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MacLane's planarity criterion for locally finite graphs

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Abstract

MacLane's planarity criterion states that a finite graph is planar if and only if its cycle space has a basis B such that every edge is contained in at most two members of B. Solving a problem of Wagner [Graphentheorie, Bibliographisches Institut, Mannheim, 1970], we show that the topological cycle space introduced recently by Diestel and Kühn allows a verbatim generalisation of MacLane's criterion to locally finite graphs. This then enables us to extend Kelmans' planarity criterion as well.

Keywords: Infinite graphs; Planarity; Cycle space

1. Introduction

For a (possibly infinite) graph G, define the *finite-cycle space* to be the set of all symmetric differences of finitely many finite circuits (the edge sets of finite 2-regular connected subgraphs). We denote this $\mathbb{Z}_2$-vector space by $\mathcal{C}_{\text{fin}}(G)$. A set (or family) $\mathcal{E}$ of edge sets $E \subseteq E(G)$ is called *simple*, if every edge of G lies in at most two elements of $\mathcal{E}$. MacLane's planarity criterion states:

Theorem 1 (*MacLane [11]*). *A finite graph G is planar if and only if $\mathcal{C}_{\text{fin}}(G)$ has a simple generating set.*

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Wagner [14] raised the question if MacLane's result could be extended so that it characterises planar graphs which are infinite. Rather than modifying the planarity criterion, Thomassen [13] describes all infinite graphs that satisfy MacLane's condition. For this, recall that a *vertex accumulation point*, abbreviated VAP, of a plane graph Γ is a point p of the plane such that every neighbourhood of p contains an infinite number of vertices of Γ.

Theorem 2 (*Thomassen [13]*). *Let G be an infinite 2-connected graph. Then G has a VAP-free embedding in the plane if and only if $\mathcal{C}_{\text{fin}}(G)$ has a simple generating set consisting of finite circuits.*

Bonnington and Richter [2] also provide a generalisation of MacLane's theorem using the *even cycle space* $\mathcal{Z}(G)$, defined as the set of all subgraphs of G with all vertex degrees even. With this space they investigate which graphs have an embedding with k VAPs.

Our main result in this paper is a verbatim generalisation of MacLane's theorem to locally finite graphs:

Theorem 3. *Let G be a countable locally finite graph. Then, G is planar if and only if $\mathcal{C}(G)$ has a simple generating set.*

Here, $\mathcal{C}(G)$ denotes the cycle space, called the *topological cycle space*, introduced for infinite graphs by Diestel and Kühn [7,8]. Its basic constituents are not just finite circuits but arbitrary topological circles (i.e. homeomorphic images of the unit circle) in the space consisting of G together with its ends.

The space $\mathcal{C}(G)$ will be formally defined in Section 2. Let us remark here only that this notion of a cycle space has allowed a number of natural or verbatim generalisations of all the following basic results for finite graphs. Namely, that the cycle space is generated by the fundamental circuits of every spanning tree; that every element of the cycle space meets every cut in an even number of edges; that every element of the cycle space is a disjoint union of circuits; and that the cycle space in a 3-connected graph is generated by the peripheral circuits (Tutte's generating theorem). See [5] for a gentle introduction and discussion of the topological cycle space.

We discuss our main result in Section 3 after introducing the necessary definitions in the next section. In Section 4 we investigate some properties of simple generating sets. The main result will be proved in the course of Sections 5 and 6. Finally, in Section 7, we extend Kelmans' planarity criterion to locally finite graphs.

2. Definitions and basic facts

The basic terminology we use can be found in [6]. All graphs in this paper are simple and undirected. Let $G = (V, E)$ be a fixed graph.

A 1-way infinite path is called a *ray*, and a 2-way infinite path is a *double ray*. The subrays of (double) rays are their *tails*. The *ends* of G are the equivalence classes of rays under the following equivalence relation: two rays in G are equivalent if no finite set of vertices separates them.

We define a topology on G together with its ends, i.e. our topological space consists of all vertices, all inner points of edges and all ends of G. On G the topology will be that of a 1-complex. Thus, the basic open neighbourhoods of an inner point on an edge are the open intervals on the edge containing that point, while the basic open neighbourhoods of a vertex x are the unions of half-open intervals containing x, one from every edge at x. For every end ω and any finite set $S \subseteq V$ there is exactly one component $C = C(S, \omega)$ of $G - S$ which contains a tail of every ray in ω. We say that ω *belongs* to C, and write $\overline{C}(S, \omega)$ for the component C together with all the ends belonging to it. Then the basic open neighbourhoods of an end ω are all sets of the form

$$\hat{C}(S, \omega) := \overline{C}(S, \omega) \cup E'(S, \omega),$$

where $E'(S, \omega)$ is any union of half-edges $(z, y] \subseteq e$, one for every edge $e = xy$ with $x \in S$, $y \in C$ (where z lies in the interior of e, that is $z \in \mathring{e}$). This topological space will be denoted by $|G|$. (For locally finite graphs G, $|G|$ is the Freudenthal compactification of G.) We shall freely view G either as an abstract graph or as a subspace of $|G|$, i.e. the union of all vertices and edges of G with the usual topology of a 1-complex. It is not difficult to see that if G is connected and locally finite, then $|G|$ is compact.

For any subset $X \subseteq |G|$, put $V(X) := X \cap V$, and let $E(X)$ be the set of edges e with $e \subseteq X$. If Z is an edge set write $\overline{Z}$ for the closure of $\bigcup Z$. A subset of $|G|$ which is homeomorphic to the unit interval $[0, 1]$ is called an *arc*. The images of 0 and of 1 are its *endpoints*. A set $C \subseteq |G|$ is a *circle* if it is a homeomorphic image of the unit circle. The next lemma will be of help when dealing with arcs and circles.

