(8.3.1)
(12.3.1)
(12.3.4)
(12.3.8)

Proof. By Lemma 12.3.4, we have $\mathrm{tw}(K^4) \geqslant 3$. By Proposition 12.4.2, therefore, a graph of tree-width < 3 cannot contain K^4 as a minor.

Conversely, let G be a graph without a K^4 minor; we assume that $|G| \geqslant 3$. Add edges to G until the graph G' obtained is edge-maximal without a K^4 minor. By Proposition 8.3.1, G' can be constructed recursively from triangles by pasting along K^2s. By induction on the number of recursion steps and Lemma 12.3.4, every graph constructible in this way has a tree-decomposition into triangles (as in the proof of Proposition 12.3.8). Such a tree-decomposition of G' has width 2, and by Lemma 12.3.1 it is also a tree-decomposition of G. □

A question converse to the above is to ask for which X (other than K^3 and K^4) the tree-width of the graphs in $\mathrm{Forb}_{\preccurlyeq}(X)$ is bounded. Interestingly, it is not difficult to show that any such X must be planar. Indeed, consider the graph on $\{1, \dots, n\}^2$ with the edge set

$$\{\,(i,j)(i',j') : |i-i'| + |j-j'| = 1\,\};$$

this graph is called the $n \times n$ *grid*. Clearly, the $n \times n$ grid is planar (for every n), and hence lies in every class $\mathrm{Forb}_{\preccurlyeq}(X)$ with non-planar X. On the other hand, it is not difficult to show that the tree-width of the $n \times n$ grid tends to infinity with n (Exercise 19). Therefore, the tree-width of the graphs in $\mathrm{Forb}_{\preccurlyeq}(X)$ cannot be bounded unless X is planar. *grid*

The following deep and surprising theorem says that, conversely, the tree-width of the graphs in $\mathrm{Forb}_{\preccurlyeq}(X)$ is bounded for every planar X:

Theorem 12.4.4. (Robertson & Seymour 1986)
The tree-width of the graphs in $\mathrm{Forb}_{\preccurlyeq}(X)$ *is bounded if and only if* X *is planar.*

The proof of Theorem 12.4.4 is too involved to be presented here. However, there is a similar result on the related notion of 'path-width', which we shall prove instead: its proof is much simpler, but it gives an indication of some of the techniques used for the proof of Theorem 12.4.4.

path-decomposition

A tree-decomposition whose tree is a path is called a *path-decomposition*. We usually denote a path-decomposition $(P, \mathcal{V})$ simply by listing the sets $V_1, \dots, V_s \in \mathcal{V}$ in the order defined by P. The least width of a path-decomposition of G is the *path-width* $\mathrm{pw}(G)$ of G.

path-width $\mathrm{pw}(G)$

The analogue of Theorem 12.4.4 for path-width is obtained simply by replacing planarity with acyclicity:

Theorem 12.4.5. (Robertson & Seymour 1983)
The path-width of the graphs in $\mathrm{Forb}_{\preccurlyeq}(X)$ *is bounded if and only if* X *is a forest.*

The forward implication of Theorem 12.4.5 is again easy. All we have to show is that trees can have arbitrarily large path-width: since

$\mathrm{Forb}_{\preccurlyeq}(X)$ contains all trees if X has a cycle, this will imply that forbidding X cannot bound the path-width unless X is a forest.

How can one show that a graph—in our case, a tree—has large path-width? Let $(V_1, \dots, V_s)$ be a path-decomposition of some connected graph G, of width $\mathrm{pw}(G)$ and such that $V_1, V_s \neq \emptyset$. Pick vertices
(12.3.2) $v_1 \in V_1$ and $v_s \in V_s$, and let Q be a v_1–v_s path in G. By Lemma 12.3.2, Q meets every V_r, $r = 1, \dots, s$. Hence, the path-decomposition
(12.3.1) $(V_1 \setminus V(Q), \dots, V_s \setminus V(Q))$ of $G - Q$ has width at most $\mathrm{pw}(G) - 1$, so $\mathrm{pw}(G - Q) < \mathrm{pw}(G)$.

