SIMPLICIAL TREE-DECOMPOSITIONS OF INFINITE GRAPHS II
The Existence of Prime Decompositions

Reinhard Diestel
Department of Pure Mathematics
University of Cambridge
16 Mill Lane
Cambridge, England

Abstract. In this second of three papers on the same subject we obtain a first characterization of the countable graphs that admit a simplicial tree-decomposition into primes.

This paper is a continuation of [4]. The problem it tackles, the existence of simplicial tree-decompositions into primes, was introduced in [4], and all the notation used is explained there. To repeat just the most basic definition, we say that if $\sigma > 0$ is an ordinal and $F = (B_\lambda)_{\lambda<\sigma}$ a family of induced subgraphs of a graph G, then F forms a *simplicial tree-decomposition* of G, if

(S1) $G = \bigcup_{\lambda<\sigma} B_\lambda$;

(S2) $\left(\bigcup_{\lambda<\mu} B_\lambda\right) \cap B_\mu =: S_\mu$ is a complete graph for each μ $(0 < \mu < \sigma)$;

(S3) no S_μ contains B_μ or any other B_λ $(0 \le \lambda < \mu < \sigma)$;

(S4) each S_μ is contained in B_λ for some $\lambda < \mu$ $(\mu < \sigma)$.

A graph is called *prime* if it has no such decomposition into more than one factor. A simplicial tree-decomposition in which all factors are prime is a simplicial tree-decomposition *into primes*, or a *prime decomposition*.

The bulk of this paper is devoted to proving one theorem: a characterization of the countable graphs that admit a simplicial tree-decomposition into primes (Theorem 1). The slightly more general problem to characterize the graphs admitting any simplicial decomposition into primes was posed by Halin in 1964 [8]. It has since stood unresolved, although a sufficient condition [8] and some necessary conditions (Dirac [7]) have been known. Our solution is independent of [8] and [7].

Section 1 contains the proof of Theorem 1. In Section 2 we give two examples illustrating the theorem. In Section 3 we briefly present a result based on Theorem 1, which will be proved in a forthcoming paper [6]. This result characterizes the decomposable graphs in a Kuratowski-like fashion, by two forbidden 'simplicial' minors.

1. The Existence of Prime Decompositions: A Characterization Theorem

The following theorem is the main result of this paper.

Theorem 1. *A countable graph G has a simplicial tree decomposition into primes if and only if G satisfies the following condition:*

(†) *If (C, S) is a side in G and C' is a component of $G \backslash S$, $C' \neq C$, then $S[C']$ has a prime extension into C.*

Before we think about proving Theorem 1, let us illustrate condition (†) by noting two properties it implies.

Lemma 2. *Let G be a graph that satisfies (†). Then G has the following two properties:*

(i) *If (C, S) is an inaccessible side of S in G, then (C, S) is the only side of S in G.*

(ii) *A simplex $S \subset G$ has an inaccessible side if and only if S is maximally prime and attached in G.*

Proof. (i) Let (C, S) be an inaccessible side in G, and C' a component of $G \backslash S$, $C' \neq C$. If (C', S) is also a side in G, i.e. if S is attached to C', then (†) implies that S has a prime extension into C. This contradicts the assumption that (C, S) is inaccessible.

(ii) If S is maximally prime and attached in G, say to C, then (C, S) is clearly an inaccessible side. Conversely, if (C, S) is inaccessible and $\overline{S} \supset S$ is maximally prime in G, then clearly $\overline{S} \cap C = \emptyset$. But $\overline{S}$ cannot have a vertex in any other component of $G \backslash S$ either, because then S would be attached to this component, contradicting (i). Hence $\overline{S} = S$, i.e. S is maximally prime in G. □

The proof that (†) is necessarily satisfied by any graph that has a prime decomposition (countable or not), is an easy application of [4, Theorem 3.2]. Let G be a graph, suppose that $(B_\lambda)_{\lambda<\sigma}$ is a prime decomposition of G, and let C, S and C' be given as in (†). Let $\overline{S} \supset S$ be a maximal simplex in G. Suppose first that $\overline{S} = S$, i.e. that S is itself a maximal simplex. Then [4, Theorem 3.2] applies directly to S. If $\Lambda(S)$ is finite, then S has a prime extension into C by (ii). If $\Lambda(S)$ is infinite, then $\lambda(C) < \sup^+\Lambda(S)$ by (iii). Moreover, S is not attached to C' (also by (iii)), so $S[C'] \subsetneqq S$ and $\Lambda(S[C'])$ is finite. Choose $\mu \in \Lambda(S)$ in such a way that $\mu > \lambda(C)$ and $\mu > \max \Lambda(S[C'])$. Then $B_\mu \supset S|_\mu \supset S[C']$ and $B_\mu \cap C \neq \emptyset$ [4, Lemma 1.6 (i)–(ii)], so B_μ is a prime extension of $S[C']$ into C.

Suppose now that $\overline{S} \neq S$, i.e. that $\overline{S} \backslash S \neq \emptyset$. Let $\overline{C}$ denote the component of $G \backslash S$ that contains $\overline{S} \backslash S$. If $\overline{C} = C$, we are done. But if $\overline{C} \neq C$, then C is also a component of $G \backslash \overline{S}$, and $\overline{S}[C] = S \subsetneqq \overline{S}$. Therefore [4, Theorem 3.2], applied to $\overline{S}$, ensures the existence of a prime extension of S into C.

This completes the necessity part of the proof.

We now show that a countable graph G has a prime decomposition if it satisfies (†).

The proof will be organized as follows. We shall consider a maximal family $F = (B_\lambda)_{\lambda<\sigma}$ of subgraphs of G, subject to a number of conditions. This family will trivially form a prime decomposition of its union G', and we shall have to show that it covers the entire graph G.

The conditions imposed on the families considered will serve two purposes. Firstly, they will ensure that each B_λ conforms to the basic requirements for factors in a prime decomposition, e.g. that each B_λ is maximally prime in G and unattached (cf. [4, Theorem 1.10]). Secondly, the conditions will have to ensure that at each 'stage' $G|_\mu$ in the construction of F such new subgraphs are indeed available and can be considered for selection as B_μ; this will restrict the choice of factors at earlier stages. From a more global point of view, we can say that the conditions have to organize the order in which certain factors are selected, so as to avoid 'traps' of the kind illustrated in [4, Section 4]. The organization of this order will depend on the structure of G as well as on a fixed enumeration of its vertices.

Before we can get down to the main part of the proof (as loosely described above), we have to make some preparations.

We begin by developing a tool for assessing the positions of potential factors within G: we define a partial order on the set of all sides in G, i.e. of all pairs (C,S) where $S \subset G$ is a simplex, C a component of $G \backslash S$, and S is attached to C.

We shall then use this concept to show that a prime extension of $S\,[\,C'\,]$ into C as in (†) can in fact always be chosen maximally prime and unattached (Lemmas 4,6,7)—which gives us a stronger version of (†) needed to find (unattached) new factors in our construction of a decomposition of G.

If G is a graph, let $\mathcal{H} := \mathcal{H}(G)$ denote the set of all sides in G, i.e. the set of all pairs (C,S) where S is a simplex in G, C a component of $G \backslash S$, and S is attached to C. For $(C,S),(C',S') \in \mathcal{H}$ write $(C,S) \leq (C',S')$ iff $C \subset C'$.

Lemma 3. (i) $\leq$ *defines a partial order on* $\mathcal{H}$*;*

(ii) $(C,S) \leq (C',S')$ *if and only if* $C \cap C' \neq \emptyset$ *and* $S' \cap C = \emptyset$*, for all* $(C,S),(C',S') \in \mathcal{H}$*;*

(iii) $(C,S) < (C',S')$ *if and only if* $S \cap C' \neq \emptyset$ *and* $S' \cap C = \emptyset$*, for all* $(C,S),(C',S') \in \mathcal{H}$*.*

Proof. (i) $\leq$ is clearly reflexive and transitive. To see the antisymmetry of $\leq$, suppose that $(C,S),(C',S') \in \mathcal{H}$ satisfy $C \subset C'$ and $C' \subset C$. We have to show that $S = S'$. Suppose this is false, say $S \backslash S' \neq \emptyset$. Since $S \backslash S'$ is attached to C and $C = C'$, the component of $G \backslash S'$ containing $S \backslash S'$ can only be C'. Thus $S \cap C = S \cap C' \neq \emptyset$, a contradiction.

(ii) If $(C,S) \leq (C',S')$, then clearly $C \cap C' = C \neq \emptyset$ and $S' \cap C \subset S' \cap C' = \emptyset$. Conversely, if C has a vertex in C' but does not meet S', then $C \subset C'$.

