 1. The first part of this subsection will be the proof the Main Lemma for ETOP. We start this subsection with a very basic lemma version of the main lemma that we will use to prove Theorem 1. We state the following lemma to give an idea about the proof of the main lemma.

**Lemma 12.** *Give a graph $G$ and an $\ell : E \to \mathbb{R}^+$ satisfying the premise of Theorem 1. Then the $\ell$-length of edges incident with any vertex $v$ of infinite degree tend to 0.*

*Proof.* Let $G$ be a 2-connected graph and $\ell$ comply with the premise of Theorem 1 and let some $\delta \in \mathbb{R}^+$ be given. Let $v$ be a vertex of infinite degree and $N$ be the set of vertices that are adjacent to $v$. Assume that there is an infinite subset $N' \subseteq N$ such that $d(v, n) \geq \delta$ for all $n \in N'$. As $G$ is 2-connected, $G - v$ is connected. We can use the Star-Comb Lemma on $G - v$ with $N'$ as the infinite set of vertices. If the Star-Comb Lemma returns a comb with a ray $R$ and leaves in $N'$, this generates an infinite fan. The length of the set of those paths from $v$ to $R$ must tend to 0 by assumption on $\ell$, so we get a contraction. If the Star-Comb Lemma returns a subdivided star with center $v'$ and leaves in $N'$, in $G$ this extends to a bundle between $v$ and $v'$ and by assumption the $\ell$-lengths of its spokes tends to 0, so we get a contraction. □

We want to extend this lemma to paths instead of edges, so we do not get a star but a subdivision of a star. We now prove the main lemma that states that the following: Let an $\varepsilon > 0$, a vertex $v$ and an infinite set $S$ of vertices such that $d(v, S) > \varepsilon$ be given. Then there is a finite edge set $F$ that separates $v$ from $S$.

**Lemma 13** (Main Lemma for Etop )**.** *Let $G$ be a graph and let $\ell : E(G) \to \mathbb{R}^+$ satisfy the premise of Theorem 1. Let $v$ be any vertex of $G$ and $S$ be an infinite set of vertices such that $d(v, s) \geq \delta$ for some $\delta \in \mathbb{R}^+$ and for all $s \in S$. Then there is a finite edge separator that separates $v$ from $S$.*

This proof is split into several claims that will be proved one at a time. For easier understanding we give the general proof concept ahead of time without the details. For that we measure distances always in metric $d$. When we say a vertex or a set of vertices is far away from another vertex or set of vertices we also mean this in terms of the metric $d$. We assume that the lemma is false and try to find a contraction. For that we chose a vertex $v$ and an infinite set of vertices that is at least some $\delta$ far away from $v$.

First we show that almost all the paths in $G$ between vertices in $S$ have to contain vertices that are in some sense close to $v$. If not we can split $G$ into two connected graphs, one whose vertices are close to $v$ and the other whose vertices are far away from $v$. We can now use the Star-Comb lemma on both graphs to find the desired contraction to the assumptions on $\ell$ in either of the four possible results.

Secondly we use the results of the first part and the 2-connectivity of $G$ to find two independent paths from the vertices in $S$ to $v$ such that for different vertices $S$ those path do not meet as long as they are close to $S$. Then we will define some auxiliary graphs that in which we replace those disjoint paths by edges to have some easier to work with structures.

*Proof.* Let $G$ be a graph and an $\ell \colon E(G) \to \mathbb{R}^+$ satisfying the premise of Theorem 1. Let $\delta \in \mathbb{R}^+$ be given and let $v$ be a vertex of $G$ and $S$ be an infinite set of vertices such that $d(v, s) \geq \delta$ for all $s \in S$. We assume that there is no finite edge separator that separates $S$ from $v$ to find a contraction.

So let $\mathcal{P}$ be an infinite set of edge disjoint $v - S$ paths in $G$. We now choose a $\delta' < \delta$.We may assume that each path in $\mathcal{P}$ only contains one vertex in $S$.

Otherwise we cut off the finite number of additional vertices from each path that contains more than one of those vertices. In addition we assume that each vertex in $S$ only gets met by a single path in $\mathcal{P}$. We may do so because if there were a vertex $s \in S$ that gets met by infinitely many paths in $\mathcal{P}$, then $d(v, s) = 0$ by the assumptions on $\ell$, which contracts the definition of $S$.

(1) *There is no infinite subset $S' \subseteq S$ such that between every two of those vertices there is a path in $G$ that does not contain any vertex with distance less than $\delta'$ from $v$.*

We assume that contrary and precede as follows. For each pair of vertices in $S'$ we pick a path such that this path does not contain any vertices with distance less than $\delta'$ from $v$. Let $\mathcal{P}'$ be the set of all those paths. We define two auxiliary graphs $H_v$ and $H_S$. Let $H_v$ be the graph that is the union of all the paths in $\mathcal{P}$. Let $H_S$ be graph that is the union of all the paths in $\mathcal{P}'$.

(2) *There is no infinite set of independent $v - S'$ paths in $H_v$.*

If there were we apply the Star-Comb lemma to $H_S$ with the vertex set being the end vertices of those infinitely many independent paths. If we get a star, say with center $c$, then $d(v, c) \geq \delta'$. But as there are infinitely many independent $v - c$ paths there is no finite edge separator separating $c$ from $v$. So we have an infinite edge bundle whose $\ell$-lengths do not tend to 0. This is a contradiction to our assumptions on $\ell$. Analog if we get a comb, say with spine $R$, then $d(v, r_i) \geq \delta'$ for all vertices $r_i \in R$. So we have an infinite fan such that the $\ell$-lengths of its spokes do not tend to 0. This also yields a contradiction to our assumptions on $\ell$. So either result yields a contraction which proves (2).

We use (2) to finish the proof of (1). If there is no infinite set of independent $v - S'$ paths in $H_v$ then each of the paths in $\mathcal{P}$ must have an infinite set of paths of $\mathcal{P}$ outside of $v$. Now we look at any vertex $h_v$ of $H_v$ that has infinite degree. By definition of $H_v$ we know that $d(v, h_v) = 0$. Now we apply the Star-Comb lemma to $H_v$ with the vertex set $S'$. Finding a star with center $c_v$ again yields a contraction because we know that $d(v, c_v) = 0$ and $d(v, s) \geq \delta'$ for all $s \in S$. We can now use this vertex instead of $v$ in the above argument and find the same contraction. So we may assume the Star-Comb lemma yields a comb in $H_v$ with spine $R$. By definition of $H_v$ and since there is no infinite set of independent $v - S'$ paths we can conclude that $R$ contains infinitely many vertices that have infinite degree in $H_v$ each of which has distance 0 to $v$. As we have seen above the distance of each of those vertices to $v$ is 0. We may assume that the $\ell$-length of the ray is finite. But as there are infinitely many vertices on $R$ that have distance 0 to $v$ we know that $R$ has a tail such that each vertex of that tail has distance least $\frac{\delta'}{2}$ from every vertex in $S'$. We assume that this is true for every vertex of $R$ by redefining $R$ as this tail. Now apply the Star-Comb lemma to $H_S$ with the vertex set that is the subset of $S'$ that gets met by the teeth of the just found comb.

As in the preceding paragraph we directly get a contraction. If we find a star with center $c_S$ in $H_S$ then have $d(c_S, r) \geq \frac{\delta'}{2}$ for all vertices $r \in R$ while there being no finite edge separator separating $c_S$ and $R$ which is a contraction. If we find a comb with spine $R_S$ then we have too rays in the same edge end such that the distance between $R$ and $R_S$ does not tend to 0 which is also a contraction. This finishes the proof of (1)

So for each vertex $s$ in $S$ there are only finitely many $s'$ such that there is a path from $s$ to $s'$ that does not contain any vertex that has distance less than $\delta'$ from $v$. Now we set $\delta'$ to be $\frac{\delta}{2}$.

Next we want to define the before mentioned auxiliary graph $H$. For that we need some definitions. Take a countable infinite subset $S' = \{s_1, s_2, \ldots\} \subseteq S$ such that from each vertex $s \in S'$ there is no path to any other vertex $s' \in S'$ that does not contains a vertex with distance less than $\delta'$ from $v$. Let $v'$ be a neighbor of $v$ with distance less than $\delta'$ from $v$. Such a vertex must exist by Lemma 12. For each $s_i \in S'$ we now define a cycle $C_i'$ that contains the vertex $s_i$ and the edge $vv'$. As $G$ is 2-connected, $G$ contains two independent $s_i - \{v, v'\}$ paths. Together with the edge $vv'$, those paths from a cycle, call that $C_i'$. Let $C_i$ be $C_i'$ without the edge $vv'$. Now let $C_i^v$ be the path in $C_i$ from $s_i$ to $v$ and let $C_i^{v'}$ be the path in $C_i$ from $s_i$ to $v'$. Let $\widetilde{C_i^v}$ be the subpath of $C_i^v$ from $s_i$ to the first vertex on $C_i^v$ that has distance less than $\delta'$ from $s_i$, and the same for $\widetilde{C_i^{v'}}$. Let $O$ be the open $\delta'$-ball around the vertex $v$. By definition of $S'$, the paths $\widetilde{C_i^{v'}}$ and $\widetilde{C_i^v}$ do not meet any $\widetilde{C_j^{v'}}$ or $\widetilde{C_j^v}$ path outside of $O$ for $i \neq j$.