Lemma 4 (*Diestel and Kühn [8]*). *For every arc A and every circle C in $|G|$ the following is true*:

(i) *the sets $A \cap G$ and $C \cap G$ are dense in A and C, respectively*;
(ii) *every arc in $|G|$ whose endpoints are vertices or ends, and every circle C in $|G|$, includes every edge of G of which it contains an inner point*; *and*
(iii) *if x is a vertex in $\mathring{A}$ (respectively in C), then A (respectively, C) contains precisely two edges or half-edges of G at x.*

Therefore, $\overline{E(C)} = C$ for a circle $C \subseteq |G|$. We call the edge set $E(C)$ a *circuit*. Clearly, this definition includes traditional finite circuits but also allows infinite circuits. Such an infinite circuit D is the disjoint union of the edge sets of double rays whose ends fit together nicely.

We say a family $\mathcal{E}$ of subsets of E is *thin*, if no vertex is incident with infinitely many of the elements in $\mathcal{E}$. Define the sum $\sum_{F \in \mathcal{E}} F$ of the thin family $\mathcal{E}$ to be the set of edges appearing in exactly an odd number of $F \in \mathcal{E}$. Note that in locally finite graphs a family is thin if and only if every edge appears in at most finitely many of its members. Thus, in locally finite graphs, thin families are exactly those for which we can decide whether an edge appears in an even or an odd number of the family members.

We define the *topological cycle space* $\mathcal{C}(G)$ of G as the set of all sums over thin families of (finite or infinite) circuits. With the symmetric difference (denoted by $\triangle$) taken as addition, $\mathcal{C}(G)$ is a $\mathbb{Z}_2$-vector space. In a finite graph G, the topological cycle space $\mathcal{C}(G)$ and the finite-cycle space $\mathcal{C}_{\text{fin}}(G)$ coincide. A theorem in [8] states that $\mathcal{C}(G)$ is closed under taking thin sums. Thus, thin sums seem to be completely natural in our context, and we shall henceforth tacitly assume that every sum is thin.

3. Discussion of main result

First let us make the notion of a generating set more precise. A *generating set* of the topological cycle space will be a set $\mathcal{F} \subseteq \mathcal{C}(G)$ such that every element of $\mathcal{C}(G)$ can be written as a thin sum of elements of $\mathcal{F}$. Thus, in contrast to a generating set in the vector space sense we allow (thin) infinite sums. There are two reasons for this. First, thin sums are integral to the topological cycle space of an infinite graph, so it seems unnatural to forbid them. Second, MacLane's criterion is false if we insist that every $Z \in \mathcal{C}(G)$ is a finite sum of elements of a simple subset of $\mathcal{C}(G)$, as we shall see in Proposition 8.

To show that, in a certain sense, our main result, Theorem 3, is as strong as possible, we need the following theorem, which is of interest on its own. It will be proved in Section 4. For a circle $C \subseteq |G|$, call the circuit $E(C)$ *peripheral* if the subgraph $C \cap G$ of the graph G is induced and non-separating.

Theorem 5. *Let G be a 3-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of circuits. Then every element of $\mathcal{F}$ is a peripheral circuit.*

First note that because of the following theorem, if the topological cycle space has a simple generating set then it also has a simple generating set consisting of circuits.

Theorem 6 (*Diestel and Kühn [8]*). *Every element of the topological cycle space of a graph is a disjoint union of circuits.*

Theorem 3 is formulated for locally finite graphs, and indeed it is false for arbitrary infinite graphs. Indeed, consider the 3-connected graph G in Fig. 1, which is not locally finite. By Theorem 5 and the remark following it, we may assume that a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consists of peripheral circuits (finite or infinite). In particular, no circuit which contains the edge e is in $\mathcal{F}$. But then such a circuit cannot be generated by any sum of circuits of $\mathcal{F}$. Thus, there is no simple generating set of $\mathcal{C}(G)$, but G is clearly planar.

Infinite circuits might seem, although topologically natural, combinatorially unwieldy. They are, however, inevitable in a certain sense: there is not always in a planar graph a simple generating set comprised of only finite circuits. Consider the graph G in Fig. 2, and

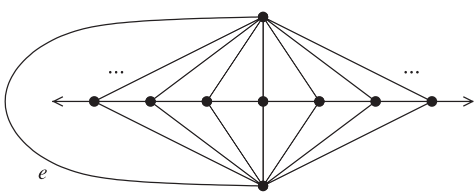

Fig. 1. A planar graph whose topological cycle space has no simple generating set.

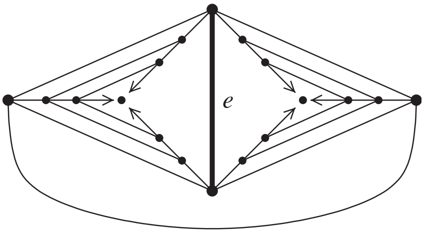

Fig. 2. A locally finite graph without a simple generating set of *finite* circuits.

suppose there is a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consisting of finite circuits. Since G is 3-connected, every $C \in \mathcal{F}$ is, by Theorem 5, a peripheral circuit. Now, if a finite circuit C contains the edge e then the subgraph consisting of the edges in C with their incident vertices is clearly separating, and thus C not peripheral. Consequently, the circuits in $\mathcal{F}$ are not even sufficient to generate every finite circuit (namely any one containing e).