(iii) If $(C,S) < (C',S')$, then $S' \cap C = \emptyset$ and $C \cap C' \neq \emptyset$ as in (ii). $S \cap C' \neq \emptyset$ holds, because otherwise $(C,S) \geq (C',S')$ by (ii). Conversely, if S has a vertex s in C' and $C \cap S' = \emptyset$, then every neighbour of s in C is also in C', giving $C \cap C' \neq \emptyset$ because S is attached to C. By (ii) and $S' \cap C = \emptyset$ this implies $(C,S) \leq (C',S')$, and therefore $(C,S) < (C',S')$. □

As an immediate consequence of Lemma 3 let us note that $(C,S) \leq (C',S')$ implies $G\,[\,C \to S\,] \subset G\,[\,C' \to S'\,]$.

Let $\mathcal{H}' \subset \mathcal{H}$, and let $\mathcal{C}$ be a strictly descending chain of elements (C,S) of $\mathcal{H}'$. We shall call $\mathcal{C}$ *maximal with respect to extension* in $\mathcal{H}'$, if $\mathcal{C}$ is not bounded below by any element of $\mathcal{H}' \backslash \mathcal{C}$. By Zorn's Lemma, every non-empty subset $\mathcal{H}'$ of $\mathcal{H}$ contains at least one such maximal chain.

The following three lemmas provide us with strengthenings of condition (†). We first show that the component C' in the condition can be replaced with any finite number of components of $G \setminus (S \cup C)$:

Lemma 4. *Let* (C,S) *be a side in a graph* G*, let* $S_1, S_2 \subset S$*, and suppose that* S_1 *and* S_2 *have prime extensions into* C*. Then* $S_1 \cup S_2$ *has a prime extension into* C*.*

Proof. Let B_1 be a prime extension of S_1 into C, and let H be the simplicial neighbourhood of S_2 in G. Since by assumption S_2 has a prime extension into C, we have $H \cap C \neq \emptyset$.

If $H \cap B_1 \cap C \neq \emptyset$, let $x \in H \cap B_1 \cap C$; then x is simplicially close in G to every vertex of $S_1 \cup S_2$. If $H \cap B_1 \cap C = \emptyset$ on the other hand, let $x \in C\,[\,B_1 \cap C \to H \cap C\,] \cap H$. Then x is again close to every vertex of S_2; let us show that x is also close to every vertex of S_1. Suppose not, and let $T \subset G$ be a simplex separating x from some $s \in S_1$. Since the vertices of S_2 are close to x as well as to s (by $G\,[\,S_2 \cup \{\,s\,\}\,] \subset S$), T must contain S_2. Then, by the definition of H, $T \subset H$. But by the choice of x and the fact that S_1 is attached to $B_1 \cap C$ (because B_1 is prime), no subgraph of H can separate x from s in G, a contradiction. Thus again x is close to every vertex of $S_1 \cup S_2$.

Therefore the vertices of $S_1 \cup S_2 \cup \{\,x\,\}$ are pairwise close in G, so the convex hull of $S_1 \cup S_2 \cup \{\,x\,\}$ in G is prime and meets C (cf. [4, Proposition 1.4]). □

As a brief diversion, let us note an immediate consequence of Lemma 4, which seems potentially useful for tackling the unsolved problem of determining which graphs admit a simplicial (tree-)decomposition into finite factors:

Corollary 5. *If (C, S) is a side in a graph G, then every finite $S' \subset S$ has a prime extension into C.*

Proof. Let $V(S') := \{\,s_1, \ldots, s_n\,\}$. As S is attached to C, each s_i has a neighbour x_i in C, and hence the trivial prime extension $G\,[\,\{\,s_i, x_i\,\}\,]$ into C. The assertion follows by induction on n. □

The next lemma essentially says that the prime extensions provided by (†) are without loss of generality maximally prime and unattached, which we need in order to use (†) for finding new factors in the construction of a prime decomposition.

Lemma 6. *Let G be a countable graph that satisfies (†), let (C_0, S_0) be an inaccessible side in G, and suppose that $S'_0 \subset S_0$ has a prime extension into C_0. Then S'_0 has an extension B into C_0 that is maximally prime and unattached in G.*

Proof. Let G, C_0, S_0 and S'_0 be given as stated, and let $V(G) = \{\,v_1, v_2, \ldots\,\}$ be a fixed enumeration of the vertices of G. Note that S_0 is maximally prime in G (by Lemma 2). Suppose the assertion fails. Then every maximal prime extension B of S'_0 into C_0 is attached in G, and therefore a simplex having exactly one (inaccessible) side (recall that B must be maximally prime in G, and use [4, Corollary 1.5] and Lemma 2). We shall construct a sequence of such extensions of S'_0 with 'nested' sides, whose 'intersection' will be the desired graph B.

To be more precise, we shall find a sequence $(S_n)_{n=0,1,\ldots}$ of extensions of S'_0 satisfying

Figure 1

(a) S_n is maximally prime in G

(b) $S_n \supset S'_{n-1}$

(c) $(C_n, S_n) < (C_{n-1}, S_{n-1})$

(d) $k(n) > k(n-1)$ $\qquad\qquad (n = 1, 2, \ldots),$

where

$$\begin{aligned} S'_n &:= G\,[\,S_{n-1} \to S_n\,] \cap S_n && (n = 1, 2, \ldots), \\ k(n) &:= \min\{\, k \mid v_k \in C_n \,\} && (n = 0, 1, \ldots), \end{aligned}$$

and C_n is the component of $G \backslash S_n$ to which S_n is attached (Fig. 1).

Before we think about constructing such a sequence $(S_n)_{n=0,1,\ldots}$, let us see how it will help us to find our desired unattached prime extension B of S'_0.

Notice first that $S'_{n-1} \subset S'_n \subset S_n$ for every $n \in \mathbb{N}$, by (b) and the definition of S'_n. Hence,

$$S'_0 \subset S'_1 \subset \ldots .$$

By (c), we further have

$$G\,[\,C_0 \to S_0\,] \supset G\,[\,C_1 \to S_1\,] \supset \ldots .$$

Now let

$$B := \bigcap_{n=0}^{\infty} G\,[\,C_n \to S_n\,].$$

Since $S'_n \subset S'_m \subset S_m \subset G\,[\,C_m \to S_m\,]$ for every $m \geq n$, and the graphs $G\,[\,C_m \to S_m\,]$ form a descending sequence, B contains every S'_n. But also conversely, every vertex of B must be in some S'_n. To see this, let any vertex $v = v_k$ of B be given. Since $(k(n))_{n=0,1,\ldots}$ is strictly increasing, v cannot be contained in every C_n and must therefore be in some S_n. Then by $(C_{n+1}, S_{n+1}) < (C_n, S_n)$, v cannot be in C_{n+1} either, so again $v \in S_{n+1}$, i.e. $v \in S_{n+1} \cap S_n \subset S'_{n+1}$. Therefore

$$B = \bigcup_{n=0}^{\infty} S'_n.$$

Note, however, that B is not contained in any one S'_n. For since S_n is attached to C_n and $S_{n+1} \cap C_n \neq \emptyset$ (by (c) and Lemma 3), $S_{n+1} \cap S_n$ does not separate $S_{n+1} \backslash S_n$ from $S_n \backslash S_{n+1}$; thus $S'_{n+1} \cap C_n \neq \emptyset$, giving $S'_{n+1} \backslash S'_n \neq \emptyset$.

Since B is a simplex, all we have to show now is that B is unattached in G—for unattached simplices are necessarily maximally prime [4, Corollary 1.9]. We show this by proving that for every $x \in G \backslash B$ there exists a vertex $y \in B$ such that every x–y path in G has an interior vertex in B. Let $x \in G \backslash B$ be given. By definition of B, we have $x \in G \setminus G\,[\,C_n \to S_n\,]$ for some n. Let y be a vertex of B satisfying $y \in S'_{n+2} \backslash S'_{n+1}$; clearly $y \in C_n$, since otherwise $y \in S_n \cap S_{n+1} \subset S'_{n+1}$ (by (c) and definition of S'_{n+1}). Let P be any x–y path in G. Since S_n separates x and y in G, P contains a subpath from S_n to y, and therefore passes through $S'_{n+1} \subset B$ (by the choice of y). Hence B is unattached in G, as claimed.

Let us now construct a sequence $(S_n)_{n=0,1,\ldots}$ that satisfies (a)–(d). Suppose we have found $S_0, \ldots, S_n$ $(n \geq 0)$ conforming to (a)–(d). Let $\mathcal{H}'$ denote the set of all inaccessible sides in $\mathcal{H}(G)$, i.e. of all sides (C, S) in G for which S is maximally prime. Let

$$\mathcal{H}_{n+1} := \{\, (C,S) \in \mathcal{H}' \mid (C,S) \leq (C_n, S_n),\ v_{k(n)} \in C,\ S'_n \subset S \,\}.$$

Since $(C_n, S_n) \in \mathcal{H}_{n+1}$, $\mathcal{H}_{n+1}$ is not empty.