We will now construct an auxiliary multigraph $H$ recursively. We will still call $H$ a graph. The graph $H$ will be 2-connected and countably infinite. To define $H$ we will define a sequence of graphs $H_i$ with $i \in \mathbb{N}$ such that $H_i \subseteq H_j$ for $j > i$. For each $C_i$ we will add some special path to $H$. In each step we modify $C_i$ by replacing some path in $C_i$ be some edge $e_i$. Those edges will be very important so we will keep track of those. Call the set of those edges $E_R$. Now we replace the path in $C_i$ that consists of $\widetilde{C_i^{v'}} \cup \widetilde{C_i^v}$ by the edge $e_i$, call the resulting graph $\widetilde{C_i}$.

We briefly summarize our definition of the $H_i$. We start with a shortened version of $C_1$ for $H_1$ and in each following step we add to $H_{i-1}$ the shortened version of $C_i$ up to the first time $C_i$ meets any vertex with distance less than $\delta'$ from $v$ that we have defined so far.

For $H_1$ we start with $\widetilde{C_1}$ union the edge $vv'$. For $\widetilde{C_i}$ with $i \geq 2$ let $v_i^1$ and $v_i^2$ be the two vertices of $\widetilde{C_i}$ such that in $G$, when we move from $s_i$ to $v$ and $v'$ using $\widetilde{C_i}$ the vertices $v_i^1$ and $v_i^2$ are the first vertices of $C_i$ that meet $C_j$ with $j < i$. As all $\widetilde{C_i}$ and all $C_i$ contain $v$ and $v'$ such vertices $v_i^1, v_i^2$ must exists for all $i \in \mathbb{N}$. Note that by definition of $C_i$, for all $i \in \mathbb{N}$ the vertex $v_i^1$ is never equal to the vertex $v_i^2$. We assume that we have defined $H_{i-1}$. We define $H_i$ to be $H_{i-1}$ union the path from $v_i^1$ to $v_i^2$ in $\widetilde{C_i}$. Let $H$ be the union of all the graphs $H_i$. Each $H_i$ is 2-connected by construction [2], so $H$ is also 2-connected.

Let $T$ be the subgraph of $H$ that contains every vertex of $H$ but no edges in $E_R$. By definition, $T$ is connected and contains no cycle, so $T$ is a tree, and as $T$ contains all the vertices of $H$, it is a spanning tree of $H$. Let $E_T$ be the edge set of $T$.

(3) *The fundamental circuits and fundamental cuts of $T$ in $H$ are finite.*

As $T$ is an ordinary spanning tree, for every edge $e$ in $E_R$, the fundamental circuit in $T + e$ is finite. Let $e$ be any edge in $E_T$, and let $F$ be the set of edges in $H$ that cross the fundamental cut of $e$. Let $K_1$ be one component of $T - e$ and $K_2$ the other component of $T - e$. We assume that $F$ is infinite. Let $N_i^F$ be the set of end vertices of the edges in $K_i$. To show that the fundamental cut are finite we make a case study. We distinguish the cases either $N_i^F$ is infinite and the case $N_i^F$ is finite. If $N_i^F$ is finite, as $F$ is infinite, there is a vertex $v_i \in N_i^F$ that is incident with infinitely many edges in $F$, we treat this case later.

So say $N_i^F$ is infinite, this means that $K_i$ is also infinite. Then we can apply the Star-Comb Lemma to $K_i$ with the vertex set $N_i^F$. Let $M_1^F$ be the set of leaves in the resulting comb or subdivided star. Now let $M_2^F$ be the subset of $N_2^F$ such that each vertex in $M_2^F$ is adjacent to a vertex in $M_1^F$ by an edge in $F$ and vice versa. Now we check what happens in $K_2$.

If $M_2^F$ is finite, they there must be a vertex $v_2$ in $M_2^F$ that is adjacent to infinitely many vertex in $M_1^F$ by edges in $F$. But in $G$ that means that either result of the Star-Comb Lemma for $K_1$ leads to a contraction. Each edge in $F$, besides $e$, corresponds in $G$ to a path of $\ell$-length at least $\delta'$. So there are either infinitely many independent $v_2 - R$ paths in $G$, each of $\ell$-length at least $\delta'$, in the case that the Star-Comb Lemma returned a comb with spine $R$, or infinitely independent $v_2 - c$ paths each of $\ell$-length at least $\delta'$ in the case that the Star-Comb Lemma returned a subdivided star with center $c$.

So we may assume that $M_2^F$ is infinite. But then we can apply the Star-Comb Lemma to $K_2$ with the vertex set $M_2^F$. If we get a subdivided star, we have the same situation as above. We may assume this does not happen.

If we get a comb with leaves, say $L$, in $M_2^F$ we also get a contraction, like follows. We look at the possible results of the Star-Comb lemma in $K_1$. Suppose we also have a comb in $K_1$. We can define a ray alternating between the spine of the combs in $K_1$ and in $K_2$, respectively. We start in the spine of the comb in $K_2$, move to $L$ by the teeth of the comb in $K_2$ and then use an edge in $E_R$ to move to $K_1$. In $K_1$ we then use the teeth of the comb in $K_1$ to move to the spine in $K_1$. We can repeat this process infinitely often without using any vertex twice. Each time we use an edge in $E_R$, this corresponds in $G$ to using a path of $\ell$-length at least $\frac{\delta}{2}$. So we can find infinitely many disjoint paths between two rays such that the $\ell$-length of the paths between them does not tend to 0, so by Lemma 4 we have a contradiction.

So we may assume that we get a comb in $K_2$ and a subdivided star in $K_1$, but by swapping the names of $K_1$ and $K_2$, we already know that this also yields a contraction.

So we may assume that $K_1$ is finite. We can assume that $K_2$ is also finite. As we have already seen, that having one $K_i$ infinite and the other finite yields a contraction.

As $K_1$ and $K_2$ are both finite we can directly find two vertices $u_1$ and $u_2$ such that there are infinitely many edges in $F$ between them. But this also directly translates to a contraction, as for those two vertices, in $G$ there infinitely many independent paths each of length at least $\delta'$ between them which in $G$ contradicts the assumptions on $\ell$. This concludes the proof of (3).

So we have shown that the fundamental circuits and cuts of $T$ are finite. Now we define a new graph $H_R$. The vertex set of $H_R$ is the set of edges of $T$ union the set of edges in $E_R$. This also gives a partition into two classes. We now define the edges of $H_R$, those will only be between vertices in different classes. Let $V_T$ be the set of vertices that are edges in $T$ and let $V_{E_R}$ be all the other vertices in $H_R$. In $H_R$ the vertex $v_1 \in V_T$ is adjacent to a vertex $v_2 \in V_{E_R}$, if and only if the fundamental cut of $T$ of the edge $v_1$ contains the edge $v_2$ in $E_R$. This is exactly the case if the fundamental circuit of $T$ for the edge $v_2$ contains the edge $v_1$. Because all the fundamental circuits and fundamental cuts are finite, every vertex in $H_R$ has finite degree. The graph $H_R$ is an infinite graph, as there are infinitely many edges in $E_R$, this means that $V_T$ also must be infinite. We now use $T$ to construct our final contraction. For that we show

(4) *$T$ does not contain of infinite degree*

(5) *$T$ does not contain ray.*

We start by proving (4). Assume that $T$ contains a vertex $w$ of infinite degree. We now define a new auxiliary graph $H_S$. Let $T_1, T_2, \dots$ be the components of $T-w$. We define the graph $H_S$ be the a star with center $w$ and whose leaves are the components of $T - w$. We now define a new graph using $H_S$, this new graph will be called $H'_S$. To define $H'_S$ we start with $H_S$ and delete $w$ from $H_S$. For each edge $e = xy \in E_R$ we now add an edge $uu'$ between the vertices $u$ and $u'$ of $H'_S$ if the component $u$ contains $x$, and the component $u'$ contains $y$.

As $H$ is 2-connected the graph $H'_S$ is connected. And as $w$ has infinite degree it follows that at least one of the following two is true for $H'_S$: Either $H'_S$ contains a vertex $u$ of infinite degree, or there is a ray $R$ in $H'_S$. If there is a such a vertex $u$, let $F$ be the set of edges in $E_R$ incident with $u$. We partition $T$ into the components $u$ and $T - u$. So there are infinitely edges between $u$ and $T - u$. As those would form an infinite fundamental cut of $T$, which we have seen before, cannot occur, we get a contraction.