There is another compelling reason for allowing infinite circuits in a generating set: for a locally finite graph G they are natural constituents of a reasonable notion of the cycle space. In Diestel [5] it is shown that, if one wants to generalise simultaneously all of the basic results for a cycle space we mentioned at the end of Section 1, one needs a cycle space which contains at least all of $\mathcal{C}(G)$.

4. Simple generating sets

As a tool, we introduce the notion of a 2-basis. For this, let $\mathcal{B} \subseteq \mathcal{C}(G)$ be a simple generating set of the topological cycle space of G. We call $\mathcal{B}$ a 2-*basis* of $\mathcal{C}(G)$ if for every element $Z \in \mathcal{C}(G)$ there is a unique (thin) subset of $\mathcal{B}$, henceforth denoted by $\mathcal{B}_Z$, with $Z = \sum_{B \in \mathcal{B}_{\mathbb{Z}}} B$. Observe that in a finite graph the 2-bases are exactly the simple bases of $\mathcal{C}(G)$, and thus conform with the traditional definition of a 2-basis in a finite graph.

Since we have left linear algebra with our definition of a 2-basis (allowing thin infinite sums), it is not clear if the properties usually expected of a basis are still retained. One of these, which we shall need later on, is that a generating set always contains a basis. For simple sets this is true:

Lemma 7. *Let G be a 2-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$. If $\mathcal{F}$ is not a 2-basis, then for any $Z \in \mathcal{F}$ the set $\mathcal{F} \setminus \{Z\}$ is a 2-basis of $\mathcal{C}(G)$.*

Proof. Observe first that it suffices to check the uniqueness required in the definition of a 2-basis for the empty set: a simple generating subset $\mathcal{B}$ of $\mathcal{C}(G)$ is a 2-basis if and only if for every $\mathcal{B}' \subseteq \mathcal{B}$ with $\sum_{B \in \mathcal{B}'} B = \emptyset$ it follows that $\mathcal{B}' = \emptyset$.

Let us assume there is a non-empty set $\mathcal{D} \subsetneq \mathcal{F}$ with $\sum_{B \in \mathcal{D}} B = \emptyset$. Since G is 2-connected every edge of G appears in a finite circuit, and thus in at least one element of $\mathcal{F}$. But as $\mathcal{F}$ is simple and $\sum_{B \in \mathcal{D}} B = \emptyset$ no edge of G can lie in an element of $\mathcal{D}$ and at the same time in an element of $\mathcal{F} \setminus \mathcal{D}$.

So, $E_1 := \bigcup \mathcal{D}$ and $E_2 := \bigcup(\mathcal{F} \setminus \mathcal{D})$ define a partition of $E(G)$ (note that both sets are non-empty). Because G is 2-connected there is, by Menger's theorem, for any two edges a finite circuit through both of them. Therefore, there is a circuit D which shares an edge e_1 with E_1 and another edge e_2 with E_2. Let $\mathcal{D}' \subseteq \mathcal{F}$ be such that $D = \sum_{B \in \mathcal{D}'} B$. Then $D' := \sum_{B \in \mathcal{D} \cap \mathcal{D}'} B \subseteq D$, since for any edge $e \in D' \setminus D$ both $\mathcal{D}' \setminus \mathcal{D}$ and $\mathcal{D} \cap \mathcal{D}'$ have an element which contains e; thus $e \in E_1 \cap E_2$, which is impossible. Therefore, D' is a subset of the circuit D, and thus either $D' = \emptyset$ or $D' = D$. Since $e_1 \in D'$ the former case is impossible; the latter, however, is so too, as $D' \subseteq E_1$ cannot contain $e_2 \in E_2$, a contradiction.

We thus have shown:

$$\sum_{B \in \mathcal{D}} B = \emptyset \text{ for } \mathcal{D} \subseteq \mathcal{F} \text{ implies } \mathcal{D} = \emptyset \text{ or } \mathcal{D} = \mathcal{F}.$$

So, if $\mathcal{F}$ is not a 2-basis, then none of its subsets but itself generates the empty set. In particular, $\mathcal{F}$ is thin. For any $Z \in \mathcal{F}$,

$$Z = \sum_{B \in \mathcal{F} \setminus \{Z\}} B,$$

thus, the thin simple set $\mathcal{F} \setminus \{Z\}$ certainly generates the topological cycle space. It also is a 2-basis, as none of its non-empty subsets generates the empty set. □

With our definition of a generating set, which allows infinite sums, we shall show that MacLane's criterion holds for locally finite graphs. Since, in a vector space context, one usually allows only finite sums for a generating set, there is one obvious question: Does Theorem 3 remain true if we consider simple generating sets in the vector space sense? The answer is a strikingly clear no:

Proposition 8. *There is no locally finite* 2*-connected infinite graph in which the topological cycle space has a simple generating set in the vector space sense* (*i.e. allowing only finite sums*).

Proof. Suppose there is such a graph G so that $\mathcal{C}(G)$ has a simple set $\mathcal{A} \subseteq \mathcal{C}(G)$ which generates every $Z \in \mathcal{C}(G)$ through a finite sum. We determine the cardinality of $\mathcal{C}(G)$ in two ways.

First, since $\mathcal{A}$ is simple, every of the countably many edges of G lies in at most two elements of $\mathcal{A}$. Therefore, $\mathcal{A}$ is a countable set, and thus, $\mathcal{C}(G)$ also.