Suppose first that $\mathcal{H}_{n+1}$ has a minimal element (C, S). By (†) and the definition of S'_n, S'_n has a maximal prime extension S' into C (consider $C' := G\,[\,S_{n-1} \to S\,] \backslash S$ in (†) if $n \geq 1$; if $n = 0$ and $S \neq S_0$, put $C' := G\,[\,S_0 \to S\,] \backslash S$; if $(C, S) = (C_0, S_0)$, the existence of S' is assumed in the assertion of the Lemma). Let H be the convex hull of $S \cup S'$ in G. Since S and S' are both maximally prime, H has a separating simplex T. By the minimality of H as a convex supergraph of S and S', every component of $H \backslash T$ contains a vertex of $S \cup S'$. Thus $H \backslash T$ has exactly two components: one containing $S \backslash T$, the other containing $S' \backslash T$ (and neither $S \backslash T$ nor $S' \backslash T$ is empty). Since S is attached to C, T cannot be contained in S, so $T \cap C \neq \emptyset$. (However, T may well be a subgraph of S'.) By the

Figure 2

convexity of H, T also separates $S\backslash T$ and $S'\backslash T$ in G; let $T' \subset T$ be a minimal $S\backslash T$–$S'\backslash T$ separator in G. Then T' is attached to $G\,[\,S \to T'\,]\backslash T'$ as well as to $G\,[\,S' \to T'\,]\backslash T'$ (Fig. 2).

By (†), we can choose S_{n+1} to be a maximal prime extension of T' into $G\,[\,S' \to T'\,]\backslash T'$ (ensuring (a) for $n+1$). Then $S_{n+1} \supset T' \supset S' \cap S \supset S'_n \supset S'_0$ (ensuring (b)), so by assumption S_{n+1} is attached in G, say to C_{n+1}. Since $S_{n+1}\,[\,S\,] = T' \subsetneqq S_{n+1}$, we have $S \cap C_{n+1} = \emptyset$. On the other hand, $S_{n+1} \cap C \supset T' \cap C \neq \emptyset$ by definition of S' and T'.Lemma 3 therefore gives $(C_{n+1}, S_{n+1}) < (C, S) \leq (C_n, S_n)$, as required in (c). Finally, $v_{k(n)}$ cannot be in C_{n+1}, because otherwise we would have $(C_{n+1}, S_{n+1}) \in \mathcal{H}_{n+1}$, contrary to the minimality of (C, S) in $\mathcal{H}_{n+1}$. Therefore $k(n+1) > k(n)$ (using $C_{n+1} \subset C_n$), establishing (d).

Hence S_{n+1} satisfies (a)–(d), as required.

Suppose now that $\mathcal{H}_{n+1}$ has no minimal element. Let $\mathcal{C}$ be a descending chain in $\mathcal{H}_{n+1}$ that is maximal with respect to extension, and define

$$H^- := \bigcap_{(C,S)\in\mathcal{C}} G\,[\,C \to S\,]\,, \qquad D^- := \bigcap_{(C,S)\in\mathcal{C}} C\,, \qquad S^- := H^-\backslash D^-.$$

Notice that $S'_n \subset S^-$ (because $S'_n \subset S$ for every $(C, S) \in \mathcal{C}$), and that $v_{k(n)} \in D^-$. Note further that the definition of S^- is such that every $s \in S^-$ is contained in S for some $(C, S) \in \mathcal{C}$. Moreover, if $s \in S$ then s cannot be in C' for any $(C', S') \in \mathcal{C}$, $(C', S') < (C, S)$, so s will be in S' for every (C', S') subsequent to (C, S) in $\mathcal{C}$. Thus any two elements s, s'

of S^- are contained in some common S, $(C,S) \in \mathcal{C}$, and are therefore adjacent. Hence, S^- is a simplex. To mention one last property of S^-, we remark that S^- separates every $x \in D^-$ from every $y \in G \backslash H^-$. This is easily checked by considering the first vertex in $\bigcup_{(C,S)\in\mathcal{C}} S$ on any (fixed) x–y path in G.

Let C^* be the component of D^- containing $v_{k(n)}$, and put $S^* := S^-[C^*]$. Notice that $S^* \cap C \neq \emptyset$ for every $(C,S) \in \mathcal{C}$: for if $S^* \subset S$ and $(C',S') \in \mathcal{C}$, $(C',S') < (C,S)$, then $S' \cap S$ $(\supset S^*)$ separates $S' \backslash S$ from $v_{k(n)}$, contradicting $v_{k(n)} \in C'$.

Let $S_{n+1} \supset S^-$ be maximally prime in G (possibly $S_{n+1} = S^-$). Since $S_{n+1} \supset S^- \supset S'_n \supset S'_0$, S_{n+1} is by assumption attached in G—say to C_{n+1}—and satisfies (a)–(b).

In order to verify (c), i.e. to check that $(C_{n+1}, S_{n+1}) < (C_n, S_n)$, we have to show that $C_{n+1} \subset D^-$. Suppose the contrary, i.e. that $C_{n+1} \backslash H^- \neq \emptyset$. Since S^- separates D^- from $G \backslash H^-$ but does not separate C_{n+1}, this means that in fact $C_{n+1} \subset G \backslash H^-$ (and $S_{n+1} \backslash S^- \subset G \backslash H^-$); in particular, $C^* \cap C_{n+1} = \emptyset$. By (†), we can therefore find a prime extension B^* of S^* into C_{n+1}; let $x \in B^* \cap C_{n+1}$. Since $x \in G \backslash H^-$, we have $x \in G \backslash G[C \to S]$ for some $(C,S) \in \mathcal{C}$. Then S separates x from $B^* \cap C$, which is non-empty, because $B^* \supset S^*$ and $S^* \cap C \neq \emptyset$. This contradicts the fact that B^* is prime, completing the proof of (c).

It remains to show (d), i.e. that $v_{k(n)} \notin C_{n+1}$. This, however, follows immediately from the maximality of $\mathcal{C}$ and the fact that $C_{n+1} \subset D^-$ (giving $(C_{n+1}, S_{n+1}) < (C,S)$ for each $(C,S) \in \mathcal{C}$). □

The reader will have noticed that Lemma 6 gives in fact more than a straightforward strengthening of (†): it provides an unattached maximal prime extension into C of *any* subsimplex S' of S that has some prime extension into C (for an inaccessible side (C,S) of G), regardless of whether S' has the form $S[C']$ for some other component C' of $G \backslash S$.

We can therefore combine Lemmas 4 and 6 to a rather powerful tool for extending partial decompositions of a graph:

Lemma 7. *Let G be a countable graph that satisfies (†), let (C,S) be a side in G, and let $C_1, \ldots, C_n$ be components of $G \backslash (C \cup S)$, $n \in \mathbb{N}$. Then*

$$S' := \bigcup_{i=1}^{n} S[C_i]$$

has an extension into C that is maximally prime and unattached in G.

Proof. Suppose first that (C,S) is inaccessible. Using (†), Lemma 4 and induction on n, we see that S' has some prime extension B into C. By Lemma 6 we may assume that B is unattached and maximally prime in G.

Suppose now that (C,S) is accessible, and let B be a maximal prime extension of S into C. Then B is maximally prime in G. If B is unattached, we are done; suppose therefore that B is attached to a component D of $G \backslash B$. Then B is a simplex [4, Corollary 1.5], and

(D, B) is an inaccessible side in G. Moreover, we have $(D, B) < (C, S)$, because $S \cap D = \emptyset$ and $B \cap C \neq \emptyset$ (Lemma 3). Therefore $D \subset C$, and hence $D \cap C_i = \emptyset$, $i = 1, \ldots, n$. Again by Lemma 4 (and induction), we can therefore find a prime extension of S' into D, which by Lemma 6 can be chosen to be maximally prime and unattached in G. □

Our last lemma looks somewhat technical, but it expresses a rather simple and useful fact. It basically gives us some leeway slightly to alter a specified part of a convex subgraph, while preserving its convexity. We shall need this lemma when, in the process of constructing a prime decomposition of a graph G, the most recent factor B^1_μ has to be replaced by an alternative unused maximally prime subgraph B^2_μ, while preserving the convexity of $G|_{\mu+1}$ already established for $G|_\mu \cup B^1_\mu$.

Lemma 8. *Let G be a graph, and let $H, B_1, B_2 \subset G$ be such that H is convex and B_2 is maximally prime in G. Let $S_i := B_i \cap H$ and $H_i := B_i \cup H$, $i = 1, 2$, and suppose that*

(i) *$B_1, B_2 \not\subset H$,*

(ii) *$S_1 \subset S_2$,*

(iii) *H_1 is convex in G,*

(iv) *S_1 does not separate $B_1 \backslash H$ from $B_2 \backslash H$ in G.*

Then H_2 is convex in G, and $S_2 = S_1$.

Proof. Let B'_2 be a component of $B_2 \backslash H$, and let $S := H\,[\,B'_2\,]$. As H is convex, S is a simplex, and since B_2 is prime, this means that $S_2 \subset S$ (and $B'_2 = B_2 \backslash H$). Hence S_1 and S_2 are both simplices.