For that we assume that we can find a ray $R$ in $H'_S$. Now this ray has to have infinitely many edges in $E_R$, as those are the only edges in $H'_S$. Now we can find a ray $R' = r'_1 r'_2 \dots$ in $H$ such that there is an infinite sequence of disjoint fundamental circuits $Z_i$ of $T$ that meet $R'$ such that if the circuit $Z_i$ meets $R'$ in the two vertices $r_k$ and $r_j$, say $k < j$, then no other circuit meets $R'$ in a vertex $r_l$, with $i < l < j$. It is easy to see, that if we find such a ray $R'$ and such a sequence, that we get a contraction: When we move along $R'$, every time we see a vertex of such a circuit, we can take that instead of $R'$ till we are back on $R'$. Each time we do that, we use at least on edge in $E_R$, and in $G$ this corresponds to moving along a path of $\ell$-length at least $\delta'$. So we found a ray of infinite $\ell$-length. To find $R'$ we start by extending the ray $R$ in $H'_S$ to a ray in $H$, say $R = r_1 r_2 \dots$. For each edge in $R$ we now pick a path in $H$. While in a component $r_i$ of $T - w$, we move along $T$ and when we cross over from a component of $T - w$ to another component of $T - w$, we use one edge in $E_R$, and then start over. For that we start by picking any vertex as $r'_1$ in $H$ that lies in the component of $r_1$. Say we have defined a finite part of the ray $R'$ and the last vertex in $R'$ is $r'_i$, which lies in the component $r_j$ of $T - w$. Let the edge $r_j r_{j+1}$ correspond to the edge $e_1 e_2$ in $H$. We may assume that $e_1$ is in $r_j$, and $e_2$ is in $r_{j+1}$. In $r_j$ there is a unique path from $r'_i$ to $e_1$. We extend $R'$ by this finite path and then add the edge from $e_1$ to $e_2$ and the vertex $e_2$ to $R'$. This defines a ray $R'$ in $H$.

Now we will find the sequence of circuits described above. Suppose that we found a finite number of those circuits. Let $r'_i$ be the last vertex of $R'$ that meets such a circuit. To show that we can find such a sequence we have to look at the graph $H_R$ again. Given a finite set of vertices $S_1$ in $V_T$ of $H_R$, there are only finitely many neighbors of $S$ in $H_R$, as $H_R$ is locally finite. Call those $S_2$. The set of of neighbors of $S_2$ in $H_R$ is also finite. Call this $S_3$. For $S_1$ we take all the edges used in the finite number of circuits that are in $V_T$ as well as all the edges $r'_k r'_{k+1}$ in $R'$ with $k + 1 \leq i$, those are always all in $V_T$. Now we move along $R'$ till we find a tail, such that no edge on this tail is in $S_3$ in $H_R$. Let $xy$ be the next edge on $R'$. By construction, for every edge in $T$ there is a circuit in $H$ containing this edge and an edge in $E_R$. Let $Z$ be a circuit in $H$ through $xy$. Suppose that this circuit $Z$ meets another circuit $Z'$ found in an earlier step. Let $e'$ be the edge in $E_R$ contained in this circuit. Then $e'$ is in $S_2$. But this means that $xy$ is in $S_3$, which is a contraction. This proves (4).

So we continue to proof (5). So we can assume that we can find a ray $R = r_1 r_2 \dots$ in $T$. Without loss of generality we may assume that $r_1 = v$. The idea for this part of the proof is to use that the fundamental cuts in $T$ are finite to construct a ray $R'$

in $H$ that uses infinitely many edges from $E_R$. As above this yields a contradiction because this $R'$ corresponds to a ray in $G$ which has infinite $\ell$-length.

For an edge $r_j r_{j+1}$ in $R$ let $F_j$ denote the fundamental cut edges to this edge. We introduce a notation for edges in $F_j$. For an edge $e$ in $F_j$ we write $e = t^- t^+$ and assume that $e^-$ is in the component of $T - r_j r_{r+1}$ that contains $v$ and hence that $t^+$ is in the other component.By definition there is a unique path in $T$ from $t^-$ to $R$, call it $P_e^+$

Let $\mathcal{P}_j^+$ be the union set of all the paths $P_e^+$ taken over all the edge $e$ in $F_j$. Because $F_j$ is a fundamental cut of $T$ and as all the fundamental cuts of $T$ are finite we know that the number paths in $\mathcal{P}_j^+$ is finite. For an edge $r_j r_{j+1}$ in $R$ let $j^* \in \mathbb{N}$ be maximal such that $r_{j^*}$ is contained in some path in $\mathcal{P}_j^+$. As $\mathcal{P}_j^+$ is finite such vertex must exist. Now we take a look at the edge $r_{j^*} r_{j^*+1}$ and the corresponding set $\mathcal{P}_{j^*}$. Note that this is also finite. So there is a natural number $j^{**}$ such that $j^{**}$ is maximal such that $r_j^{**}$ is contained in a path in $\mathcal{P}_{j^*}$. We call the edge $r_{j^{**}} r_{j^{**}+1}$ the *conclusion edge* of the edge $r_j r_{j+1}$. Given any edge $e$ on the ray $R$ we define the *conclusion set* $\mathcal{C}_e = \{c_1, c_2, \ldots\}$ to be set of edges on $R$ such that $c_1$ is the conclusion edge of $e$ and for $i \geq 2$ the edge $c_i$ is the conclusion edge of $c_{i-1}$.

We now look at the fundamental cut of an edge $r_l r_{l+1}$ of $R$ with $l \geq (j^{**} + 1)$. Let $e$ be any edge of $F_l$. Let $Q$ be the path from $t^-$ to $R$. We know by definition of $j^{**}$ that $Q$ does not meet any path in $\mathcal{P}_{j^*}$. If $Q$ were to meet any path in $\mathcal{P}_{j^*}$ then the edge $e$ would be in $F_{j^*}$, which it is not.

Now we can construct a ray $R'$ in $H$ that contains infinitely many edges in $E_R$. In the first step we want to select the infinite set of edges in $E_R$ that will be contained in $R'$. For that we chose any edge $e$ of $R$. Let $\mathcal{C}_e = \{c_1, c_2, \ldots\}$ be the conclusion set of $e$. For every edge $c_i = r_j r_{j+1}$ we now use any edge in its fundamental cut, call it $f_i = f_i^- f_i^+$. Again we assume that $f_i^-$ is in the component of $T - c_i$ that contains $v$. Let the paths from $f_i^-$ and $f_i^+$ to $R$ be $P_i^-$ and $P_i^+$, respectively. To define the ray $R'$ we start on the vertex $v$. We move along $R$ till we find the first vertex contained on a path $P_i^-$. We follow $P_i^-$ till we meet the vertex $f_i^-$. Now we take the edge $f_i$ and follow the path $P_i^+$ back to $R$ where we move up along $R$ till we find the next vertex that is contained on a path $P_j^-$ and repeat this process. Call the resulting ray $R'$. As $R'$ contains infinitely many edges of $E_R$ replacing all those edges by the paths that they represent in $G$ yields a ray in $G$ that contains infinitely many paths that each have at least constant $\ell$-lengths. This proves (5) and hence this finishes our proof. □

With the help of Lemma 13 and the Lemma 6 we can now prove that for any ray $R$ and any $\delta \in \mathbb{R}^+$ there is no infinite set $B$ of vertices such that every vertex in $B$ has distance more than $\delta$ from $R$ unless there is a finite edge set $F$ that separates $B$ from $R$. The following lemma makes that more precise.

**Lemma 14.** *Let $G$ be a graph and let $\ell : E(G) \to \mathbb{R}^+$ satisfy the premise of Theorem 1. Furthermore let $R = r_1 r_2 \ldots$ be ray in $G$ and $\delta \in \mathbb{R}^+$ and let $B \subseteq V(G)$ such that $d(b, r_i) \geq \delta$ for all $b \in B$ and all $i \in \mathbb{N}$. Then there is a finite edge set $F$ that separates $B$ from $R$, i.e. $F$ separates each vertex in $R$ from $B$.*

*Proof.* Let $G$ be a graph and an $\ell : E(G) \to \mathbb{R}^+$ satisfying the premise of Theorem 1. In addition let $R = r_1 r_2 \ldots$ and $\delta \in \mathbb{R}^+$ and an infinite $B \subseteq V(G)$ be given such that $d(b, r_i) \geq \delta$ for all $b \in B$ and all $i \in \mathbb{N}$. We assume that there is no finite edge separator that separates $R$ from $B$. We now want to find a contraction to Lemma 6. The premise of Lemma 6 is fulfilled by the assumptions of Lemma 14 on $\ell$ and $G$. To find the contraction we need to show that there is an infinite set of vertex disjoint $B - R$ paths in $G$. Using Lemma 13 we can conclude the following structures of $G$: We may assume that there is no subdivided star in $G$ with center in $B$ and leaves in $R$. Even stronger we may assume that there is no vertex $x$ in $G$ such that there is an $N \in \mathbb{N}$ such that $d(x, r_i) \geq \delta'$ for some $\delta' \in \mathbb{R}^+$ for all $i \geq N$ and such that there is no finite edge separator separating $x$ from $R$. This means that we may assume that $B$ is infinite. On the other hand we also know, that no vertex $r_i$ on $R$ is center of a subdivided star with leaves in $B$. And even stronger than that we know that there is no vertex $x$ in $G$ such that $d(x, b) \geq \delta'$for some $\delta' \in \mathbb{R}^+$ and for infinitely many $b \in B$ and such that there is no finite edge separator separating $x$ from the set of all those $b \in B$. Together this yields the following: There is no vertex $v$ in $G$ such that $d(v, b) \geq \varepsilon$ and such $d(v, r_j) \geq \varepsilon'$ for infinitely many $b \in B$ and infinitely many $r_j \in R$ and some $\varepsilon, \varepsilon' \in \mathbb{R}^+$ unless there is a finite edge separator separating $v$ from those infinitely many $b \in B$ and all those infinitely many $r_j \in R$.