Thus every connected graph G contains a path whose deletion reduces the path-width of G. If we may further assume (e.g. by some suitable induction hypothesis) that $G - Q$ has large path-width for every path $Q \subseteq G$, then G has even larger path-width.

We now use this idea to show that trees can have arbitrarily large path-width. Let T_3^k denote the tree in which one specified vertex r has degree 3, all other vertices (except the leaves) have degree 4, and all leaves have distance k from r. If $T = T_3^{k+1}$ and Q is any path in T, then Q contains at most two of the three edges at r; hence, $T - Q$ contains a component of $T - r$, which is a copy of T_3^k. Induction on k thus shows that $\mathrm{pw}(T_3^k) \geqslant k$ for all k.

For the proof of the backward implication of Theorem 12.4.5 we
$G = (V, E)$ need some definitions and two lemmas. Let $G = (V, E)$ be a graph. For
∂X $X \subseteq V$, we denote by ∂X the set of all vertices in X with a neighbour
$\mathcal{B}_n$ in $G - X$. For every integer $n \geqslant 0$ we define a set $\mathcal{B}_n = \mathcal{B}_n(G)$ of subsets of V by the following recursion:

(i) $\emptyset \in \mathcal{B}_n$;

(ii) if $X \in \mathcal{B}_n$, $X \subseteq Y \subseteq V$ and $|\partial X| + |Y \setminus X| \leqslant n$, then $Y \in \mathcal{B}_n$ (Fig. 12.4.1).

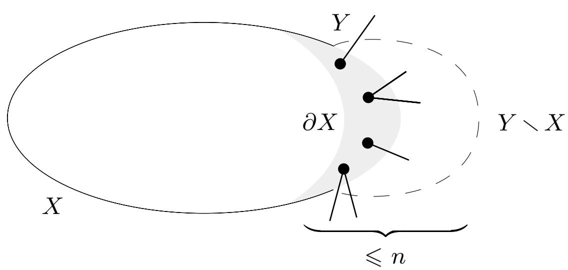

Fig. 12.4.1. If X lies in $\mathcal{B}_n$, then so does Y

Thus, a set $X \subseteq V$ lies in $\mathcal{B}_n$ if and only if there is a sequence

$$\emptyset = X_0 \subseteq \dots \subseteq X_s = X$$

such that $|\partial X_r| + |X_{r+1} \setminus X_r| \leqslant n$ for all $r < s$. For example, if $(V_1, \dots, V_s)$ is a path-decomposition of G of width $< n$, then all its 'initial segments' $V_1 \cup \ldots \cup V_r$ ($r \leqslant s$) lie in $\mathcal{B}_n$, including V for $r = s$ (exercise). Conversely, we have the following:

Lemma 12.4.6. *If* $V \in \mathcal{B}_n$, *then* $\mathrm{pw}(G) < n$.

Proof. If $V \in \mathcal{B}_n$, then there is a sequence $\emptyset = X_0 \subseteq \ldots \subseteq X_s = V$ such that $|\partial X_r| + |X_{r+1} \setminus X_r| \leqslant n$ for all $r < s$. We set

$$V_{r+1} := \partial X_r \cup (X_{r+1} \setminus X_r)$$

and show that $(V_1, \ldots, V_s)$ is a path-decomposition of G (Fig. 12.4.2).