If $H \subset B_1$, then $H = S_1 = S_2$ and H_2 coincides with B_2, which is convex by [4, Corollary 1.5]. Let us therefore assume that $H \not\subset B_1$, i.e. that $H \backslash S_1 \neq \emptyset$.

Since H_1 is convex, S_1 separates $B_1 \backslash S_1$ from $H \backslash S_1$ in G. As B_2 is prime and therefore not separated by S_1, this implies by (iv) that S_1 separates $B_2 \backslash S_1$ from $H \backslash S_1$. In particular $(B_2 \cap H) \backslash S_1 = \emptyset$, so by (ii) $S_1 = S_2$. Furthermore, if P is any H_2–H_2 path in G, the endvertices x, y of P are either both in H or both in B_2. As H and B_2 are each convex, xy must be an edge of G, so H_2 is convex as claimed. □

We are now ready to begin on the central part of our proof. Let G be a countable graph, $V(G) = \{\, v_1, v_2, \ldots \}$ its vertex set, and suppose that G satisfies (†). As earlier, let $\mathcal{H}$ denote the set of all sides in G.

In view of the discussion in [4, Section 4] of the graphs H^1 and H^2, and the traps they contained for the construction of a prime decomposition, let us say that a subgraph B of G *defuses* a side $(C, S) \in \mathcal{H}$ if either $B \cap C \neq \emptyset$ or $B \supset S$. Whenever $F = (B_\lambda)_{\lambda<\sigma}$ is a family of induced subgraphs of G and $\mu \leq \sigma$, call $(C, S) \in \mathcal{H}$ *defused at* $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda$ if some B_λ with $\lambda < \mu$ defuses (C, S), and *undefused* (*at* $G|_\mu$) otherwise.

Let $\mathcal{F}$ be the set of all well-ordered families F of induced subgraphs of G ($F = (B_\lambda)_{\lambda<\sigma}$ say) satisfying the following seven conditions:

(i) every B_λ is maximally prime in G $(\lambda < \sigma)$,

(ii) every B_λ is unattached in G $(\lambda < \sigma)$,

(iii) $B_\mu \backslash G|_\mu \neq \emptyset$ (where $G|_\mu := \bigcup_{\lambda<\mu} B_\lambda$) $(\mu < \sigma)$,

(iv) each S_μ $(:= B_\mu \cap G|_\mu)$ is contained in some B_λ, $\lambda < \mu$ $(0 < \mu < \sigma)$,

(v) each $G|_\mu$ is convex in G $(\mu \leq \sigma)$,

(vi) if $\mathcal{H}^1_\mu \neq \emptyset$ then $B_\mu \subset H_\mu$ $(\mu < \sigma)$,

(vii) if $\mathcal{H}^1_\mu = \emptyset$ and $\mathcal{H}^2_\mu \neq \emptyset$ then $B_\mu \supset S$ for some $(C, S) \in \mathcal{H}^2_\mu$ $(\mu < \sigma)$,

where $\mathcal{H}^1_\mu$, $\mathcal{H}^2_\mu$ and H_μ are defined as follows.

An induced subgraph B of G is called *eligible at* $G|_\mu$ $(\mu \leq \sigma)$ if selecting it as B_μ is compatible with (i)–(v), i.e. if B is maximally prime and unattached, $B \backslash G|_\mu \neq \emptyset$, $B \cap G|_\mu \subset B_\lambda$ for some $\lambda < \mu$, and $B \cup G|_\mu$ is convex in G. $(C, S) \in \mathcal{H}$ is called *defusable at* $G|_\mu$ $(\mu \leq \sigma)$ if (C, S) is undefused at $G|_\mu$ and G has a subgraph B that is eligible at $G|_\mu$ and defuses (C, S). If this B can be chosen such that $B \cap C \neq \emptyset$ (or, equivalently, $B \subset G\,[\,C \to S\,]$), we call (C, S) 1-*defusable*; otherwise (C, S) is 2-*defusable (at* $G|_\mu$*)*. Then $\mathcal{H}^1_\mu$, $\mathcal{H}^2_\mu$ and H_μ are defined by setting

$$\mathcal{H}_\mu := \big\{\, (C, S) \in \mathcal{H} \;\big|\; (C, S) \text{ is defusable at } G|_\mu \,\big\}$$

$$k(\mu) := \begin{cases} \infty & \text{if } \mathcal{H}_\mu = \emptyset \\ \min \big\{ i \;\big|\; \exists\, (C, S) \in \mathcal{H}_\mu :\; v_i \in C \big\} & \text{if } \mathcal{H}_\mu \neq \emptyset \end{cases}$$

$$\mathcal{H}^1_\mu := \big\{ (C, S) \in \mathcal{H}_\mu \;\big|\; (C, S) \text{ is 1-defusable at } G|_\mu \text{ and } v_{k(\mu)} \in C \big\}$$

$$\mathcal{H}^2_\mu := \big\{ (C, S) \in \mathcal{H}_\mu \;\big|\; (C, S) \text{ is 2-defusable at } G|_\mu \text{ and } v_{k(\mu)} \in C \big\}$$

$$H_\mu := \bigcap_{(C,S) \in \mathcal{H}^1_\mu} G\,[\,C \to S\,] \qquad (\mu \leq \sigma).$$

Thus if we view F as being created by selecting its members B_λ inductively, conditions (i)–(v) ensure that F is a prime decomposition of $\bigcup_{\lambda<\sigma} B_\lambda$, while conditions (vi) and (vii) express certain preferences in choosing the factors. Broadly speaking, we can think of each B_μ as being selected by the following procedure.

Having arrived at $G|_\mu$, we first determine whether B_μ can be chosen such as to defuse any undefused side at all. If not, we let B_μ be any induced subgraph of G that is eligible at $G|_\mu$ (note that this is the case when $\mathcal{H}_\mu$ is empty, so (vi) and (vii) do not apply). On the other hand, if there are sides defusable at $G|_\mu$, we try to select B_μ in such a way that it defuses a side (C, S) among these for which C contains a vertex of smallest possible index k. Additional preference is given to those factors B that defuse some such side (C, S) by satisfying $B \cap C \neq \emptyset$; moreover, if such B exist at all, we even insist that B satisfy

$B \cap C \neq \emptyset$ for *every* 1-defusable (C, S) with $v_k \in C$, i.e., that $B \subset H_\mu$. If, on the other hand, the only way an eligible B can defuse some such side (C, S) is by satisfying $B \supset S$, we take B_μ to be any of these B.

For two families $F, F' \in \mathcal{F}$, say $F = (B_\lambda)_{\lambda<\sigma}$ and $F' = (B'_\lambda)_{\lambda<\sigma'}$, let us write $F \leq F'$ if $\sigma \leq \sigma'$ and $B_\lambda = B'_\lambda$ for all $\lambda < \sigma$. Then $\leq$ defines a partial order on $\mathcal{F}$. By Zorn's Lemma $\mathcal{F}$ has a maximal element, for since the union of a nested sequence of convex subgraphs of G is again convex in G (cf. (v)), every chain in $\mathcal{F}$ is bounded by the union of its members.

Let $F^* = (B_\lambda)_{\lambda<\sigma}$ be a fixed maximal element of $\mathcal{F}$. We shall prove that F^* is a prime decomposition of G.

It is easily seen that F^* is a prime decomposition of $G' := \bigcup_{\lambda<\sigma} B_\lambda$ $(= G|_\sigma)$. Indeed, by assumption each B_λ is prime, and F^* satisfies (S1) and (S4). Thus all we have to check is that every S_μ is a simplex, and that no S_μ contains any B_λ, $\lambda < \mu$. (The other requirement of (S3) is met because of (iii).) To see this, let $\mu < \sigma$, and consider a component C of $B_\mu \backslash G|_\mu$. As $G|_\mu$ is convex in G, $S := G|_\mu [\, C \,]$ is a simplex. But B_μ, being prime, is not separated by any simplex, so $B_\mu \backslash G|_\mu$ is in fact equal to C, and $S_\mu \subset S$. Moreover, S_μ is attached to $B_\mu \backslash G|_\mu$, so S_μ cannot contain any B_λ, $\lambda < \mu$ (by (ii)).

It remains to show that F^* is a decomposition of the entire graph G, i.e. that $G' = G$. Suppose $G \backslash G' \neq \emptyset$. If $\mathcal{H}_\sigma = \emptyset$, let v be any vertex of $G \backslash G'$; otherwise set $v := v_{k(\sigma)}$. Let $C_\sigma := G [\, v \to G' \,] \backslash G'$ and $S_\sigma := G' [\, v \,]$. Since G' is convex in G (by (v)), S_σ is a simplex. We shall prove that F^* can be extended by a new factor $B_\sigma \subset G [\, C_\sigma \to S_\sigma \,]$, in contradiction to the maximality of F^* in $\mathcal{F}$.