Now we can find an infinite set of vertex disjoint $B - R$ paths in $G$. Let $\mathcal{P}$ be any infinite set of edge-disjoint $B - R$ paths in $G$. We can assume that for each vertex of any path $P$ in $\mathcal{P}$ there are only finitely many other paths in $\mathcal{P}$ containing this vertex: Suppose there was a vertex $v$ such that $v$ was contained in infinitely many path in $\mathcal{P}$. Call the set of those paths $\mathcal{P}'$. Let $B' \subseteq B$ be the set of vertices in $B$ that get met by paths in $\mathcal{P}'$ and let $R'$ be the set of vertices in $R$ that gets met by paths in $\mathcal{P}'$. By definition there is no finite edge separator $v$ from $B'$ or $R'$, as $v$ is contained in infinitely many edge disjoint $B' - R'$ paths. As each path in $\mathcal{P}'$ has $\ell$-length at least $\delta$, there must be a $\delta'$ with $0 < \delta' < \delta$ such that $d(v, b) \geq \delta'$ for infinitely many $b \in B'$ or $d(v, r_j) \geq \delta'$ for infinitely many $r_j \in R'$. If neither were to happen, then for $\delta' = \frac{\delta}{3}$ we get that there are infinitely many vertices in $b \in B$ and $r_j \in R$ such that $d(b, r_j) \leq 2\delta' < \delta$, which contradicts $d(b, r_j) \geq \delta$ for all $b \in B$ and all $r_j \in R$. So we conclude that there is such a $\delta'$ such that $d(v, b) \geq \delta'$ or $d(v, r_j) \geq \delta'$ for infinitely many $b \in B$ or $r_j \in R$, respectively. By our above assumptions this cannot happen, as by definition there is no finite edge separator separating $v$ from all those $b \in B$ or all those $r_j \in R$. It follows that there is no such vertex $v$. In the above argument we have seen that each vertex in a path in $\mathcal{P}$ only gets met by finitely many other paths in $\mathcal{P}$. So we can always choose the paths in $\mathcal{P}$ to be vertex disjoint while keeping $\mathcal{P}$ infinite. We now apply the Lemma 6 to $B$ and $R$ and get the desired contraction. □

The two above lemmas have the corollary that given some $\delta \in \mathbb{R}^+$, a vertex $v$ and any infinite set $S$ of vertices such that $d(v, S) \geq \delta$ for all $s \in S$ we find a finite edge separator separating $S$ from $v$.

**Corollary 1.** *Let $G$ be a graph and let $\ell : E(G) \to \mathbb{R}^+$ satisfy the premise of Theorem 1. For every $\delta \in \mathbb{R}^+$ and for every vertex $v$ and every infinite set of vertices $S$ in $G$ such that $d(v, s) \geq \delta$ for every $s \in S$ there is a finite edge set $F$ that separates $v$ from $S$.*

*Proof.* We can assume that $s$ is a vertex of infinite degree, otherwise taking all the edges incident with $v$ for $F$ would be sufficient. We assume that there are infinitely many edge disjoint $v - S$ paths in $G$, but only infinitely many independent $v - S$ paths. Let $\mathcal{P}$ be a set of infinitely many edge disjoint $v - S$ paths. Define $H$ to be the graph that is the union of all the paths in $\mathcal{P}$. Note that $H$ is connected, as every path in $\mathcal{P}$ contains $v$. So we apply the Star-Comb Lemma to $H$ with the vertex set $S$. We cannot get a subdivided star, as for the center of that star, say $c$. There is no finite edge separator separating $c$ from $v$ as $c$ is contained in infinitely many edge disjoint $v - S$ paths, so $d(c, v) = 0$ by assumptions on $\ell$. By triangle inequality $d(v, s) \leq d(v, c) + d(c, s)$ and with $d(c, v) = 0$ this means that $d(c, s) \geq \delta$ for all $s \in S$ which cannot happen by Lemma 13. We cannot get a comb with spine, say $R = r_1 r_2 \ldots$, as there is no finite edge separator separating $R$

from $v$ as $R$ consists of parts of infinitely many edge disjoint $v-S$ paths and each of those paths eventually never again contains another vertex on $R$. This means that $\inf_{i\in\mathbb{N}} d(v, r_i$ is equal to zero by Lemma 9. This means that for $\delta' \leq \frac{\delta}{2}$ there is an $N \in \mathbb{N}$ such that $d(r_i, v) \leq \delta'$. By the triangle inequality follows that $d(r_i, s) \geq \delta'$ for all $i \geq N$ and all $s \in S$, because $d(r_i, s) + d(r_i, v) \geq d(v, s) \geq \delta$ for all $s \in S$. This is a contraction to Lemma 14. □

As corollary we can extend the statement of Corollary 1 to finite equivalence classes of vertices in $||G||$.

**Corollary 2.** *Let $G$ be a graph and let $\ell : E(G) \to \mathbb{R}^+$ satisfy the premise of Theorem 1. Furthermore let $[x]$ in $||G||$ be an equivalence class of $V(G)$ that is containing finitely many vertices and let $\delta \in \mathbb{R}^+$ be given. Let $B$ be an infinite set of vertices such that $d(x, b) \geq \delta$ for all $b \in B$. Then there is a finite edge separator $F$ such that $F$ separates all the vertices in $[x]$ from $B$ in $G$.* □

We will extend Corollary 2 to all equivalence classes in of $G$ in the following lemma. For an equivalence class $[x]$ of $G \cup \Omega(G)$ we say an edge set $F$ separates $[x]$ from some set of vertices in $B$ when one of the following is true: If $[x]$ only contains one end then for every tail of every ray in $\omega$ separates $F$ a tail from that ray from $B$ or if $[x]$ contains vertices, then $F$ separates all the vertices contained in $[x]$ from $B$.

**Lemma 15.** *Let $G$ be a graph and let $\ell : E(G) \to \mathbb{R}^+$ satisfy the premise of Theorem 1. Let $[x]$ in $||G||$ be an equivalence class of $G \cup \Omega(G)$ and let $\delta \in \mathbb{R}^+$ be given. Let $B$ be a set of vertices in $G$ with $d(x, b) \geq \delta$ for all $b \in B$. Then there is a finite edge separator that separates $[x]$ from $B$ in $G$.*

*Proof.* We only need to prove this lemma for equivalence classes $[x]$ that contains infinitely many vertices and the case that $[x]$ is an undominated edge-end $\omega$ by Corollary 2. So first let $[x]$ be an infinite equivalence class and let $X$ be the set of vertices in $[x]$. Let $B$ be a set of vertices such that $d(x, b) \geq \delta$ for all $b \in B$. We may assume that $B$ is infinite by Lemma 13. We have to show that there is a finite edge separator $F$ in $G$ that separates $X$ from $B$. We assume that there is no finite edge separator separating $X$ from $B$ and conclude a contraction. We now set $\delta' = \frac{\delta}{2}$. Let $O_X$ be the union of all open $\delta'$-ball around vertices $x \in X$. Then we add every edge $e = v_1v_2$ such that $e$ is only partially in $O_X$ but $v_1$ and $v_2$ are in $O_X$ to $O_X$.

Let $\mathcal{P}'$ be an infinite set of edge disjoint $B - X$ paths. For each path $P'$ in $\mathcal{P}'$ let $P$ be the subpath of $P'$ from $B$ to the first vertex of $P'$ that is in $O_X$, call the set of those first vertices $N$ and let $\mathcal{P}$ be the set of all those paths $P$. We have seen in Lemma 5 that $O_X$ is connected. We may assume that $N$ is infinite by Corollary 1. So we can apply the Star-Comb Lemma to $O_X$ and with the vertex set $N$. By Lemma 13 we cannot get a subdivided star, with center say $c$, because $d(c, b)$ would be greater than $\delta'$ for all $b \in B$ and as $c$ meets infinitely many edge disjoint $B - X$ paths, there is no finite edge separator separating $c$ from $B$. And analogous by Lemma 14 we cannot get a comb with spine $R = r_1r_2\ldots$ as $d(r_i, b)$ would be greater than $\delta'$ for all $b \in B$ and all $i \in \mathbb{N}$ and as $R$ meets infinitely many edge disjoint $B - X$ paths, there is no finite edge separator separating $R$ from $B$.