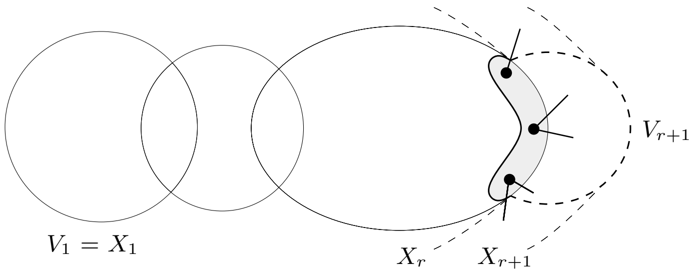

Fig. 12.4.2. Constructing a path-decomposition from $\mathcal{B}_n$

Induction on r shows that $X_r = V_1 \cup \ldots \cup V_r$ for all $r \leqslant s$; in particular, $V = X_s = V_1 \cup \ldots \cup V_s$. Hence (T1) holds. For the proof of (T2), let $xy \in E$ be given. Let $r(x)$ be minimum with $x \in X_{r(x)}$, and $r(y)$ minimum with $y \in X_{r(y)}$. We assume that $r(x) \leqslant r(y) =: r$, and show that x, like y, lies in V_r. This is clear if $r(x) = r$. Yet if $r(x) < r$, then x lies in X_{r-1}, and hence in $\partial X_{r-1} \subseteq V_r$ since $xy \in E$. For the proof of (T3), finally, let $p < q < r$ and $x \in V_p \cap V_r$ be given. Then $x \in V_p \subseteq V_1 \cup \ldots \cup V_{q-1} = X_{q-1} \subseteq X_{r-1}$, so $x \in X_{r-1} \cap V_r$. By definition of V_r this implies $x \in \partial X_{r-1}$, so $x \in \partial X_{r-1} \cap X_{q-1} \subseteq \partial X_{q-1} \subseteq V_q$. $\square$

Lemma 12.4.7. *Let* $Y \in \mathcal{B}_n$ *and* $Z \subseteq Y$. *If there is a family* $(P_z)_{z \in \partial Z}$ *of disjoint* Z–∂Y *paths in* G *with* $z \in P_z$ *for all* $z \in \partial Z$, *then* $Z \in \mathcal{B}_n$ *(Fig. 12.4.3).*

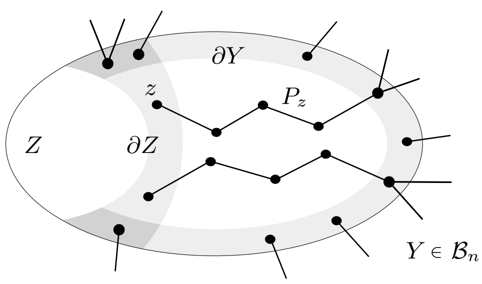

Fig. 12.4.3. Five paths P_z; three of them trivial

Proof. By definition of $\mathcal{B}_n$, there are sets $\emptyset = Y_0 \subseteq \ldots \subseteq Y_s = Y$ such that

$$|\partial Y_r| + |Y_{r+1} \setminus Y_r| \leqslant n$$

Z_r for all $r < s$. We shall deduce from this that, setting $Z_r := Y_r \cap Z$, we also have

$$|\partial Z_r| + |Z_{r+1} \setminus Z_r| \leqslant n$$

for all $r < s$; then $Z = Z_s \in \mathcal{B}_n$.

Fix r. Since $Z_{r+1} \setminus Z_r = Z_{r+1} \setminus Y_r \subseteq Y_{r+1} \setminus Y_r$, it suffices to show that $|\partial Z_r| \leqslant |\partial Y_r|$. We prove this by constructing an injective map $z \mapsto y$ from $\partial Z_r \setminus \partial Y_r$ to $\partial Y_r \setminus \partial Z_r$ (Fig. 12.4.4).