To spare the reader the task of monitoring over all stages of the proof what happens if $\sigma = 0$ (i.e. if $G' = \emptyset$), let us deal with this case first.

Let S be a maximally prime subgraph of G containing v. If S is unattached, set $B_\sigma := S$. If S is attached, to the component C of $G \backslash S$ say, then (C, S) is an inaccessible side of G. Let x be any neighbour of v in C. Then $\{\, v \,\}$ has the prime extension $(\{\, v, x \,\}, \{\, vx \,\})$ into C, so by Lemma 6 we can select as B_σ an unattached maximally prime subgraph of G containing v.

In either case we have $v \in B_\sigma$ and therefore $B_\sigma \subset H_\sigma$ (notice that $(C_\sigma, \emptyset) \in \mathcal{H}^1_\sigma$ and hence $\mathcal{H}^1_\sigma \neq \emptyset$, because B_σ exists as chosen). Hence B_σ is as desired.

Let us from now on assume that $\sigma > 0$. In order to comply with (v), our desired new factor B_σ must be chosen in such a way that $G' \cup B_\sigma$ is convex. Since S_σ is attached to C_σ and C_σ is connected, this means that B_σ will have to contain the entire S_σ (cf. [4, Corollary 1.7 (iii)]), i.e. S_σ will be the simplex of attachment of B_σ. Our first objective therefore is to show that S_σ is contained in B_λ for some $\lambda < \sigma$, so that $B_\sigma \cap G|_\sigma$ can satisfy (iv).

The following proposition (A) serves this purpose.

(A) $\quad S_\sigma \subset B_\lambda$, for some $\lambda < \sigma$.

Figure 3

The idea underlying the proof of (A) is central to the whole proof of Theorem 1: using conditions (vi) and (vii), we show that S_σ was created in at most finitely many steps, unlike the simplex S that caused problems in our examples H^1 and H^2 in [4, Section 4]. More precisely, we shall prove that $\Lambda(S_\sigma)$ is finite; then (A) will be immediate from [4, Corollary 1.7 (i)]. Suppose $\Lambda(S_\sigma)$ is infinite, and let $S \supset S_\sigma$ be a maximal simplex in G'.

We first show that S must be unattached in G'. Suppose S is attached in G', say to the component C' of $G' \backslash S$. Then $(G\,[\,C' \to S\,] \backslash S\,, S)$ is a side in G. Applying (†) to this side with C_σ assuming the role of C' in (†), we may deduce that S_σ has a prime extension B into $G\,[\,C' \to S\,] \backslash S$ (but note that B is not necessarily a subgraph of G').

Let us find a vertex $x \in C'$ that is simplicially close to every vertex of S_σ (Fig. 3). If $B \cap C' \neq \emptyset$, we simply pick $x \in B \cap C'$. Suppose now that $B \cap C' = \emptyset$. Then $B \backslash S \subset G \backslash G'$; notice that, by the convexity of G' in G, $G\,[\,C' \to S\,]$ contains no vertices from components of $G' \backslash S$ other than C'. Since G' is convex, $G'\,[\,B \backslash S\,]$ is a simplex. As B is prime, this means that $G'\,[\,B \backslash S\,]$ does not separate any vertices of B, so $G'\,[\,B \backslash S\,] \supset S_\sigma$. But $G'\,[\,B \backslash S\,]$ also has a vertex in C', since, by definition of B, S does not separate $B \backslash S$ from C'. Choosing $x \in G'\,[\,B \backslash S\,] \cap C'$, we have therefore again found a vertex $x \in C'$ that is close to every vertex of S_σ.

Having shown that C' contains a vertex x which is simplicially close to every vertex of S_σ, we may infer by [4, Proposition 1.4], that S_σ has a prime extension into C': the convex hull of $S_\sigma \cup \{\,x\,\}$ in G'. This, however, contradicts [4, Theorem 3.2 (iii)] (put $S' := S_\sigma$). Therefore S is unattached in G'.

Figure 4

Having shown that S is unattached in G', we may deduce that the position of S in G' is as stated in [4, Theorem 3.2 (iv)]; let $\Lambda \subset \Lambda(S)$ and $(C_\lambda)_{\lambda\in\Lambda}$ be as provided, and let λ' denote $\lambda(C_\lambda)$ (for $\lambda \in \Lambda$).

Let us prove that $\{ k(\lambda') \mid \lambda \in \Lambda \}$ is not bounded by any $n \in \mathbb{N}$. Suppose $\{ k(\lambda') \mid \lambda \in \Lambda \}$ is bounded. Then $\mathcal{H}_{\lambda'} \neq \emptyset$ for every $\lambda \in \Lambda$, and for some $k \in \mathbb{N}$ there are infinitely many $\lambda \in \Lambda$ with $k(\lambda') = k$. We can therefore find $\lambda_1, \lambda_2 \in \Lambda$ satisfying $\lambda_1 < \lambda_2$, $k(\lambda'_1) = k(\lambda'_2) = k$, and either $\mathcal{H}^1_{\lambda'_1} = \mathcal{H}^1_{\lambda'_2} = \emptyset$ or $\mathcal{H}^1_{\lambda'_1}, \mathcal{H}^1_{\lambda'_2} \neq \emptyset$. Notice that condition (c) of [4, Theorem 3.2 (iv)] and our assumption that $\lambda_1 < \lambda_2$ imply $\lambda'_1 \leq \lambda_1 \in \Lambda(S|_{\lambda_2}) = \Lambda(S_{\lambda'_2})$, so $\lambda'_1 < \lambda'_2$.

Let us write B^i for $B_{\lambda'_i}$ $(i = 1, 2)$. We shall prove that B^2 was eligible at $G|_{\lambda'_1}$, and use this to deduce that the choice of B^1 contradicts (vi). To see that B^2 was eligible at $G|_{\lambda'_1}$, note that $B^2 \cap G|_{\lambda'_1} = S|_{\lambda_1} = S_{\lambda'_1}$, which is by (S4) contained in some B_λ, $\lambda < \lambda'_1$. Thus all we have to check is that $G|_{\lambda'_1} \cup B^2$ is convex in G. This, however, follows from Lemma 8 (put $H := G|_{\lambda'_1}$ and $B_i := B^i$; Fig. 4).

For the proof that the choice of B^1 was inconsistent with condition (vi), suppose first that $\mathcal{H}^1_{\lambda'_1} = \mathcal{H}^1_{\lambda'_2} = \emptyset$. Then $\mathcal{H}^2_{\lambda'_1}$ and $\mathcal{H}^2_{\lambda'_2}$ are non-empty, so by (vii) there are sides $(C^i, S^i) \in \mathcal{H}^2_{\lambda'_i}$ satisfying $S^i \subset B^i$ and $v_k \in C^i$ (and $B^i \cap C^i = \emptyset$, because $\mathcal{H}^1_{\lambda'_i} = \emptyset$), $i = 1, 2$.

Since $v_k \in C^1 \cap C^2$, we have $C^1 \cap C^2 \neq \emptyset$. Moreover, as (C^2, S^2) is still undefused at $G|_{\lambda'_2}$,

$$S^1 \cap C^2 \subset B^1 \cap C^2 = \emptyset\,.$$

Therefore $(C^2, S^2) \le (C^1, S^1)$ by Lemma 3 (ii). Since B^1 defuses (C^1, S^1) but not (C^2, S^2), these sides are not identical, so even $(C^2, S^2) < (C^1, S^1)$. Applying Lemma 3 (iii), we thus obtain

$$B^2 \cap C^1 \supset S^2 \cap C^1 \neq \emptyset\,,$$

so B^2 is an extension of $S_{\lambda'_1} \subset S^1$ into C^1. Since B^2 is eligible at $G|_{\lambda'_1}$, this implies that (C^1, S^1) is 1-defusable at $G|_{\lambda'_1}$. Hence $\mathcal{H}^1_{\lambda'_1} \neq \emptyset$, contrary to our assumption.

Suppose now that neither of $\mathcal{H}^1_{\lambda'_1}$, $\mathcal{H}^1_{\lambda'_2}$ is empty. Then by (vi) there are sides $(C^i, S^i) \in \mathcal{H}^1_{\lambda_i}$ that satisfy $B^i \cap C^i \neq \emptyset$, $i = 1, 2$. Since B^2 is eligible at $G|_{\lambda'_1}$, (C^2, S^2) is 1-defusable at $G|_{\lambda'_1}$, so $(C^2, S^2) \in \mathcal{H}^1_{\lambda'_1}$. On the other hand, (C^2, S^2) is still undefused at $G|_{\lambda'_2}$, so $B^1 \not\subset G\,[\,C^2 \to S^2\,]$. Therefore $B^1 \not\subset H_{\lambda'_1}$, contradicting (vi). This completes the proof that $\{\, k(\lambda') \mid \lambda \in \Lambda \,\}$ is unbounded in $\mathbb{N}$.