Now let $[x]$ only contain a single undominated end $\omega$ in $G$. Let $\delta \in \mathbb{R}^+$ be given and let $B$ be an infinite set of vertices such that $d(b, \omega) \geq \delta$ for all $b \in B$. Pick any ray $R = r_1r_2\ldots$ in $\omega$ such that $d(r_i, \omega) \leq \frac{\delta}{2}$ for each vertex $r_i \in R$. This is possible because every ray has finite $\ell$-lengths. Now every vertex in $B$ has distance at least $\frac{\delta}{2}$ from every vertex in $R$. By Lemma 14, there is a finite edge separator $F$ separating this ray $R$ from each vertex in $B$. Because every other ray in $\omega$ is also equivalent to $R$ and because $\omega$ was undominated this $F$ separates a tail of those other rays from each vertex of $B$. So this $F$ also separates $\omega$ from each vertex of $B$. $\square$

5.3. **Main proof.** In this section we will prove Theorem 1. Before we start the proof we will recall the basic concept of the proof that is already mentioned in Section 4. Let $\widetilde{\Omega}$ be the set of edge-ends of $G$ which are not represented by a vertex in $||G||$. We will use the following steps to prove the theorem. First we will show that $||G|| \setminus \widetilde{\Omega}$ and $(G, \ell)$ have the 'same' points. Secondly we will define a metric $d$ on $||G|| \setminus \widetilde{\Omega}$ by using the metric from $(G, \ell)$. And then thirdly expanding the metric $d$ to all of $||G||$ in a way that makes $d$ induce the existing topology of $||G||$ and makes $||G||$ complete and $||G|| \setminus \widetilde{\Omega}$ dense in it. By the uniqueness of completion this proves the theorem. We use the phrase *'same'* in the following sense: A set of points in $||G||$ and $(G, \ell)$ is the 'same' if some homeomorphism maps those points from $||G||$ to $(G, \ell)$. We now prove Theorem 1.

*Proof.* Let $G = (V, E)$ be a 2-connected graph and $\ell : E \to \mathbb{R}^+$ satisfy: every ray has finite $\ell$-length and the $\ell$-lengths of the spokes of any infinite fan or bundle tend to 0. Let $\widetilde{\Omega} \subseteq \Omega$ be the set of edge-ends of $G$ in $||G||$ whose points are not represented by a vertex of $G$. We have seen in Lemma 10 that there is a bijection between $||G|| \setminus \widetilde{\Omega}$ and $(G, \ell)$ that extends the identity on $G$. Let $d$ be the metric defined in Section 4. Now we the following:

(6) *The metric $d$ induces the same topology on $G$ as* ETOP.

It is easy to see that $d$ and ETOP induce the same topology on inner points of edges. In the following we will assume that $x$ is a vertex or an end. Technically $x$ is an equivalents class of points with distance 0 that got identified, but there is no need for a distinction here for the following reason: We showed that in ETOP and in $||G||$ the same points get identified, that means that when we take an $\varepsilon$-ball around a point within an equivalent class of points, the ball will always contain all elements of that class, since they have distance 0. In addition, in ETOP. there is no way to separate points within such a class by a finite edge set as $d(x, y) = 0$ with $x, y \in V$ means that there are infinitely many edge disjoint $x - y$ paths in $G$. Deleting any finite edge set $F$ will always put all those points within a single component of $G$.

For the first direction we show that:

(7) *For any basic open neighborhood $U$ around a point $x$ in $||G||$ there is a $\varepsilon$-ball contained in $U$ that contains $x$.*

Let any basic open neighborhood $U$ around a point $x$ in $||G||$ be given. Any basic open set in ETOP comes from a finite set $F$ of edges. We choose $\varepsilon' \geq 0$ to be the smaller than $\min_{e \in F} \ell(e)$. It follows that the $\varepsilon'$-ball around $x$ will be disjoint to any point outside of $U$ besides inner points of the edges in $F$ that are outside of $U$. Here we need to be a little more precise and because, as stated above, in ETOP open sets are created by inner points of edges, rather then the edges themselves. So we choose $\varepsilon \leq \varepsilon'$ so that the $\varepsilon$-ball around $x$ will not leave $U$, which can only happen on the edges $F$. But this can be achieved by simply taking $\varepsilon \leq \varepsilon'$ smaller than the minimal distance from $x$ to any inner point of $F$ that was chosen to create $U$ which is possible as $F$ is finite. This finishes the proof of (7). For the other direction we show:

(8) *For any open $\varepsilon$-ball $U$ around a point in $[x]$ in $||G||_d$ there is open set in $||G||$ contained within $U$ that contains $x$.*

So let $U$ be any $\varepsilon$-ball around any point $[x]$ in $||G||_d$. Remember, that $[x]$ is a point in $||G||_d$, but it might be an equivalent class of points before identification. As above, we assume $[x]$ does not contain inner points of edges. We need to find a finite edge set $F$ such that all edges of $F$ are at least partially in $U$ and the component after deletion of $F$ must contain $[x]$ and be contained in $U$. We know by Lemma 15, that there is a finite edge separator separating $[x]$ from every vertex outside of $U$. Now we need to check if there could be edges with both end vertices in $U$ that contain points outside of $U$ that cause problems.

Let $e = v_1v_2$ be an edge which contains a point outside of $U$ but $v_1$ and $v_2$ are both in $U$. To avoid having to make special cases for this kind of edge, we do the following: Let $y$ be an inner point of $e$ outside of $U$. We know change the graph from $G$ to some $G'$ by subdividing $e$ creating a new vertex at the point of $y$, joining $y$ to $v_1$ and $v_2$, let the $e_i$ be the edge from $y$ to $v_i$ for $i \in \{1, 2\}$. We also change $\ell$ such that $\ell(e_1) + \ell(e_2) = \ell(e)$ and $\ell(e_i) = d(y, v_i)$. We do this once for every such edge. In this new graph, there are no edges that have both end vertices in $U$ but contain a point outside of $U$.

We show that $G'$ also complies with the premise of Theorem 1. The graph $G'$ is 2-connected by definition, subdividing edges does not change 2-connectivity. Also by definition there is no ray $R$ in $G'$ of infinite $\ell$-length. Let $v_1$ and $v_2$ be two vertices in $G'$ such that there are infinitely many edge disjoint $v_1 - v_2$ paths in $G'$. First note that because every vertex in $G'$ that is not a vertex in $G$ has degree two, call those vertices *new.* So $v_1$ and $v_2$ are vertices in $G$. So for every $v_1 - v_2$ path $P'$ in $G'$ there is a $v_1 - v_2$ path $P$ in $G$ with $\ell(P) = \ell(P')$, this is again, because all the new vertices have degree two. Similarly, for every infinite fan, the vertex of infinite degree must be in $G$. Let $T$ be the set of vertices on the ray of the fan that gets met by the paths from the vertex of infinite degree. As each vertex of $T$ has degree three, every vertex in $T$ must also be in $G$. So for each path in $G'$ from the vertex of infinite degree to $T$ there is a path in $G$ connecting the same vertices such that the $\ell$-length of those paths is identical.

As the premise of Theorem 1 is fulfilled we can find a finite edge separator $F$ by Lemma 15.

Now we need to take a look at this finite edge separator $F$ we used to separate every vertex and every edge-end outside of $U$ from $[x]$ and find inner points on each of those edges to define an open set in $||G||$. We can assume that every edge $e$ in $F$ has at least one of its end vertices in $U$, say $v_e$. As $U$ is an open $\varepsilon$-ball around $[x]$, we know that $d(v_e, x) < \varepsilon$. Define $\delta := \varepsilon - d(v_e, x)$. Now pick any inner point $y$ of $e$ with $d(v_e, y) < \delta$. Doing this for every edge in $F$ gives us the inner points of finitely many edges needed for the open set in $||G||$. Note that, if an edge $e$ has both of its end vertices in $U$, we do this for both of its end vertices. As $F$ is finite, this will still give us a finite set of inner points of edges. This finished the proof of (8) and hence proves (6).

So we have now shown that the topologies of $||G||$ and $|G|_\ell$ coincide. It is obvious that $(G, \ell)$ is dense in $||G||$ as there is a homeomorphism between the points in $(G, \ell)$ and $||G|| \setminus \widetilde{\Omega}$ with the usual definition of $\widetilde{\Omega}$, and the topological closure of $||G|| \setminus \widetilde{\Omega}$ is exactly $||G||$. It remains to show that $||G||$ is complete. As ETOP is compact [4]. any Cauchy sequence $(x_n)$ in $||G||$ has a limit point $x$ in $||G||$ which is also compact [4, 5, 6]. As no Cauchy sequence can have more than one limit, the sequence converges to $x$. □

## 6. Inducing Mtop

In this section we prove Theorem 2. This will follow the same general structure as the proof for ETOP. Throughout this section an end will always be a vertex-end of $G$. We will write $|G|_{\mathrm{M}}$ for the topology on the point set of $|G|_{\mathrm{M}}$ given by MTOP and $|G|_d$ for the topological space also on the point set of $|G|_{\mathrm{M}}$ which is given by the metric $d$ introduced in Section 4.