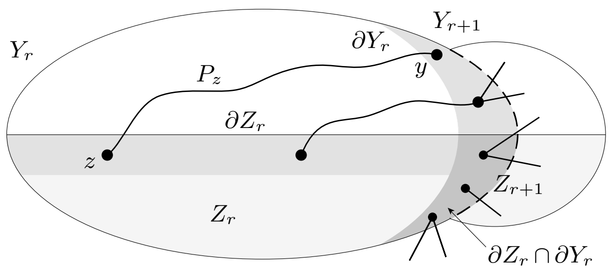

Fig. 12.4.4. An injective path linkage between $\partial Z_r \setminus \partial Y_r$ and $\partial Y_r \setminus \partial Z_r$

z Consider a vertex $z \in \partial Z_r \setminus \partial Y_r$. Then z has a neighbour in $Y_r \setminus Z_r = Y_r \setminus Z$, so $z \in \partial Z$. Now P_z is a path from $(Z_r \subseteq)\ Y_r$ to ∂Y,
y so P_z has a vertex y in ∂Y_r; note that $y \neq z$ by the choice of z. As z is the only vertex of P_z in Z, we have $y \in \partial Y_r \setminus \partial Z_r$. Since the paths P_z are disjoint, these vertices y are distinct for different z, so $|\partial Z_r| \leqslant |\partial Y_r|$ as claimed. □

(1.5.2)
(3.3.1) **Proof of Theorem 12.4.5.** The forward implication of the theorem was proved earlier. For the converse, we prove the following:

n, F *If* $\mathrm{pw}(G) \geqslant n \in \mathbb{N}$, *then* G *contains every forest* F *with* $|F| - 1 = n$ *as a minor.* $(*)$

Clearly, by $(*)$, if X is any forest then every graph in $\mathrm{Forb}_{\preccurlyeq}(X)$ has path-width less than $|X| - 1$.

So let $\mathrm{pw}(G) \geqslant n$, and assume without loss of generality that F is a
$v_1, v_2, \ldots$ tree. Let $(v_1, \ldots, v_{n+1})$ be an enumeration of $V(F)$ as in Corollary 1.5.2, i.e. so that v_{i+1} has exactly one neighbour in $\{ v_1, \ldots, v_i \}$, for all $i \leqslant n$.

For every $i = 0,\dots,n$, we shall define a family $\mathcal{X}^i = (X_0^i,\dots,X_i^i)$ of disjoint subsets of V, such that $X_j^k \subseteq X_j^\ell$ whenever $j \leqslant k \leqslant \ell$ and all X_j^i with $j > 0$ are connected in G. We then write $\mathcal{X}^i$

$$X^i := X_0^i \cup \ldots \cup X_i^i\,.$$ X^i

For each i, the following three statements will hold:

(i) G contains an X_j^i–X_k^i edge whenever $1 \leqslant j < k \leqslant i$ and $v_j v_k \in E(F)$ (so $F[v_1,\dots,v_i]$ is a minor of $G[X_1^i \cup \ldots \cup X_i^i]$);

(ii) $|X_j^i \cap \partial X^i| = 1$ for all $1 \leqslant j \leqslant i$;

(iii) X^i is maximal in $\mathcal{B}_n$ with $|\partial X^i| \leqslant i$.

Note that (ii) and (iii) together imply $|\partial X^i| = i$.