Let us now finish our proof of (A). Suppose first that S is attached in G, say to the component C of $G \backslash S$. (If $S = S_\sigma$, C may coincide with C_σ.) Since S is unattached in G' and G' is convex, we have $C \cap G' = \emptyset$. Hence, for each $\lambda \in \Lambda$, $S\,[\,C_\lambda\,]$ has an unattached and maximally prime extension into C (by (†) and Lemma 6), which is eligible at $G|_{\lambda'}$ by Lemma 8 (as earlier). Hence $(C, S) \in \mathcal{H}_{\lambda'}$ and consequently $k(\lambda') \le \min\{\, i \mid v_i \in C \,\}$ for every $\lambda \in \Lambda$, contrary to the unboundedness of $k(\lambda')$ established above.

Suppose now that S is unattached in G. Then S itself is eligible at $G|_{\lambda'}$ for every $\lambda \in \Lambda$ [4, Corollary 1.9]. Since S contains S_σ and therefore defuses (C_σ, S_σ), this implies that $k(\lambda') \le \min\{\, i \mid v_i \in C_\sigma \,\}$, again contradicting the unboundedness of $k(\lambda')$. This completes the proof of (A).

As an immediate consequence of (A) let us note that

(B) S_σ is attached in G'.

Let us now determine a subgraph B_σ that extends F^* and contradicts the assumed maximality of F^* in $\mathcal{F}$. All graphs we consider as candidates for the role of B_σ will be maximally prime and unattached extensions of S_σ into C_σ. Then the only conditions we shall have to verify for $\bigcup_{\lambda \le \sigma} B_\lambda$ will be (vi) and (vii); notice that (iv) will hold by (A), and (v) will be satisfied because B_σ and G' are convex and S_σ separates $B_\sigma \backslash S_\sigma$ from $G' \backslash S_\sigma$ in G.

If $\mathcal{H}_\sigma = \emptyset$, (vi)–(vii) are trivially satisfied, so we may take B_σ to be any unattached maximally prime extension of S_σ into C_σ (by (B), (†) and Lemma 7).

If $\mathcal{H}^1_\sigma = \emptyset$ but $\mathcal{H}^2_\sigma \neq \emptyset$, the existence of a suitable B_σ is guaranteed by the definition of $\mathcal{H}^2_\sigma$. (We remark that any undefused (C, S) with $v \in C$ must satisfy $S \neq S_\sigma$ (by (A)) and hence $(C, S) < (C_\sigma, S_\sigma)$, so B_σ $(\supset S)$ will be a subgraph of $G\,[\,C_\sigma \to S_\sigma\,]$.)

Figure 5

Suppose now that $\mathcal{H}_\sigma^1 \neq \emptyset$. In our search for B_σ we now have to ensure that B_σ meets the rather strict condition (vi), i.e. that B_σ intersects with C for every 1-defusable side (C, S). This will require some work, which is the price we now pay for having been able to use the full strength of (vi) in the proof of (A).

Since $v \in C$ for every $(C, S) \in \mathcal{H}_\sigma^1$, Lemma 3 (ii) gives

(C) $\quad (C, S) \leq (C', S')$ iff $S' \cap C = \emptyset\,, \qquad$ for all $(C, S), (C', S') \in \mathcal{H}_\sigma^1\,.$

Similarly, we have $(C, S) < (C_\sigma, S_\sigma)$ for all $(C, S) \in \mathcal{H}_\sigma^1$, and consequently $G\,[\,C \to S\,] \subset G\,[\,C_\sigma \to S_\sigma\,]$. (The inequality is strict, because since (C_σ, S_σ) is '2-defused' by some B_λ, $\lambda < \sigma$, (C_σ, S_σ) is not in $\mathcal{H}_\sigma$.) Moreover,

(D) $\quad S \cap G' = S_\sigma$ for every $(C, S) \in \mathcal{H}_\sigma^1\,,$

that is, every such S contains the entire S_σ. To see this, recall that any eligible $B \subset G\,[\,C_\sigma \to S_\sigma\,]$ must contain S_σ. Since (C, S) is 1-defusable at $G|_\sigma$, we have $S_\sigma \subset B \subset G\,[\,C \to S\,]$ for some such B, giving $S_\sigma \subset S$ by $C \cap G' = \emptyset$.

In order to determine B_σ, it will be convenient to find a prime and convex subgraph B^* of G that satisfies $B^* \cap C_\sigma \neq \emptyset$, $\;B^* \supset S_\sigma$, $\;v \notin B^*$ and $S^* \not\supset S_\sigma$, where $S^* := B^*\,[\,v\,]$ (Fig. 5).

If $\mathcal{H}^1_\sigma$ contains an unbounded descending chain $\mathcal{C}$ (i.e. one that is maximal in $\mathcal{H}^1_\sigma$ with respect to extension and has no minimal element), it is not difficult to find such a graph B^*. As in the proof of Lemma 6, we let

$$H^- := \bigcap_{(C,S)\in\mathcal{C}} G\,[\,C \to S\,]\,, \qquad D^- := \bigcap_{(C,S)\in\mathcal{C}} C\,, \qquad S^- := H^-\backslash D^-.$$

Moreover, we let C^* be the component of D^- containing v, and put $S^* := S^-\,[\,C^*\,]$. By (D), we have $S_\sigma \subset S$ for every $(C,S) \in \mathcal{C}$, and hence $S^- \supset S_\sigma$. As in the proof of Lemma 6 it is shown that S^- is a simplex, and that $S^* \cap C \neq \emptyset$ for each $(C,S) \in \mathcal{C}$, giving $S^- \cap C_\sigma \neq \emptyset$. Since $C^* \subset D^-$, we further have $(C^*, S^*) \leq (C,S)$ for every $(C,S) \in \mathcal{C}$. As $\mathcal{C}$ is by assumption unbounded in $\mathcal{H}^1_\sigma$, this means that $(C^*, S^*) \notin \mathcal{H}^1_\sigma$. Hence $S^* \not\supset S_\sigma$, because otherwise (C^*, S^*) would be 1-defusable at $G|_\sigma$ (by (†) and Lemma 7), implying $(C^*, S^*) \in \mathcal{H}^1_\sigma$. Therefore $B^* := S^-$ is as required.

Let us now assume that every descending chain in $\mathcal{H}^1_\sigma$ is bounded. Let $\mathcal{H}^0_\sigma$ denote the set of all minimal elements of $\mathcal{H}^1_\sigma$. Then

(E) $\quad S$ is maximally prime in G, $\quad$ for every $(C,S) \in \mathcal{H}^0_\sigma$.

To prove (E), it suffices to show that every $(C,S) \in \mathcal{H}^0_\sigma$ is inaccessible (Lemma 2). Suppose $(C,S) \in \mathcal{H}^0_\sigma$ is accessible, and let $\overline{S}$ be a maximal prime extension of S into C.

If $v \notin \overline{S}$, we may assume that $S^* := \overline{S}\,[\,v\,]$ contains S_σ, for otherwise we can put $B^* := \overline{S}$ and have B^* as desired. ($\overline{S}$ is convex, because it is maximally prime in G.) But assuming $S^* \supset S_\sigma$ and putting $C^* := G\,[\,v \to \overline{S}\,]\backslash\overline{S}$, we find that (C^*, S^*) is a side which is 1-defusable at $G|_\sigma$ (by (†) and Lemma 7) and satisfies $(C^*, S^*) < (C,S)$. This contradicts the minimality of (C,S) in $\mathcal{H}^1_\sigma$.

If $v \in \overline{S}$ on the other hand, we can easily find a new factor B_σ extending F^*. If $\overline{S}$ is unattached in G, we put $B_\sigma := \overline{S}$. If $\overline{S}$ is attached, however, say to $\overline{C}$, then $\overline{C} \cap C_\sigma \neq \emptyset$ and hence $\overline{C} \subset C_\sigma$, because $\overline{S} \supset S \supset S_\sigma$ and v has a neighbour in $\overline{C}$. We can therefore use (†) (and Lemmas 4,6) to select as B_σ an unattached maximally prime extension of $\overline{S}\,[\,S_\sigma \cup \{\,v\,\}\,]$ into $\overline{C}$, once more exploiting the fact that $\{\,v\,\}$ has a trivial prime extension into $\overline{C}$ because it has a neighbour in $\overline{C}$. Then, in either case, $v \in B_\sigma$ implies that $B_\sigma \subset H_\sigma$, so $(B_\lambda)_{\lambda \leq \sigma}$ satisfies (i)–(vii). This completes the proof of (E).

If $\mathcal{H}^0_\sigma$ consists of a single element (C,S), we let B_σ be any unattached maximally prime extension of S_σ into C (by (D), (†) and Lemma 6). Then $B_\sigma \subset G\,[\,C \to S\,] = H_\sigma$, so $\bigcup_{\lambda \leq \sigma} B_\lambda$ satisfies (i)–(vii), contradicting the maximality of F^*. As $\mathcal{H}^1_\sigma$ is by assumption non-empty, $\mathcal{H}^0_\sigma$ therefore has at least two elements (C,S) and (C',S').

