10 Hamilton Cycles

In Chapter 1.8 we briefly discussed the problem of when a graph contains an Euler tour, a closed walk traversing every edge exactly once. The simple Theorem 1.8.1 solved that problem quite satisfactorily. Let us now ask the analogous question for vertices: when does a graph G contain a closed walk that contains every vertex of G exactly once? If $|G| \geqslant 3$, then any such walk is a cycle: a *Hamilton cycle* of G. If G has a Hamilton cycle, it is called *hamiltonian*. Similarly, a path in G containing every vertex of G is a *Hamilton path*.

Hamilton cycle

Hamilton path

To determine whether or not a given graph has a Hamilton cycle is much harder than deciding whether it is Eulerian, and no good characterization is known[1] of the graphs that do. In the first two sections of this chapter we present the standard sufficient conditions for the existence of a Hamilton cycle, as well as a more recent non-standard one. The third section is devoted to the proof of another classic: Fleischner's theorem that the 'square' of every 2-connected graph has a Hamilton cycle. We shall present this theorem with an ingenious short proof due to Georgakopoulos.

10.1 Sufficient conditions

What kind of condition might be sufficient for the existence of a Hamilton cycle in a graph G? Purely global assumptions, like high edge density, will not be enough: we cannot do without the local property that every vertex has at least two neighbours. But neither is any large (but constant) minimum degree sufficient: it is easy to find graphs without a Hamilton cycle whose minimum degree exceeds any given constant bound.

[1] . . . or indeed expected to exist; see the notes for details.

The following classic result derives its significance from this background:

Theorem 10.1.1. (Dirac 1952)
Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ has a Hamilton cycle.

Proof. Let $G = (V, E)$ be a graph with $|G| = n \geqslant 3$ and $\delta(G) \geqslant n/2$. Then G is connected: otherwise, the degree of any vertex in the smallest component C of G would be less than $|C| \leqslant n/2$.

Let $P = v_0 \dots v_k$ be a longest path in G. Let us call v_i the *left* end of the edge $v_i v_{i+1}$, and v_{i+1} its *right* end. By the maximality of P, each of the $d(v_0) \geqslant n/2$ neighbours of v_0 is the right end of an edge of P, and these $d(v_0)$ edges are distinct. Similarly, at least $n/2$ edges of P are such that their left end is adjacent to v_k. Since P has fewer than n edges, it has an edge $v_i v_{i+1}$ with both properties (Fig. 10.1.1).

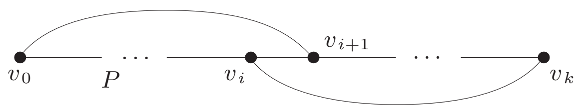

Fig. 10.1.1. Finding a Hamilton cycle in the proof Theorem 10.1.1

We claim that the cycle $C := v_0 v_{i+1} P v_k v_i P v_0$ is a Hamilton cycle of G. Indeed, since G is connected, C would otherwise have a neighbour in $G - C$, which could be combined with a spanning path of C into a path longer than P. □

Theorem 10.1.1 is best possible in that we cannot replace the bound of $n/2$ with $\lfloor n/2 \rfloor$: if n is odd and G is the union of two copies of $K^{\lceil n/2 \rceil}$ meeting in one vertex, then $\delta(G) = \lfloor n/2 \rfloor$ but $\kappa(G) = 1$, so G cannot have a Hamilton cycle. In other words, the high level of the bound of $\delta \geqslant n/2$ is needed to ensure, if nothing else, that G is 2-connected: a condition just as trivially necessary for hamiltonicity as a minimum degree of at least 2. It would seem, therefore, that prescribing some high (constant) value for κ rather than for δ stands a better chance of implying hamiltonicity. However, this is not so: although every large enough k-connected graph contains a cycle of length at least $2k$ (Ex. 22, Ch. 3), the graphs $K_{k,n}$ show that this is already best possible.

Slightly more generally, a graph G with a separating set S of k vertices such that $G - S$ has more than k components is clearly not hamiltonian. Could it be true that all non-hamiltonian graphs have such a separating set, one that leaves many components compared with its size? We shall return to this question at the end of this section.

For now, just note that such graphs as above also have relatively large independent sets: pick one vertex from each component of $G - S$ to obtain one of order at least $k+1$. Might we be able to force a Hamilton cycle by forbidding large independent sets?