6.1. **Lemmas for** MTOP**.** In this subsection we prove the lemmas which help us to prove Theorem 2. Like in Section 5, we start this section by proving that the metric $d$ defined in Section 4 is indeed a metric. We state the definition of $d$ again in Lemma 17. To show that $d$ is a metric we need the lemma that $(G,\ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$ have the same points, i.e. that there is a bijection between $(G,\ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$ that extends the identity on $G$, where $\Omega(G)$ is the set of ends of $G$.

**Lemma 16.** *Let $G$ be a 2-connected graph and $\ell\colon E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 2. Then the identity on $G$ extends to a bijection between $(G,\ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$.*

*Proof.* As both $(G,\ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$ contain all the edges and vertices and no edge or vertex gets identified and neither $(G,\ell)$ nor $|G|_{\mathrm{M}} \setminus \Omega(G)$ contains points that are not in $G$, considered as a 1-complex, this lemma is true. □

The next lemma shows that $d$ is indeed a metric.

**Lemma 17.** *Let $G$ be a 2-connected graph and $\ell\colon E(G) \to \mathbb{R}^+$ a function complying with the premise of Theorem 2. Furthermore let $d$ be the metric on $|G|_M \setminus \Omega(G)$ inherited by the metric $d_\ell$ from $(G,\ell)$. Given an $x \in \Omega(G) \cup V(G)$ and $y \in \Omega(G)$ we define*

$$d(x,y) = \inf \{\ell(R) \mid R \textit{ is an } x-y \textit{ ray or double ray}\},$$

*where $\ell(R) := \sum_{e\in E(R)} \ell(e)$ for any ray or double ray $R$. Then $d$ is a metric on $|G|_{\mathrm{M}}$.*

*Proof.* We first show that:

(9) $d(x,y) = 0$ *if and only if* $x = y$.

By the of definition $d(x,y)$ is 0 if $x = y$. In the following we assume that $x \neq y$ and we show that $d(x,y) \neq 0$. We show that we may assume $x$ and $y$ are not inner points on the same edge as each edge $e$ is a copy of the interval $[0,\ell(e)]$. Suppose that $x$ is an inner point of an edge $e = v_1v_2$ and $y$ is not contained in $e$. Then $d(x,y)$ is at least equal to $\min\{d(x,v_1), d(x,v_2)\}$ by definition of $d$. And as $x$ was an inner point of $e$ we know $\min\{d(x,v_1), d(x,v_2)\} > 0$ which we had to show. Now we assume that neither $x$ nor $y$ are inner points of edges. We may also assume that neither $x$ nor $y$ are vertices. By the premise of Theorem 2 if either $x$ or $y$ is a vertex, say $x$ is a vertex, $d(x,y) \neq 0$, as there is some $\delta_v \in R^+$ such that $\ell(e) \geq \delta_v$ for each edge $e$ incident with $x$.

The only case that is left is that $x$ and $y$ are different ends of $G$. If $x$ and $y$ are different ends of $G$, then there is a finite vertex separator $S$ separating $x$ from $y$. By the premise of Theorem 2, for every vertex $v \in V(G)$ there is $\delta_v \in \mathbb{R}^+$ such that for each edge $e$ incident with $v$ the $\ell$-length of $e$ is greater than $\delta_v$. Let $\delta'$ be the minimum of all $\delta_v$ for this finite vertex separator $S$. As $S$ is finite, this minimum does exist. Each double ray with tails in $x$ and tails in $y$ must go through $S$, so each such double ray has $\ell$-length at least $\delta'$. It follows that $d(x,y) \geq \delta' > 0$. This concludes the proof of (9). Next we show that:

(10) $d(x,y) = d(y,x)$ *for all* $x, y$ *in* $|G|_{\mathrm{M}}$

This follows directly by the definition of $d_\ell$, since any $x - y$ path, ray or double ray is also a $y - x$ path, ray or double ray, respectively. So lastly we show that:

(11) *The triangle inequity holds for $d$.*

This is analogous to the proof for $||G||$, found in Lemma 11. We only consider the case $x, y, z \in \Omega(G)$ as the other cases are easier version of the same proof or follow directly from the fact that in the infimum over all the $x - z$ paths, every path contained in every $x - y - z$ walk[3] is considered.

We show that $d(x, z) \leq d(x, y) + d(y, z) + \varepsilon$ for all $\varepsilon > 0$. Because of the infimum in the definition of $d$ this suffices to show that $d(x, z) \leq d(x, y) + d(y, z)$. Without loss of generality we assume $d(x, z) \neq 0$. Given any $\varepsilon > 0$. By the definition of $d$ we can pick a double ray from $x$ to $y$[4] and one from $y$ to $z$ whose $\ell$-lengths exceed $d(x, y)$ and $d(y, z)$ by at most $\frac{\varepsilon}{3}$, respectively. Otherwise $d(x, y)$ or $d(y, z)$ would be larger, by at least $\frac{\varepsilon}{3}$. Call such double rays $R_{xy}$ and $R_{yz}$. We will find a path with total $\ell$-length less than $\frac{\varepsilon}{3}$ that connects these two double rays. The union of this path and $R_{xy} \cup R_{yz}$ will contain the desired double ray. By Lemma 3 we know that the infimum over the $\ell$-length of all $R_{xy} - R_{yx}$ paths is 0. Hence there is a path $P_{xy}$ with $\ell$-length at most $\frac{\varepsilon}{3}$ that connects $R_{xy}$ and $R_{yz}$. It follows that there is a double ray $R_{xz}$ from $x$ to $z$ with $\ell$-length less than $\varepsilon + d(x, y) + d(y, z)$.[5] So $d$ is indeed a metric on $|G|_{\mathrm{M}}$. □

In the next lemma we use Lemma 6 to prove that for every end $\omega$ and every infinite set $B$ of vertices such that all vertices in $B$ has distance at least some $\delta \in \mathbb{R}^+$ from $\omega$, there is a finite vertex separator $S$ that separates $\omega$ from $B$. As long as the separator $S$ is finite, $S$ may contain all of $B$.

**Lemma 18.** *Let $G$ be a 2-connected graph and $\ell$ be a length function complying with the premise of Theorem 2 and let $\omega$ be an end of $G$. For every set $B$ of vertices and every $\delta \in \mathbb{R}^+$ such that $d(\omega, b) \geq \delta$ for all $b \in B$ there is a finite $S \subset V(G)$ such that $S$ separates $\omega$ from $B$.*

*Proof.* Let $\delta \in \mathbb{R}^+$, an end $\omega$ of $G$ be given and let $B$ be a set of vertices such that there is no finite vertex separator separating $\omega$ from $B$ and such that $d(b, \omega) \geq \delta$ for all $b \in B$. We may assume that $B$ is infinite, as if $B$ were a finite set of vertices, taking every vertex in $B$ for $S$ would be sufficient. Now we need to find a finite set $S$ of vertices that separates $B$ from $\omega$. We assume that there is no finite vertex separator separating $B$ from $\omega$ and find a contraction by using Lemma 6. If there is no finite vertex separator separating $B$ from $\omega$, then for every ray $R$ in $\omega$ there also is no finite vertex separator separating every tail of $R$ from $B$. As every ray in $\omega$ as finite $\ell$-length, it has a tail such for each vertex $r$ in this tail $d(r, b) \geq \frac{\delta}{2}$ for all $b \in B$. Let $R = r_1 r_2 \ldots$ be such a ray and suppose that there is no finite vertex separator separating $B$ from $\omega$. This means there are infinitely many vertex disjoint $R - B$ paths in $G$. The premise of Lemma 6 is fulfilled as graph $G$ is 2-connected and $\ell$ complies by our initial assumptions. So we can apply Lemma 6 with $B$ and $R$ as by assumption there are infinitely vertex disjoint $B - R$ paths in $G$. This yields the desired contraction. □

We now state a corollary that later, when we are checking whether $|G|_{\mathrm{M}}$ and $|G|_d$ are inducing the same open sets around ends ensures that inner points of edges are not a problem. We do this the same way we did in the ETOP case by changing $G$ a little by subdividing edges that might cause a problem.

**Corollary 3.** *Let $G$ be a 2-connected metrizable graph and $\ell$: $E(G) \to \mathbb{R}^+$ comply with the premise of Theorem 2. For an end $\omega$ of $|G|_{\mathrm{M}}$ and any $\delta \in \mathbb{R}^+$ and any set of edges such that each of these edges contains an inner point $x$ such that $d(x, \omega) \geq \delta$ there is a finite vertex separator that separates all those edges from $\omega$.*

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[3]A *walk* is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges such that $e_i = v_i v_{i+1}$ for all $i < k$, an $x - z - y$ walk is s a walk containing $x, y, z$ in that order[2].

[4]A double ray from $x$ to $y$ is a double ray such that it has tails that belongs to $x$ and $y$, respectively

[5]Start in $x$ follow $R_{xy}$ to the first vertex on $P_{xy}$, use $P_{xy}$ to $R_{yz}$ and continue on $R_{yz}$ towards $z$.