Second, there is, by Lemma 7, a 2-basis $\mathcal{B} \subseteq \mathcal{A}$. As $\mathcal{C}(G)$ is an infinite set (since G is infinite and 2-connected), so is $\mathcal{B}$. Hence, there are distinct $B_1, B_2, \ldots \in \mathcal{B}$. Also, as G is locally finite and $\mathcal{B}$ simple, all subsets of $\mathcal{B}$ are thin. Therefore, all the sums

$$\sum_{i \in I} B_i \text{ for } I \subseteq \mathbb{N}$$

are distinct elements of $\mathcal{C}(G)$. Since the power set of $\mathbb{N}$ has uncountable cardinality, it follows that $\mathcal{C}(G)$ is uncountable, a contradiction. □

The rest of this section is devoted to the proof of Theorem 5, which we restate:

Theorem 5. *Let G be a 3-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of circuits. Then every element of $\mathcal{F}$ is a peripheral circuit.*

A basic tool when dealing with finite circuits are bridges, see for instance Bundy and Murty [1]. As our circuits may well be infinite, we need an adaption of the notion of a bridge, which we introduce together with a number of related results before proving the theorem.

Definition 9 (*Bruhn [3]*). Let $C \subseteq |G|$ be a circle in a graph G. We call the closure B of a topological component of $|G| \setminus C$ a *bridge* of C. The points in $B \cap C$ are called the *attachments* of B in C.

There is a close relationship between bridges and peripheral circuits. Indeed, in a 3-connected graph a circuit D is peripheral if and only if the circle $\overline{D}$ has a single bridge [3].

For the subgraph $H := C \cap G$, the following can be shown: a set $B \subseteq |G|$ is a bridge of C if and only if it is induced by a chord of H or if there is a component K of $G - H$ such that B is the closure of K plus the edges between K and H together with the incident vertices. Thus, our definition coincides with the traditional definition of a bridge in a finite graph.

Lemma 10 (*Bruhn [3]*). *Let $C \subseteq |G|$ be a circle in a graph G, and let B be a bridge of C. Let x be an attachment of B. Then*:

(i) *x is a vertex or an end*;
(ii) *if x is an end then every neighbourhood of x contains attachments of B that are vertices*;
(iii) *every edge of which B contains an inner point lies entirely in B*; *and*
(iv) *either B is induced by a chord of C or the subgraph $(B \cap G) - V(C)$ is non-empty and connected.*

We define a *residual arc* of the bridge B in the circle C to be the closure of a topological component of $C \setminus B$. Note that if B has at least two attachments every residual arc is indeed an arc (if not then the circle C itself is a residual arc, and it is the only one).

Lemma 11 (*Bruhn [3]*). *Let G be a 2-connected graph, and let $C \subseteq |G|$ be a circle with a bridge B. Then*:

(i) *the endpoints of a residual arc L of B in C are attachments of B*; *and*
(ii) *for a point $x \in C \setminus B$ there is exactly one residual arc L of B in C containing x.*

We say a bridge B of C *avoids* another bridge B' of C if there is a residual arc of B that contains all attachments of B'. Otherwise, they *overlap*. Note that overlapping is a symmetric relation. Two bridges B and B' of C are called *skew* if C contains four (distinct) points v, v', w, w' in that cyclic order such that v, w are attachments of B and v', w' attachments of B'. Clearly, if two bridges B and B' are skew, they overlap. On the other hand, in a 3-connected graph, overlapping bridges are either skew or 3-*equivalent*, i.e. they both have only three attachments which are the same:

Lemma 12. *Let G be a 3-connected graph. Let $C \subseteq |G|$ be a circle, and let B and B' be two overlapping bridges of C. Then B and B' are either skew or 3-equivalent.*

Proof. First, if either B or B' is induced by a chord, it is easy to see, that they are skew because they overlap. Thus, by Lemma 10(iv), we may assume that each of the bridges has three attachments. Next, assume that $B \cap C = B' \cap C$. If $|B \cap C| = 3$ then B and B' are 3-equivalent, otherwise they are clearly skew.

So, suppose there is an attachment u of B with $u \notin B'$. The attachment u is contained in a residual arc L of B'. Its endpoints u', v' are attachments of B'. Since B and B' are overlapping, not all attachments of B may lie in L. Thus, there is an attachment $v \in C \setminus L$ of B. Then, the sequence u, u', v, v' shows that B and B' are skew. □

For a set $X \subseteq |G|$, an X-path is a path that starts in X, ends in X and is otherwise disjoint from X.

Lemma 13. *Let B and B' be two skew bridges of a circle $C \subseteq |G|$ in a graph G. Then there are two disjoint C-paths $P = u \ldots v$ and $P' = u' \ldots v'$ such that u, u', v, v' appear in that order on C.*

Proof. Since B and B' are skew there are points x, x', y, y' appearing in that cyclic order on C such that x, y are attachments of B and x', y' are attachments of B'. If x is a vertex put $u := x$. If not, then there is a whole arc $A \subseteq C$ around x disjoint from any of the other points. In A we find, by Lemma 10(ii), an attachment u of B that is a vertex. Doing the same for x', y and y', if necessary, we end up with vertices u, u', v, v' appearing in that cyclic order on C such that $u, v \in B$ and $u', v' \in B'$. As $(B \cap G) - V(C)$ is connected, by Lemma 10(iv), we find an u–v path P through B, and analogously an u'–v' path P' through B'. Since bridges meet only in attachments, P and P' are disjoint. □

We need that in a 3-connected graph, for any circle, there are always two overlapping bridges (if there is more than one bridge at all). For this, we define for a circle C in the graph G the *overlap graph* of C in G as the graph on the bridges of C such that two bridges are adjacent if and only if they overlap. The next lemma ensures that there are always overlapping bridges.

Lemma 14 (*Bruhn [3]*). *For every circle C in a 3-connected graph G the overlap graph of C in G is connected.*

The next simple lemma will be used repeatedly in the proof of Theorem 5.