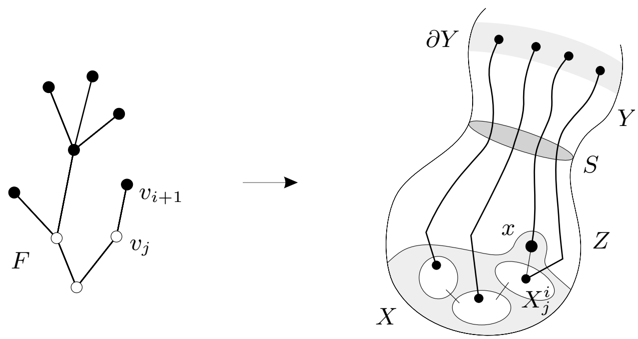

Fig. 12.4.5. Constructing an F minor in G

Let $X_0^0 \in \mathcal{B}_n$ be maximal with $|\partial X_0^0| = 0$ (possibly $X_0^0 = \emptyset$). Then (i)–(iii) hold for $i = 0$. Assume now that $\mathcal{X}^i$ has been defined so that (i)–(iii) hold, for given $i \leqslant n$. If $i = 0$, let x be any vertex of $G - X^0$; note that $G - X^0 \neq \emptyset$, since $X^0 \in \mathcal{B}_n$ but $V \notin \mathcal{B}_n$ by Lemma 12.4.6. If $i > 0$, consider the unique $j \leqslant i$ with $v_j v_{i+1} \in E(F)$, and let $x \in G - X^i$ be a neighbour of the unique vertex in $X_j^i \cap \partial X^i$; cf. (ii). Set X_0^0 x x

$$X := X^i \cup \{\,x\,\}\,.$$ X

If $i = n$, we have $F \preccurlyeq G[X]$ by (i) and the choice of x, so we are done. Assume then that $i < n$. Then $X \in \mathcal{B}_n$ and $|\partial X| > i$, by (iii) and the definition of $\mathcal{B}_n$. Since $\partial X \cap X^i \subseteq \partial X^i$, this means that

$$|\partial X| = i + 1$$

and

$$\partial X = \partial X^i \cup \{\,x\,\}\,.$$

Y Let $Y \in \mathcal{B}_n$ be maximal with $X \subseteq Y$ and

$$|\partial Y| = i + 1\,;$$

this set Y will later become X^{i+1}.

$\mathcal{P}$ By Menger's theorem (3.3.1), there exist a set $\mathcal{P}$ of disjoint X–∂Y
S paths in $G[Y]$ and a set $S \subseteq Y$ which separates X from ∂Y in $G[Y]$ and contains exactly one vertex from each path in $\mathcal{P}$ (but no other vertices).
Z Let Z denote the union of S with the vertex sets of the components of $G - S$ that meet X. Clearly,

$$\partial Z \subseteq S$$

and $X^i \subsetneq X \subseteq Z$; let us show that even

$$X \subseteq Z \subseteq Y.$$

Let $z \in Z$ be given. If $z \in S$, then $z \in Y$ by the choice of S. If $z \notin S$, then z can be reached from X by a path avoiding S. If $z \notin Y$, then by $X \subseteq Y$ this path contains an X–∂Y path in $G[Y]$, contradicting the definition of S.

Thus $Z \subseteq Y \in \mathcal{B}_n$, so $Z \in \mathcal{B}_n$ by Lemma 12.4.7 applied to the Z–∂Y paths contained in the paths from $\mathcal{P}$. By (iii), $i < |\partial Z| \leqslant |S| = |\mathcal{P}|$. As every path in $\mathcal{P}$ meets ∂X, this gives $i < |\mathcal{P}| \leqslant |\partial X| = i + 1$ and hence

$$|\mathcal{P}| = i + 1\,,$$

so $\mathcal{P}$ links ∂X to ∂Y bijectively.

We now define $\mathcal{X}^{i+1}$. For $1 \leqslant k \leqslant i$ let $X_k^{i+1} := X_k^i \cup V(P_k)$, where P_k is the path in $\mathcal{P}$ containing the unique vertex of ∂X^i in X_k^i; cf. (ii). Similarly, let X_{i+1}^{i+1} be the vertex set of the path in $\mathcal{P}$ that contains x. Finally, put $X_0^{i+1} := Y \setminus (X_1^{i+1} \cup \ldots \cup X_{i+1}^{i+1})$. Clearly,

$$X^{i+1} = Y.$$

Condition (i) for $i+1$ holds by choice of x; (ii) holds by $X^{i+1} = Y$ and definition of $\mathcal{P}$; (iii) holds by $X^{i+1} = Y$ and the choice of Y, combined with $X^i \subseteq Y$ and (iii) for i.

As remarked earlier, $F \preccurlyeq G$ follows from the definition of X when $i = n$. $\square$