*Proof.* Let $G$ and $\ell$ fulfill the premise of this lemma and let $\omega$ be an end in $|G|_{\mathrm{M}}$. Let $F$ be a set of edges such that each edge contains a point $x$ with $d(x, \omega) \geq \delta$ for some $\delta \in \mathbb{R}^+$. We may assume that $F$ is infinite, otherwise we can just take all the vertices incident with the edges in $F$ for the finite separator. In addition we may assume that each of the edges in $F$ contains atleast one vertices in with distance less than $\delta$ from $\omega$ by Lemma 18. We now set $\varepsilon$ to be equal to $\frac{\delta}{2}$. Let $F_0 \subseteq F$ be the set of edges in $F$ such that each edge in $F_0$ does not have contain a vertex with distance greater $\varepsilon$ from $\omega$. We now pick a first finite vertex separator $S_0$ by Lemma 18 that separates all the edges in $F_0$ from $\omega$. We do this by applying Lemma 18 to $\omega$ with the distance $\varepsilon$ and for the vertex set we use all vertices contained in edges in $F_0$. Let $F_1$ be all the edges in $F$ that are not in $F_0$. By definition of $F_0$ all the edges in $F_1$ have $\ell$-length at least $\varepsilon$. We set $\delta'$ to be equal to $\frac{\delta}{4}$.

Let $e \in F_1$ and let $\delta_e \in \mathbb{R}^+$ be the minimum of the lower bounds of $\ell$-lengths of edges incident with the end vertices of $e$[6], We now know the following:

(1) There is a point $x_{e_0}$ on $e$ such that $d(x_e, \omega) \geq \delta$.
(2) There is a point $x_e$ on $e$ such that $d(x_e, \omega) \geq \frac{3\delta}{4}$.
(3) The minimum distance[7] from $x_e$ to the end vertices of $e$ is at least $\delta'\delta_e$[8].

We get 2. by using 1. There is a point $x_{e_0}$ on $e$ with distance at least $\delta$ from $\omega$ and as the $\ell$-length of $e$ is at least $\frac{\delta}{2}$ we can conclude 2. We now show that:

(12) *There is finite vertex set $S_1$ that separates $F_1$ from $\omega$.*

We may assume that $F_1$ is infinite, otherwise taking all the end vertices of the edges in $F_1$ will be a viable choice for $S_1$. To proof (12) we define a 2-connected graph $G'$ with the same ends as $G$ and a length function $\ell'$ which comply with the premises of Theorem 2 by subdividing the edges in $F_1$ at those points $x_e$ and then using Lemma 18 to find a vertex separator $S_1'$ that separates the new vertices from $\omega$. As $F_1$ is infinite $S_1'$ can only contain finitely many new vertices so we can use $S_1'$ to find the desired set $S_1$ in $G$. Let $G'$ be the graph we obtain by subdividing every edge in $F_1$ at the point $x_e$. Let the subdividing vertices for the edge $e$ of $G$ be $v_e$, we call all those vertices $v_e$ *new* and all the other vertices of $G'$ *old*. As subdividing edges does not change 2-connectivity $G'$ is 2-connected. Note that every new vertex has degree two. For every edge $e = v_1v_2$ of $G$, with say $d(v_1, x) \leq d(x, v_2)$, we set $\ell'(v_1v_x) = d(v_1, x)$ and $\ell'(v_xv_2) = \ell(v_1v_2) - \ell(v_1v_x)$. It is straight forward to check that every ray in $G'$ has finite $\ell'$-length. In addition it is clear that $G'$ and $G$ have the same ends. By the definition of $F_1$ we know that for every vertex $v$ of $G'$ that is also a vertex of $G$ the product $\delta'\delta_v$ is a lower bound of the $\ell$-lengths of all the edges incident with $v$ in $G'$ where $\delta_v$ is the lower bound of the $\ell$-lengths of all the edges incident with $v$ in $G$ which exists by the initial assumptions on $\ell$. Lastly we check if the $\ell'$-lengths of the open spokes of any infinite fan in $G'$ tend to 0. Let any infinite fan in $G'$ be given and let $y$ be the vertex of infinite degree in this fan and $R'$ be the ray of this fan. Note that the vertex $y$ is also a vertex of $G$ as it has more than degree two. The same is true for the vertices in $R'$ that get met by spokes of this infinite fan. Because $G'$ is a subdivision of $G$ this means that paths in $G'$ between old vertices directly correspond to paths in $G$ between the same vertices. In addition the the $\ell'$ length of any path between old vertices in $G'$ has by definition of $\ell'$ is at most equal to the $\ell$-length of the path before the subdivision. Together this yields that the $\ell'$-lengths of the open spokes of any infinite fan in $G'$ tend to 0. By Lemma 18 there is a finite vertex separators in $G'$ that separates all new vertices from $\omega$, call it $S_1'$. We now use $S_1'$ to define a

[6]Such exists by our initial assumptions on $\ell$
[7]Measured by the metric $d$
[8]Note that $\delta'$ is fixed for all edges in $F$

separator $S_1$ in $G$. For $S_1$ we star by taking all the old vertices of $S_1'$. For every new vertex in $S_1'$ we add both its neighbors in $G'$ to $S_1$, as those are old vertices by definition of $G'$. Now $S_1$ separates all the edges in $F_1$ from $\omega$, which proves (12) and hence also finishes this proof. □

6.2. **Main proof.** In this section we prove Theorem 2.

*Proof.* Let $G = (V, E)$ be a a 2-connected metrizable graph and $\ell : E \to \mathbb{R}^+$ be a length function complying with the premise of Theorem 2. By Lemma 16 we know that the identity on $G$ extends to a bijection between $(G, \ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$. Let $d$ be the metric induced by $d_\ell$ on $|G|_{\mathrm{M}} \setminus \Omega(G)$. We extend this metric like in Lemma 17 to all of $|G|_{\mathrm{M}}$. Now we show that

(13) *The metric $d$ induces the same topology on $G$ as* MTOP.

It is easy to see, that $d$ and MTOP induce the same topology on inner points of edges, as this is just the standard topology on intervals. In the following we will assume that $x$ is a vertex or an end.

(14) *The basic open neighborhoods given by $|G|_{\mathrm{M}}$ around any point $x$ contain an open $\varepsilon$-ball that also contains $x$.*

Let any basic open neighborhood $U$ around a point $x$ in $|G|_{\mathrm{M}}$ be given.. In MTOP. there are two types of basic open set. The first one are the open stars with a radius $\varepsilon' \in (0, 1)$ around vertices, the others are open sets around ends that come from a finite vertex separator. If $U$ is in the first class of open sets, as the $\ell$-length of all edges incident with any given vertex are bounded by below by some $\delta \in \mathbb{R}^+$, we can chose an $\varepsilon$ small enough such that in $|G|_d$, an $\varepsilon$-ball around $x$ is contained in $U$. If $U$ is of the second class of basic open sets, then there is a finite vertex separator $S$ that separates $U$ from every point in $|G|_{\mathrm{M}}$ outside of $U$. Let $\varepsilon'$ be the length of the constant used to define $U$, such that for every edge from $U$ to $S$ the first open $\varepsilon'$ interval belongs to $U$. Let $\delta \in \mathbb{R}^+$ be such that the $\ell$-length of all the edges incident with the vertices of $S$ is at least $\delta$. This must exist by assumption. Now let $\varepsilon$ be smaller than $\delta$ and small enough, such that the $\varepsilon$-ball in $|G|_d$ is in $U$. This concludes the proof of (14). For the other direction

(15) *The basic open $\varepsilon$-balls around a point $x$ contain an open set of $|G|_{\mathrm{M}}$ which also contains $x$.*

Let $U$ be any $\varepsilon$-ball around any point $x$ in $|G|_d$. As above, we assume $x$ is not an inner points of an edges. We start by showing that we may assume that $x$ is not an end. If $x$ is an end then by Lemma 18 and Corollary 3 there is a finite vertex separator that separates every vertex and every inner points of an edge outside of $U$ from $x$. Let $\delta$ be the minimum of all the lower bounds for those finitely many vertices such that each edge incident with those vertices has $\ell$-length at least $\delta$. There must be an $\varepsilon$ small enough, that taking an $\varepsilon$-collar of all the edges leaving the component containing $x$, after the deletion of this finite vertex separator, such that all points in this $\varepsilon$-collar are in $U$. So the component that contains $x$ after deletion of this finite vertex separator together with this $\varepsilon$-collar forms an open set contained in $U$.