Lemma 15. *Let G be a 3-connected graph, and let $\mathcal{B}$ be a 2-basis of $\mathcal{C}(G)$ consisting of circuits. Let C and D be circuits in G such that $\overline{C} \cap \overline{D}$ is an arc. Suppose that $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Then, either $\mathcal{B}_C \subseteq \mathcal{B}_D$ or $\mathcal{B}_D \subseteq \mathcal{B}_C$.*

Proof. Put $K := \sum_{B\in\mathcal{B}_C\cap\mathcal{B}_D} B$ and consider an edge $e \notin C \cup D$. Then both $\mathcal{B}_C$ and $\mathcal{B}_D$ contain either both or none of the at most two circuits $B \in \mathcal{B}$ with $e \in B$. Thus, both or none of them is in $\mathcal{B}_C \cap \mathcal{B}_D$, and hence $e \notin K$. Therefore, K is an element of the topological cycle space contained in $C \cup D$. These are precisely $\emptyset$, C, D and $C + D$ (since $\overline{C} \cap \overline{D}$ is an arc). Note that $K \neq \emptyset$ as $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Also, $K \neq C + D$, since otherwise

$$\mathcal{B}_C \cap \mathcal{B}_D = \mathcal{B}_K = \mathcal{B}_{C+D} = \mathcal{B}_C \triangle \mathcal{B}_D,$$

which is impossible. Consequently, we obtain either $K = C$ and thus, $\mathcal{B}_C \subseteq \mathcal{B}_D$, or $K = D$ and $\mathcal{B}_D \subseteq \mathcal{B}_C$. □

Proof of Theorem 5. Note that it suffices to prove the theorem for a 2-basis $\mathcal{B}$. Indeed, if $\mathcal{F}$ is not a 2-basis, consider two distinct elements Z_1 and Z_2 of $\mathcal{F}$. By Lemma 7, both $\mathcal{F} \setminus \{Z_1\}$ and $\mathcal{F} \setminus \{Z_2\}$ are a 2-basis of $\mathcal{C}(G)$, and, if Theorem 5 holds for these, it clearly also holds for $\mathcal{F}$.

Consider a non-peripheral circuit C. Then, the circle $\overline{C}$ has more than one bridge [3]. Two of these, B and B' say, are, by Lemma 14, overlapping. By Lemma 12, they are either skew or 3-equivalent. We show that $C \notin \mathcal{B}$ for each of the two cases.

(i) Suppose that B and B' are skew. By Lemma 13, there are two disjoint $\overline{C}$-paths $P = u \dots v$ and $P' = u' \dots v'$ such that u, u', v, v' appear in this order on $\overline{C}$. Denote by $L_{uu'}, L_{u'v}, L_{vv'}, L_{v'u}$ the closures of the topological components of $\overline{C} \setminus \{u, u', v, v'\}$ such that x, y are the endpoints of L_{xy}. Define the circuits

$$\begin{aligned} &C_1 := E(L_{uu'} \cup L_{u'v} \cup P), \qquad &&C_2 := E(L_{vv'} \cup L_{v'u} \cup P),\\ &D_1 := E(L_{u'v} \cup L_{vv'} \cup P') \text{ and } &&D_2 := E(L_{v'u} \cup L_{uu'} \cup P'). \end{aligned}$$

Observe that $C_1 + C_2 = C = D_1 + D_2$, and additionally, that $\overline{C_i} \cap \overline{D_j}$ is an arc for any $i, j \in \{1, 2\}$.

Suppose $C \in \mathcal{B}$. Since

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} = \mathcal{B}_{C_1+C_2} = \mathcal{B}_C = \{C\},$$

not both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{C_2}$ may contain C. As the same holds for D_1 and D_2 we may assume that

$$C \notin \mathcal{B}_{C_1} \text{ and } C \notin \mathcal{B}_{D_1}. \tag{1}$$

Consider an edge $e \in C_1 \cap D_1 \subseteq C$. Both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ must contain a circuit which contains e. By (1), this cannot be C. Therefore, and since $\mathcal{B}$ is simple, $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ contain the same circuit K with $e \in K$. Consequently, $\mathcal{B}_{C_1} \cap \mathcal{B}_{D_1} \neq \emptyset$, and applying Lemma 15 we may assume that

$$\mathcal{B}_{C_1} \subseteq \mathcal{B}_{D_1}. \tag{2}$$

Now, consider an edge $e' \subseteq L_{uu'}$, hence $e \in C_1 \cap D_2$. There is a circuit $K' \in \mathcal{B}_{C_1}$ with $e' \in K' \neq C$. By (2), $K' \in \mathcal{B}_{D_1}$, but since e' lies in $L_{uu'}$ we have $e' \notin D_1$. Thus, $\mathcal{B}_{D_1}$ also contains the other circuit in $\mathcal{B}$ that contains e', which is C, a contradiction to (1). Therefore, $C \notin \mathcal{B}$.

(ii) Suppose that B and B' are 3-equivalent. Let v_1, v_2, v_3 be their attachments, which then are vertices (by Lemma 10(ii)). Then there is a vertex $x \in V(B \setminus C)$ and three x–$\overline{C}$ paths $P_i = x \dots v_i \subseteq B, i = 1, 2, 3$ whose interiors are pairwise disjoint. Let $Q_i = y \dots v_i$ be analogous paths in B'. The closures of the topological component of $\overline{C} \setminus \{v_1, v_2, v_3\}$ are three arcs; denote by $L_{i,i+1}$ the one that has v_i and v_{i+1} as endpoints (where indices are taken mod 3). For $i = 1, 2, 3$, define the circuits

$$C_i := E(L_{i,i+1} \cup P_i \cup P_{i+1}) \text{ and } D_i := E(L_{i,i+1} \cup Q_i \cup Q_{i+1}).$$

Note that $C_1 + C_2 + C_3 = C = D_1 + D_2 + D_3$.