By (C), we have $S' \cap C \neq \emptyset$ as well as $S \cap C' \neq \emptyset$, so C contains $S'\backslash S$ and C' contains $S\backslash S'$ (and these graphs are not empty). Let H be the convex hull of $S \cup S'$ in G, and define B^* the way T' was defined in the proof of Lemma 6. Then B^* is a simplex that satisfies $B^* \cap C \neq \emptyset$ and $B^* \cap C' \neq \emptyset$ (and hence $B^* \cap C_\sigma \neq \emptyset$), contains $S \cap S'$ (and therefore S_σ; cf. (D)), separates $S\backslash B^*$ ($\neq \emptyset$) from $S'\backslash B^*$ ($\neq \emptyset$) in G, and is attached to $D := G\,[\,S \to B^*\,]\backslash B^*$ as well as to $D' := G\,[\,S' \to B^*\,]\backslash B^*$.

If $v \in B^*$, we simply let B_σ be an unattached maximally prime extension of B^* into D or D' (by (†) and Lemma 7). Then $B_\sigma \subset H_\sigma$ because $v \in B_\sigma$, so $(B_\lambda)_{\lambda \leq \sigma}$ conforms to (i)–(vii), contradicting the maximality of F^*.

Therefore $v \notin B^*$; let C^* denote the component of $G \backslash B^*$ containing v. Since D and D' are also components of $G \backslash B^*$, we may assume without loss of generality that $C^* \neq D$. Then $S \cap C^* = \emptyset$, which implies $(C^*, S^*) < (C, S)$ by Lemma 3 (ii) (with $S^* := B^*[v]$, as usual). But (C, S) is minimal in $\mathcal{H}^1_\sigma$, so this means that $(C^*, S^*) \notin \mathcal{H}^1_\sigma$, and therefore $S^* \not\supset S_\sigma$ (by (†) and Lemma 7). Hence B^* is as desired.

Let us now finish our proof by using the properties of B^* and S^* to find an additional factor $B_\sigma \subset H_\sigma$. Notice that S^* is a simplex, because B^* is convex. Put $C^* := G[v \to B^*] \backslash B^*$, and let (C, S) be any element of $\mathcal{H}^1_\sigma$. Since S contains S_σ but $G[C^* \to S^*]$ does not, $S \cap C^*$ must be empty. Hence $(C^*, S^*) < (C, S)$, by $v \in C^* \cap C$ and Lemma 3 (ii). Thus $S^* \cap C \neq \emptyset$ for every $(C, S) \in \mathcal{H}^1_\sigma$, giving $B^* \subset H_\sigma$.

If B^* is unattached in G, we let $B_\sigma := B^*$ and are done (B_σ will be maximally prime by [4, Proposition 1.8]); suppose therefore that B^* is attached in G, say to the component D^* of $G \backslash B^*$. Since B^* is convex, this means that B^* is a simplex, and (D^*, B^*) is a side in G. Since B^* is not attached to C^*, clearly $D^* \cap C^* = \emptyset$. Furthermore, we have $(D^*, B^*) < (C_\sigma, S_\sigma)$ because $B^* \supset S_\sigma$ and $B^* \cap C_\sigma \neq \emptyset$ (Lemma 3 (iii)), giving $D^* \subset C_\sigma$. Hence, $D^* \cap (C^* \cup G') = \emptyset$. Let B_σ be an unattached maximally prime extension of $S_\sigma \cup S^*$ into D^* (by (†) and Lemma 7). Then $B_\sigma \cap C \supset S^* \cap C \neq \emptyset$ for every $(C, S) \in \mathcal{H}^1_\sigma$, giving $B_\sigma \subset H_\sigma$. Hence $\bigcup_{\lambda \leq \sigma} B_\lambda$ satisfies (i)–(vii), contrary to the assumed maximality of F^*.

This completes the proof of Theorem 1.

As an immediate corollary of the proof of Theorem 1 we see that, for any countable graph G that admits a simplicial tree-decomposition into primes, not only every prime factor of G is maximally prime, minimally convex and unattached [4, Theorem 1.10], but conversely any subgraph of G with these properties is a factor in some simplicial tree-decomposition of G into primes:

Corollary 9. *Let G be a countable graph admitting a prime decomposition, and let $B \subset G$. Then the following statements are equivalent.*

(i) *B is minimally convex and unattached in G;*

(ii) *B is maximally prime and unattached in G;*

(iii) *B is a factor in some prime decomposition of G.*

Proof. As (i) and (ii) are equivalent by [4, Proposition 1.8], all we have to show is that (ii) implies (iii). By assumption G has a prime decomposition, so G satisfies (†). Choose an enumeration of $V(G)$, beginning with a vertex in B. Then $F = (B)$ is a family that satisfies conditions (i)–(vii) from the proof of Theorem 1, so F can be extended to a prime decomposition of G. □

2. Two Examples

As the examples discussed in part one of this paper already indicated, a main difficulty that had to be overcome in the proof of Theorem 1 was the possible existence of inaccessible sides in the graph G considered: the algorithm used to construct a prime decomposition of G had to detect and defuse any inaccessible side (C, S) before S was completely covered by factors $B_\lambda \not\subseteq G[C \to S]$. The way this was achieved was by ensuring that the set of inaccessible sides defused by each new factor was in a rather strong sense maximal (condition (vi)), and the difficulty in proving the theorem lay in showing that such new factors can indeed always be found.

Bearing this in mind, it may not be too surprising that the proof of Theorem 1 becomes substantially easier if all the graphs considered have at most countably many simplices with inaccessible sides, which can then be defused according to a much simpler priority rule. On the other hand, it is not clear at first sight whether a countable graph can have uncountably many such simplices at all, i.e. whether such a simplification of the proof would perhaps still cover all countable graphs.

In this section we shall answer this question in the negative by giving an example of a graph that satisfies (†), and in which uncountably many simplices have an inaccessible side. The graph we construct will also provide some illustration for the proof of Theorem 1.

The second graph exhibited in this section is an uncountable variation of the first. It still satisfies (†), but it has no prime decomposition. This second example therefore shows that Theorem 1 cannot be extended to uncountable graphs, and is in this respect best possible.

Let us construct a graph T_0 as follows. Let the vertices of T_0 be all finite 0-1 sequences, i.e. let $V(T_0) := \{0,1\}^{[\omega]}$, and join $(a_0, \ldots, a_n)$ to $(b_0, \ldots, b_m)$ whenever $n < m$ and $a_i = b_i$ for $i = 0, \ldots, n$. Thus if we write $\alpha < \beta$ for $\alpha, \beta \in \{0,1\}^{[\omega]}$ iff β is an extension of α, then T is simply the comparability graph of its vertex set.

Let T_1 be the graph obtained from T_0 by adding all edges of the form $(a_0, \ldots, a_{n-1}, 0)(a_0, \ldots, a_{n-1}, 1, 0, \ldots, 0)$, i.e. by additionally joining $(a_0, \ldots, a_n)$ to $(b_0, \ldots, b_m)$ whenever $n \le m$, $a_i = b_i$ for $i = 0, \ldots, n-1$, $a_n = 0$, $b_n = 1$, and $b_{n+1} = \ldots = b_m = 0$ (Fig. 6).

T_1 has maximally prime subgraphs of two different types. First, there are the simplices spanned by those maximal sets V of pairwise comparable vertices that satisfy

$$\forall\, i \in \mathbb{N} : \exists\, (a_0, \ldots, a_n) \in V : (n \ge i \text{ and } a_n = 1).$$

Every such simplex S has an inaccessible side (C, S), where C is the component of $T_1 \backslash S$ spanned by all vertices of $T_1 \backslash S$ that are lexicographically smaller than some vertex in S (or 'left' of S in Fig. 6). It is also clear that T_1 has $2^{\aleph_0}$ of these simplices.

The maximally prime subgraphs of the other type are extensions (by one additional vertex) of simplices spanned by those maximal sets of pairwise comparable vertices that

Figure 6

are not among the vertex sets of the simplices with inaccessible sides defined above. Each of these simplices S has the property that

$$\exists\, i \in \mathbb{N} : \forall\, (a_0, \ldots, a_n) \in V(S) : \big(i \leq j \leq n \;\Rightarrow\; a_j = 0\big)\,.$$

The additional vertex $x(S)$ is $(a_0, \ldots, a_{n-1}, 0)$ if $(a_0, \ldots, a_n) \in S$ and

$$n \;=\; \max\{\, i \in \mathbb{N} \mid (a_0, \ldots, a_i) \in S \text{ and } a_i = 1 \,\}\,;$$

so S is in fact uniquely determined by $x(S)$, and $S_x := T_1\,[\, S \cup \{\, x(S) \,\}\,]$ is an unattached and hence maximally prime simplex in T_1.