So we can assume that $x$ is a vertex. If $x$ is a vertex then $U$ is an open $\varepsilon$-ball around $x$. As the $\ell$-lengths of all edges incident with $x$ are bounded from below by some $\delta$, we can find an $\varepsilon' > 0$ small enough, such that an open $\varepsilon'$-star around $x$ is contained in $U$. This concludes the proof of (15). Together (14) and (15) proof (13). So we have seen, that $d$ induces the existing topology of $|G|_{\mathrm{M}}$. We have also shown, that the topologies of $|G|_{\mathrm{M}}$ and $|G|_d$ coincide. To follow the same steps as the ETOP proof we have to show that $(G, \ell)$ is dense in $|G|_{\mathrm{M}}$. This is obvious as there is a homeomorphism between the points in $(G, \ell)$ and $|G|_{\mathrm{M}} \setminus \Omega(G)$ and the topological closure of $|G|_{\mathrm{M}} \setminus \Omega(G)$ is exactly $|G|_{\mathrm{M}}$. It remains to show that

(16) *The space $|G|_{\mathrm{M}}$ is complete.*

For that we call two Cauchy sequences $(x_i)_{i\in\mathbb{N}}$ and $(v_j)_{j\in\mathbb{N}}$ equivalent if for every $\delta \in \mathbb{R}^+$ there is an $N \in \mathbb{N}$ such that $|x_i - v_i| \leq \delta$. If one Cauchy sequence of two equivalent Cauchy sequence converges to some point $a$, then the other Cauchy sequence also converges to the same point $a$.

To show (16), let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence of points in $|G|_{\mathrm{M}}$. We may assume that $(x_i)_{i\in\mathbb{N}}$ does not become constant at some point, as then this point is the converging point of this sequence. As $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence, for every $\varepsilon \in \mathbb{R}^+$ there is an $N \in \mathbb{N}$ such that $d(x_i, x_j) \leq \varepsilon$ for all $i, j \geq N$. We now show that we can make some assumptions on this sequence.

We may assume that the sequence $(x_i)_{i\in\mathbb{N}}$ does not contain infinitely many inner points of the same edge, since then this edge would contain an accumulation point as every edge is a homeomorphic copy of $[0, 1]$, which is compact. And as each Cauchy sequence only contains a single accumulation point, the sequence would converge.

Next we may assume that the sequence contains at most one point within each edge, so there is no edge such that there are two or more points of the sequence of that edge. We may do so, as every subsequence of a Cauchy sequence is a Cauchy sequence, and if this subsequence converges, so does the main sequence, as it is a Cauchy sequence.

(17) *We may assume that all the $x_i$ are vertices or ends.*

Suppose that infinitely many $x_i$ are inner points of edges, say $x_i$ lies on the edge $v_i w_i$, and say $d(v_i, x_i) \leq d(x_i, w_i)$. Now we look at the sequence of $(v_i)_{i\in\mathbb{N}}$. We show that $d(v_i, v_j) \leq 2d(x_i, x_j)$. Say $d(x_i, x_j) = \delta$, then

$$d(v_i, v_j) \leq d(v_i, x_i) + d(x_i, x_j) + d(v_j, x_j)$$

This is the 'path' $v_i$ to $x_i$, then going from $x_i$ to $x_j$ and then from $x_j$ to $v_j$. We have $d(v_i, x_i) + d(v_j, x_j) \leq \delta$ by definition of $(v_i)_{i\in\mathbb{N}}$, so $d(v_i, v_j) \leq 2\delta$. This shows that $(v_i)_{i\in\mathbb{N}}$ is a Cauchy sequence and it shows that this sequence is equivalent to $(x_i)_{i\in\mathbb{N}}$. This finishes our argument for (17).

If $(x_i)_{i\in\mathbb{N}}$ contains infinitely many vertices, we let $(v_i)_{i\in\mathbb{N}}$ be the subsequence of $(x_i)_{i\in\mathbb{N}}$ that only contains vertices. As $(x_i)_{i\in\mathbb{N}}$ and $(v_i)_{i\in\mathbb{N}}$ are equivalent, it is sufficient to show that $(v_i)_{i\in\mathbb{N}}$ converges. So we may assume that $(x_i)_{i\in\mathbb{N}}$ either contains no ends or only finitely many vertices. We now make a case study of the two above possibilities.

(1) Suppose that $(x_i)_{i\in\mathbb{N}}$ contains only finitely many vertices. We define another sequence $(v_i)_{i\in\mathbb{N}}$ that is equivalent to $(x_i)_{i\in\mathbb{N}}$ but only contains vertices. For that we may assume that $(x_i)_{i\in\mathbb{N}}$ only contains ends. We may do so because deleting a finite segment of a Cauchy sequence does not change whether the new sequence is a Cauchy sequence nor if it converges or to what point it converges. We define the sequence $(v_i)_{i\in\mathbb{N}}$ as follows: For every $x_i$ in $(x_i)_{i\in\mathbb{N}}$ we choose a ray $R$ in $G$ that belongs to $x_i$. By definition of $d$ there is a vertex $v_i$ on $R$ such that $d(v_i, x_i) \leq \frac{1}{i}$. If there were an end $x_j$ such that this fails to happen, then every vertex on the ray $R$ has distance at least $\frac{1}{j}$ from $x_j$. But this means one of the following two has to occur: Either that there is another ray $R' = r'_1 r'_2 \dots$ in $x_j$. In which case there is an $K \in \mathbb{N}$ such that each vertex $d(r'_k, r) \geq \frac{1}{2j}$ for

all $k \geq K$ and all $r \in R$, which yields a contraction by Lemma 3. Or that $d(x_j, r_i) \geq \frac{1}{2j}$ for all rays in $x_j$, but this cannot happen by definition of $d$. It follows that $d(x_i, v_i) \leq \frac{1}{i}$. So let some $\varepsilon' \in \mathbb{R}^+$ be given. We choose $K_1$ greater than $\frac{1}{2\varepsilon'}$, this yields that $d(x_i, v_i) \leq \frac{\varepsilon'}{2}$ for all $i \geq K_1$. Now we choose $K_2 \geq K_1$ such that $d(x_i, x_j) \leq \frac{\varepsilon'}{2}$ for all $i \geq K_2$. Together this yields that $d(x_i, v_j) \leq \varepsilon'$ for all $i, j \geq K_2$ by the triangle inequality. This means that $(v_i)_{i\in\mathbb{N}}$ is a Cauchy sequence that is equivalent to $(x_i)_{i\in\mathbb{N}}$.

(2) We assume that $(x_i)_{i\in\mathbb{N}}$ only contains vertices. We now path $P_i$ in $G$ from $x_i$ to $x_{i+1}$ such that $\ell(P_i) \leq 2d(x_i, x_{i+1})$ for every $i \in \mathbb{N}$. This is possible by the definition of $d$. Let $V_i$ be the set of vertices contained in all the paths $P_1, \ldots, P_i$. As those are finitely many, there is a $\delta_i \in \mathbb{R}^+$ such that each edge incident with any of those vertices has $\ell$-length at least $\delta$ by our assumptions on $\ell$. For for every $i \in \mathbb{N}$ we choose an $N_i \in \mathbb{N}$ such that all paths $P_j$ with $j \geq N_i$ never meet $V_i$ again. We can do this because the $\ell$-length of all the edges incident with vertices in $V_i$ is bounded from below and the fact that $(x_i)_{i\in\mathbb{N}}$ is a Cauchy sequence and $\ell(P_i) \leq 2d(x_i, x_{i+1})$. Let $H$ be the graph that is the union of all the path $P_i$. There is no vertex of infinite degree in $H$ by the argument that for each vertex $v_H$ in $H$ there are only finitely many paths $P_i$ that meet $v_H$, but $H$ is connected by construction. We apply the Star-Comb Lemma to $H$ with the vertex set $X$, where $X$ is all the vertices in $(x_i)_{i\in\mathbb{N}}$. We now show, that if the spine of the comb is $R = r_1 r_2 \ldots$ then for every $\varepsilon \in \mathbb{R}^+$ there is a $K \in \mathbb{N}$ such that $d(r_i, x_i) \leq \varepsilon$ for all $i \geq K$. Let $\varepsilon$ be given, we choose $K_1$ to be large enough such that $d(x_i, x_j) \leq \frac{\varepsilon}{4}$ for all $i, j \geq K_1$. We now choose $K_2 \geq K_1$ such that for each vertex $r_j, j \geq K_2$, the ray $r_j r_{j+1} \ldots$ has $\ell$-length less than $\frac{\varepsilon}{4}$. By construction of $R$, there is a $K_3 \geq K_2$ such that each path in $P_i$ that meets the ray $r_j r_{j+1} \ldots$, with $j \geq K_3$, has $\ell$-length less than $\frac{\varepsilon}{4}$. This means that $d(r_i, x_j) \leq \varepsilon$ for all $i, j \geq K_3$. The path from $x_j$ to $R$ has $\ell$-length at most $\frac{\varepsilon}{4}$, moving up along $R$ means moving at most $\frac{\varepsilon}{4}$ and by definition of $P_i$, the path $P_i$ has $\ell$-length at most $\frac{\varepsilon}{4}$, in sum this means that $d(r_i, x_j) \leq \varepsilon$ for all $i, j \geq K_3$. We know that every ray $R$ has finite $\ell$-length that there is a unique point in $|G|_{\mathrm{M}}$ to which $R$ converges. So the sequence $(x_i)_{i\in\mathbb{N}}$ converges to the same point.

This finishes up the proof of Theorem 2. □

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