Now suppose $C \in \mathcal{B}$. As

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} \triangle \mathcal{B}_{C_3} = \mathcal{B}_{C_1+C_2+C_3} = \mathcal{B}_C = \{C\},$$

either C lies in all of the $\mathcal{B}_{C_i}$ or in only one of them, in $\mathcal{B}_{C_3}$, say. In both cases, we have $C \notin \mathcal{B}_{C_1+C_2}$. We obtain the same result for the D_i: either C lies in all of the $\mathcal{B}_{D_i}$ or in only one of them. In any case, we can define D as either D_1 or $D_2 + D_3$ such that $C \notin \mathcal{B}_D$. Put $D' := C + D$, and note that $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$.

Then, since $C_1 + C_2$ shares an edge in C with D, and neither $\mathcal{B}_{C_1+C_2}$ nor $\mathcal{B}_D$ contains C, we have $\mathcal{B}_{C_1+C_2} \cap \mathcal{B}_D \neq \emptyset$. Applying Lemma 15, we obtain that one of the two sets $\mathcal{B}_{C_1+C_2}, \mathcal{B}_D$ is contained in the other.

First assume that $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_D$, and consider an edge $e \in C$ that lies in both $C_1 + C_2$ and D'. Such an edge exists since $D' = D_1$ or $D' = D_2 + D_3$. Since $C \notin \mathcal{B}_{C_1+C_2}$, e lies in a circuit $K \neq C$ in $\mathcal{B}_{C_1+C_2}$, and thus also $K \in \mathcal{B}_D$. On the other hand, $e \in C \in \mathcal{B}_{D'}$ contradicts $e \in D'$.

So, we may assume that $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2}$. Because $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$ we even have $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \cap \mathcal{B}_{D'}$. Thus, by Lemma 15, either $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'}$ or $\mathcal{B}_{D'} \subseteq \mathcal{B}_{C_1+C_2}$. The latter is impossible as $C \notin \mathcal{B}_{C_1+C_2}$. Therefore, we obtain

$$\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}.$$

Now, from $C \notin \mathcal{B}_{C_1+C_2}$ follows that $\mathcal{B}_{C_1+C_2} = \mathcal{B}_D$, contradicting $C_1 + C_2 \neq D$. Thus, $C \notin \mathcal{B}$. □

5. The backward implication

In this section, we show the backward implication of our main result, namely that if the topological cycle space has a simple generating set then G is planar. But first, let us remark that it is sufficient to show Theorem 3 for 2-connected graphs. Indeed, the Kuratowski planarity criterion for countable graphs below asserts that a countable graph is planar if and only if its blocks are planar.

Theorem 16 (*Dirac and Schuster [10]*). *Let G be a countable graph. Then, G is planar if and only if G contains neither a subdivision of K_5 nor a subdivision of $K_{3,3}$.*

The backward direction will follow from the next lemma.

Lemma 17. *Let G be a* 2*-connected graph such that* $\mathcal{C}(G)$ *has a* 2*-basis, and let* $H \subseteq G$ *be a finite* 2*-connected subgraph. Then* $\mathcal{C}(H)$ *has a* 2*-basis.*

Proof. Let $\mathcal{B}$ be the 2-basis of $\mathcal{C}(G)$. Since H is finite, there are $Z \in \mathcal{C}(H)$ with a non-empty generating set $\mathcal{B}_Z \subseteq \mathcal{B}$ which is $\subseteq$-minimal among all B_Z with $Z \in \mathcal{C}(H)$. Let us denote these by $Z_1, \ldots, Z_k$.

Consider a $D \in \mathcal{C}(H)$ with $\mathcal{B}_D \cap \mathcal{B}_{Z_i} \neq \emptyset$ for some i. We claim that $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$. First, note that

$$C := \sum_{B \in \mathcal{B}_D \cap \mathcal{B}_{Z_i}} B \subseteq E(H).$$

Indeed, consider an edge $e \notin E(H)$. Since $Z_i, D \subseteq E(H)$, and since $\mathcal{B}$ is simple, e either lies on exactly two or on none of the elements of $\mathcal{B}_{Z_i}$, and the same holds for $\mathcal{B}_D$. Furthermore, if e lies on two elements of $\mathcal{B}_{Z_i}$ and on two of $\mathcal{B}_D$, these must be the same. So, $e \notin C$.

Therefore, $C \subseteq E(H)$, and thus $C \in \mathcal{C}(H)$. As $\mathcal{B}_C \subseteq \mathcal{B}_{Z_i}$ we obtain, by the minimality of $\mathcal{B}_{Z_i}$, that $C = Z_i$. Consequently, $\mathcal{B}_{Z_i} = \mathcal{B}_C \subseteq \mathcal{B}_D$, as claimed.

This result also implies $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for all $1 \leqslant i < j \leqslant k$. Thus, every edge of H appears in at most two of the Z_i. Furthermore, we claim that $\{Z_1, \ldots, Z_k\}$ is a generating set for $\mathcal{C}(H)$. Then, $\{Z_1, \ldots, Z_k\}$ contains a 2-basis of $\mathcal{C}(H)$, and we are done.