There is only one maximally prime subgraph of T_1 that is not of one of these two types. This is the unattached simplex $S_0 := T_1\,[\,\emptyset, (0), (00), (000), \ldots\,]$.

It is not difficult to check that T_1 satisfies (†), and that T_1 has a prime decomposition whose factors are simplices of the form S_x and possibly S_0.

Let us now turn to the second of the two examples in this section. Define the graph T_2 from T_0 by adding to it new vertices $v(S)$, one for each maximal simplex S in T_0, joining $v(S)$ to every vertex of S for each S. T_2 has order $2^{\aleph_0}$ (recall that the maximal simplices of T_0 are precisely those subgraphs of T_0 that are spanned by maximal sets of 'pairwise comparable' vertices, so there are $2^{\aleph_0}$ of them), and the set $\mathcal{S}$ of maximally prime subgraphs of T_2 is precisely the set of all subgraphs of the form $T_2 [\, S \cup \{\, v(S) \,\} \,]$.

Since every attached simplex of T_2 can be extended to a simplex of the form $T_2 [\, S \cup \{\, v(S) \,\} \,]$ in each of its sides, no side in T_2 is inaccessible. Hence T_2 satisfies (†).

However, T_2 has no prime decomposition. To see this, let us suppose that $(B_\lambda)_{\lambda<\sigma}$ is a prime decomposition of T_2. Since prime factors are always maximally prime and the only maximally prime subgraph of T_2 containing the vertex $v(S)$ is $T_2 [\, S \cup \{\, v(S) \,\} \,]$, the set $\{\, B_\lambda \mid \lambda < \sigma \,\}$ of factors coincides with $\mathcal{S}$. Hence σ is uncountable. But $T_2 [\, S \cup \{\, v(S) \,\} \,]$ is also the only maximally prime subgraph of T_2 containing S. Since by assumption every $\mu < \sigma$ is such that $S_\mu \subset B_\lambda$ for some $\lambda < \mu$ (S4), this means that, whenever $v \in T_2 \backslash T_0$, $S_{\lambda(v)}$ cannot contain the entire S, i.e. $\lambda(s) = \lambda(v)$ for some $s \in S$. Therefore $\Lambda(T_0) = \sigma$, which contradicts the countability of T_0.

3. Simplicial Minors

The purpose of this section is to announce another characterization of the countable graphs that have simplicial tree-decompositions into primes. The characterization is based on Theorem 1, and it will be pesented in detail in a forthcoming paper [6].

The basic idea of this result is to show that there are essentially only two different non-decomposable graphs: similarly to Kuratowski's classical theorem on planar graphs, it characterizes the decomposable graphs in terms of two forbidden minors, the usual notion of a minor being slightly restricted to match the purpose. Both forbidden minors are variations of Halin's graph H_0 (see [4]).

Let G, G' be graphs, and let $f : V(G) \to V(G')$ be surjective. Halin [10] defines f to be a *homomorphism* from G onto G' if

$$vw \in E(G) \quad \Rightarrow \quad \Big(f(v)f(w) \in E(G') \vee f(v) = f(w) \Big)$$

and

$$v'w' \in E(G') \quad \Rightarrow \quad \exists\, vw \in E(G) : \Big(f(v) = v' \wedge f(w) = w' \Big);$$

f is called *contractive* if $G [\, f^{-1}(v) \,]$ is connected for every $v \in V(G')$. Notice that homomorphisms between graphs map simplices to simplices.

If $H \subset G$ and $H' \subset G'$ are induced subgraphs and $f : V(G) \to V(G')$ is a homomorphism, we shall abbreviate $G' [\, f(V(H)) \,]$ to $f(H)$ and $G [\, f^{-1}(V(H')) \,]$ to $f^{-1}(H')$. Then $f|_{f^{-1}(H')}$ is a homomorphism from $f^{-1}(H')$ onto H', which is contractive if f is. Conversely, $f|_H$ is a contractive homomorphism from H onto $f(H)$ if H is convex in G and f is contractive, in which case $f(H)$ is also convex in G'.

Let us call a contractive homomorphism f from G onto G' *simplicial* if f preserves simplicial closeness, i.e. if f satisfies the implication

$$v, w \in V(G) \text{ are close in } G \quad \Rightarrow \quad f(v),\, f(w) \text{ are close in } G'.$$

Simplicial homomorphisms are well compatible with simplicial decompositions: they map minimally convex subgraphs to minimally convex subgraphs and maximally prime subgraphs to maximally prime subgraphs or attached simplices, and the restriction of a simplicial homomorphism to a convex subgraph is again a simplicial homomorphism.

A graph H' is often called a *minor* of a graph G if G has a subgraph H from which there exists a contractive homomorphism f onto H'. We shall call H' a *simplicial minor* of G (and write $G \succcurlyeq_s H'$), if H and f can be chosen in such a way that H is convex in G and f is simplicial.

It is not difficult to show that if $G_1 \succcurlyeq_s G_2$ and $G_2 \succcurlyeq_s G_3$, then $G_1 \succcurlyeq_s G_3$. Or in other words, if $\mathcal{H}$ is a set of graphs, then the graph property

$$\mathcal{G}(\mathcal{H})_{\succcurlyeq_s} := \{\, G \mid H \in \mathcal{H} \Rightarrow G \not\succcurlyeq_s H \,\}$$

is closed under taking simplicial minors.

In order to be characterized in this way, the class of decomposable graphs must of course match this feature, i.e. simplicial minors of decomposable graphs must again be decomposable. And indeed, it can be shown that this is so: if a countable graph G has a simplicial tree-decomposition into primes and H is a simplicial minor of G, then H has a simplicial tree-decomposition into primes.

Let $\mathcal{G}$ be the class of countable graphs that have a simplicial tree-decomposition into primes, and let $\mathcal{H}$ be the class of all other countable graphs. Then trivially $\mathcal{G} \supset \mathcal{G}(\mathcal{H})_{\succcurlyeq_s}$, and since simplicial minors of decomposable graphs are again decomposable, we even have $\mathcal{G} = \mathcal{G}(\mathcal{H})_{\succcurlyeq_s}$. Moreover, this assertion remains valid if we replace $\mathcal{H}$ with any set $\mathcal{H}' \subset \mathcal{H}$ in which every graph of $\mathcal{H}$ has a simplicial minor. We are therefore left with the challenge to find a minimal such $\mathcal{H}'$.

Let H_1 be the graph obtained from H_0 by contracting the path Q to one vertex q, and let H_2 be obtained from H_1 by joining up all vertices inside P, thus turning P into a simplex (Fig. 7). Neither H_1 nor H_2 has a prime decomposition (by Theorem 1; put $C' := \{\, q \,\}$), and neither of the two graphs is a simplicial minor of the other. It will be shown in [6] that $\mathcal{H}' = \{\, H_1, H_2 \,\}$ solves the above problem:

Theorem. [6] *A countable graph G has a simplicial tree-decomposition into primes if and only if neither of H_1, H_2 is a simplicial minor of G.*

A similar characterization will be obtained for the countable graphs that admit a tree-decomposition into primes.

Figure 7

References

1. R. Diestel, On the problem of finding small subdivision and homomorphism bases for classes of countable graphs, *Discrete Math.* **55** (1985), 21–33.
2. R. Diestel, A separation property of planar triangulations, *J. Graph Theory* **,** to appear.
3. R. Diestel, Simplicial decompositions of graphs—some uniqueness results, *J. Combin. Theory B* **,** to appear.
4. R. Diestel, Simplicial tree-decompositions of infinite graphs I, submitted.
5. R. Diestel, Simplicial tree-decompositions of infinite graphs III—the uniqueness of prime decompositions, submitted.
6. R. Diestel, Simplicial minors and decompositions of graphs, submitted.
7. G. A. Dirac, Simplicial decompositions of infinite graphs, *Symp. Math. Roma* XVIII (1986), 159–195.
8. R. Halin, Simpliziale Zerfällungen beliebiger (endlicher oder unendlicher) Graphen, *Math. Ann.* **156** (1964), 216–225.
9. R. Halin, Zur Klassifikation der endlichen Graphen nach H. Hadwiger und K. Wagner, *Math. Ann.* **172** (1967), 46–78.
10. R. Halin, *Graphentheorie II*, Wissenschaftliche Buchgesellschaft, Darmstadt 1981.
11. R. Halin, On the representation of triangulated graphs in trees, *Europ. J. Combinatorics* **5** (1984), 23–28.
12. R. Halin, Simplicial decompositions and triangulated graphs, in: (B. Bollobás, ed.) *Graph Theory and Combinatorics*, Academic Press, London 1984.
13. N. Robertson and P. D. Seymour, Graph minors I–XVIII, *J. Combin. Theory B* **,** to appear.
14. K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.* **114** (1937), 570–590.