So consider a $D \in \mathcal{C}(H)$, and let I denote the set of those indices i with $\mathcal{B}_{Z_i} \cap \mathcal{B}_D \neq \emptyset$. We may assume $I = \{1, \ldots, k'\}$ for a $k' \leqslant k$. Then, by $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$ and $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for $i, j \in I$, it follows that $\mathcal{B}_D$ is the disjoint union of the sets $\mathcal{B}_{Z_1}, \mathcal{B}_{Z_2}, \ldots, \mathcal{B}_{Z_{k'}}$ and

$$\mathcal{B}' := \mathcal{B}_D \setminus \bigcup_{i=1}^{k'} \mathcal{B}_{Z_i}.$$

Consequently,

$$\sum_{B \in \mathcal{B}'} B = \sum_{B \in \mathcal{B}_D} B + \sum_{B \in \mathcal{B}_{Z_1}} B + \cdots + \sum_{B \in \mathcal{B}_{Z_{k'}}} B = D + Z_1 + \cdots + Z_{k'} \subseteq E(H)$$

as all the summands lie in H. Now, if $\mathcal{B}' \neq \emptyset$ then there is a $Z \in \mathcal{C}(H)$ with a non-empty and minimal $\mathcal{B}_Z \subseteq \mathcal{B}'$ which then must be one of the Z_i, a contradiction. Thus, $\mathcal{B}'$ is empty and we have $D = \sum_{i=1}^{k'} Z_i$. $\square$

For the backward implication of Theorem 3, we use the well-known fact that the cycle space of every subdivision of K_5 or of $K_{3,3}$ fails to have a 2-basis (see, for instance Diestel [6]).

Lemma 18. *Let G be a locally finite* 2*-connected graph such that* $\mathcal{C}(G)$ *has a simple generating set. Then G is planar.*

Proof. Suppose not. Then G contains, by Theorem 16, a subdivision H of K_5 or of $K_{3,3}$ as subgraph. By Lemma 7, $\mathcal{C}(G)$ has a 2-basis. Then, by Lemma 17, $\mathcal{C}(H)$ also has a 2-basis, which is impossible. $\square$

6. The forward implication

To show the forward implication of Theorem 3, i.e. that the topological cycle space of a planar graph has a simple generating set, we proceed as in the finite case: we embed our graph G in the sphere and then show that the set of the face boundaries' edge sets is a simple generating set. So, our first priority is to ensure that every face is indeed bounded by a circle of $|G|$. As for the backward direction we may assume that G is 2-connected.

This, however, is certainly not the case when a VAP of the embedded graph coincides with a vertex or an inner point of an edge. To avoid this problem we consider topological embeddings of the space $|G|$ in the sphere (rather than graph embeddings of G), which, in our context, is no restriction:

Theorem 19 (*Richter and Thomassen [12]*). *Let G be a locally finite 2-connected planar graph. Then $|G|$ embeds in the sphere.*

We call a topological space 2-*connected* if it is connected and remains so after the deletion of any point. Thus, any embedding of the (standard) compactification $|G|$ of a 2-connected graph G in the sphere clearly is 2-connected. Note that also, any such embedding is compact if G is locally finite and connected. A *face* of a compact subset K of the sphere is a component of the complement of K. A face boundary $\partial f \subseteq K$ of a face f is simply the boundary of f. If K is the image of $|G|$ under an embedding, then it can be shown in a similar way as for finite plane graphs (see for instance [6]) that if an inner point of an edge lies in a face boundary then the whole edge lies in it.

Theorem 20 (*Richter and Thomassen [12]*). *Every face of a compact 2-connected locally connected subset of the sphere is bounded by a simple closed curve.*

Another result of Richter and Thomassen [12] states that $|G|$ is locally connected if G is locally finite and connected. [1] As a simple closed curve by definition is homeomorphic to the unit circle, we obtain:

Corollary 21. *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \to S^2$. Then the face boundaries of $\varphi(|G|)$ are circles of $|G|$.*

Showing the forward implication, we now complete the proof of Theorem 3.

Lemma 22. *Let G be a locally finite 2-connected planar graph. Then, $\mathcal{C}(G)$ has a simple generating set.*

Proof. By Theorem 19, $|G|$ has an embedding $\varphi : |G| \to S^2$ in the sphere. Put $\Gamma := \varphi(|G|)$. We show that the set $\mathcal{F}$ which we define to consist of the edge sets of the face boundaries of Γ, is a simple generating set of $\mathcal{C}(G)$. Certainly, $\mathcal{F}$ is simple, and, by Corollary

[1] They show this to be true for all *pointed* compactifications of G, which are those obtained from the standard compactification by identifying some ends.

21, a subset of $\mathcal{C}(G)$. So, we only have to prove that every element of the topological cycle space is the sum of certain elements of $\mathcal{F}$.

Fix a face f^* of Γ. First, consider a circuit C in G. Then for the circle $\overline{C}$, $\varphi(\overline{C})$ is homeomorphic to the unit circle and, thus, bounds two faces (by the Jordan-curve theorem). Let f_C be the face not containing f^*. As G is 2-connected, every edge e lies on a finite circuit, and therefore on the boundaries of exactly two faces of Γ, which we denote by f_e and f'_e. Hence, the set

$$\mathcal{B}^C := \{E(\partial f) \ : \ f \subseteq f_C \text{ is a face of } \Gamma\}$$

is thin. Moreover, as we have $f_e, f'_e \subseteq f_C$ or $f_e, f'_e \nsubseteq f_C$ if and only if $e \notin C$, it follows that

$$\sum_{B \in \mathcal{B}^C} B = C. \tag{3}$$

Now, consider an arbitrary element Z of the topological cycle space. By definition, there is a thin family $\mathcal{D}$ of circuits with $Z = \sum_{C \in \mathcal{D}} C$. If none of the elements of $\mathcal{F}$ appears in $\mathcal{B}^C$ for infinitely many $C \in \mathcal{D}$, then the family $\mathcal{B}$, which we define to be the (disjoint) union of all $\mathcal{B}^C$ with $C \in \mathcal{D}$, is thin (since every edge lies on exactly two face boundaries). Then, $Z = \sum_{B \in \mathcal{B}} B$, and we are done. Therefore, if $F(\Gamma)$ is the set of faces of Γ, it suffices to show that the set

$$F := \{f \in F(\Gamma) \ : \ f \subseteq f_C \text{ for infinitely many } C \in \mathcal{D}\}$$

is empty.

So suppose $F \neq \emptyset$. By definition of f_C, we have $f^* \nsubseteq f_C$ for all $C \in \mathcal{D}$, and thus also $F \neq F(\Gamma)$. Hence, there is an edge e such that one of its adjacent faces, say f_e, lies in F and the other, f'_e, in $F(\Gamma) \setminus F$. Then, $E(\partial f_e)$ appears in infinitely many $\mathcal{B}^C$ while e lies on only finitely many $C \in \mathcal{D}$. Thus, also $E(\partial f'_e)$ lies in infinitely many $\mathcal{B}^C$, which implies $f'_e \in F$, a contradiction. □

Let us finally remark that with dual graphs as defined in [4] it is possible to extend MacLane's criterion to graphs that satisfy

no two vertices are joined by infinitely many edge-disjoint paths. (*)

which is a slightly larger class of graphs than the locally finite graphs. This extension necessitates that we work in an amended version of the cycle space $\mathcal{C}(G)$, which is described in Diestel and Kühn [9]. Indeed, using duality the forward direction of Theorem 3 for graphs that satisfy (*) can be shown in a similar way as for finite graphs, while the backward direction can be proved as detailed in Section 5.

7. Kelmans' planarity criterion

For finite 3-connected graphs there is another well-known planarity criterion, namely Kelmans' criterion [2] (see Kelmans [15]). It follows from MacLane's criterion together with Tutte's generating theorem. The latter is known to be true for locally finite graphs:

Theorem 23 (*Bruhn [3]*). *Let G be a locally finite* 3*-connected graph. Then the peripheral circuits generate the topological cycle space.*

We know from Corollary 21 that the face boundaries of a locally finite 2-connected planar graph are circles. When G is 3-connected then, as for finite graphs (see [6]), the Jordan curve theorem implies that these circles are precisely the closures in $|G|$ of the peripheral circuits of G:

Lemma 24. *Let G be a locally finite* 3*-connected graph with an embedding* $\varphi : |G| \to S^2$ *in the sphere. Then, the face boundaries are precisely the closures in* $\varphi(|G|)$ *of the peripheral circuits of G.* □

Now, we easily obtain a verbatim generalisation of Kelmans' criterion for locally finite graphs.

Theorem 25. *Let G be a locally finite 3-connected graph. If G is planar then every edge appears in exactly two peripheral circuits. Conversely, if every edge appears in at most two peripheral circuits then G is planar.*

Proof. If G is planar then there is, by Theorem 19, also an embedding of $|G|$, in which, by Lemma 24, the closure of every peripheral circuit is a face boundary. Since G is 2-connected every edges lies in exactly two face boundaries, hence in exactly two peripheral circuits of G.

For the backward implication let $\mathcal{F}$ be the set of all peripheral circuits of G, which then is simple. Thus, $\mathcal{F}$ is, by Theorem 23, a simple generating set, and hence G planar, by Theorem 3. □

As MacLane's planarity criterion, Kelmans', too, fails when infinite circuits are prohibited. Indeed, there are 3-connected non-planar graphs in which every edge lies on at most two finite peripheral circuits. The graph G shown in Fig. 3 is such an example. It consists of a $K_{3,3}$ (bold) to which three disjoint infinite 3-ladders are added. First observe that any finite peripheral circuit that contains edges of $G - \{u, w\}$ cannot contain any edge incident with either one of u, w, as otherwise it also contains (the edges of) a finite $\{x, y, z\}$–$\{x, y, z\}$ path in $G - \{u, w\}$, and thus is separating. Therefore, every finite peripheral circuit of G

[2] This criterion is sometimes also known as Tutte's planarity criterion, as it easily follows from Tutte's generating theorem. However, this seems to have gone unnoticed until Kelmans published a direct proof of the criterion neither using MacLane's nor Tutte's theorem.

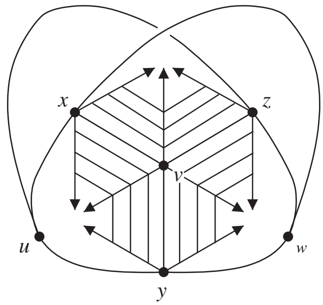

Fig. 3. Infinite circuits are necessary for Kelmans' criterion.

has either none or all of its edges incident with $\{u, w\}$; in the latter case, it is a circuit of $G[u, w, x, y, z]$.

Now, assume that there is an edge of G that appears in three finite peripheral circuits. All of these circuits then lie either in $G - \{u, w\}$ or in $G[u, w, x, y, z]$, where they are also peripheral. Now, it is easy to check that none of the edges of the finite graph $G[u, w, x, y, z]$ lies on three peripheral circuits, and by Theorem 25 this is also impossible for any edge of the planar 3-connected graph $G - \{u, w\}$. This shows that Kelmans' criterion fails if only finite circuits are admitted.

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