By itself, the assumption of $\alpha(G) \leqslant k$ guarantees a cycle of length at least $|G|/k$ (Ex. 15, Ch. 5). But a Hamilton cycle cannot be forced even by assuming $\alpha \leqslant 2$. (Example?) Yet making α small compared with κ, even just small enough to kill our earlier $K_{k,n}$ counterexample (where $n > k$ and hence $K_{k,n} = K_{\kappa,\alpha}$), it does indeed imply hamiltonicity:

Proposition 10.1.2. *Every graph G with $|G| \geqslant 3$ and $\alpha(G) \leqslant \kappa(G)$ has a Hamilton cycle.*

Proof. Let C be a longest cycle in G. Enumerate the vertices of C cyclically, say as $V(C) = \{ v_i \mid i \in \mathbb{Z}_n \}$ with $v_i v_{i+1} \in E(C)$ for all $i \in \mathbb{Z}_n$. C

Suppose C is not a Hamilton cycle. Let $\{ v_i \mid i \in I \}$ be the set of neighbours on C of a component D of $G - C$. No two of these neighbours are adjacent on C, since we could then extend C through D to form a longer cycle (Fig. 10.1.2, left). Similarly, the maximality of $|C|$ implies that $\{ v_{i+1} \mid i \in I \}$ is independent (Fig. 10.1.2, right). I, D

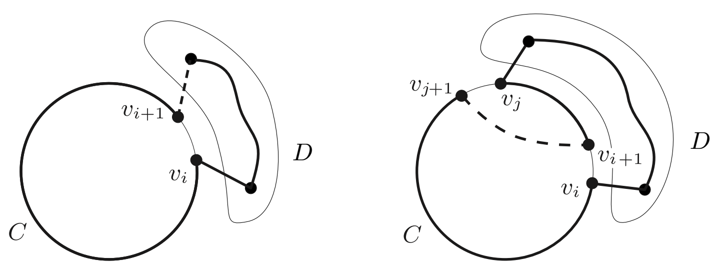

Fig. 10.1.2. Two cycles longer than C

Pick a vertex $v \in D$; then $\{ v_{i+1} \mid i \in I \} \cup \{v\}$ is still independent and has size $|I|+1$. But $\{ v_i \mid i \in I \}$ separates G, so $|I| \geqslant \kappa(G)$. This contradicts our assumption that $\alpha(G) \leqslant \kappa(G)$. $\square$

Our next result uses the ideas from the proof of Proposition 10.1.2 to establish a *local* degree condition for hamiltonicity, considerably strengthening Dirac's theorem and several similar results proved later in its wake:

Theorem 10.1.3. (Asratian & Khachatrian 1990)
A connected graph G of order at least 3 is hamiltonian if

$$d(u) + d(w) \geqslant |N(u) \cup N(v) \cup N(w)|$$

for every induced path uvw.

Proof. Suppose G is not hamiltonian. Consider any induced path uvw. Since $d(u)+d(w) = |N(u)\cup N(w)|+|N(u)\cap N(w)|$, our degree assumption implies that

$$\begin{aligned}|N(u)\cap N(w)| &\geqslant |N(u)\cup N(v)\cup N(w)| - |N(u)\cup N(w)|\\ &= |N(v)\setminus N(\{u,w\})| \geqslant |\{u,w\}| \geqslant 2\,. \end{aligned}\tag{1}$$

In particular, G contains a cycle.

C u V V^+

Let C be a longest cycle in G. Since G is not hamiltonian, there is a vertex $u \notin C$ that has a neighbour on C; let $V := N(u)\cap V(C)$. For vertices $v \in V$ let v^+ denote the successor of v on C in some fixed orientation of C, and put $V^+ := \{\, v^+ \mid v \in V \,\}$.

Since C is a longest cycle, we have $V \cap V^+ = \emptyset$, and

no two vertices of $V^+\cup\{u\}$ are adjacent or have a common neighbour outside C (2)

(compare Fig. 10.1.2). In particular, the paths uvv^+ are induced. Hence every $v \in V$ satisfies

$$|N(u)\cap N(v^+)| \underset{(1)}{\geqslant} |N(v)\setminus N(\{u,v^+\})| \underset{(2)}{\geqslant} |N(v)\cap V^+|+1\,;$$

the second inequality comes from the fact that $N(\{u,v^+\})\cap V^+ = \emptyset$ and u lies in neither of these sets. The number $\|V,V^+\| = \|V^+,V\|$ of edges between V and V^+ therefore satisfies

$$\|V,V^+\| = \sum_{v\in V} |N(v)\cap V^+| \leqslant \sum_{v\in V}\big(|N(u)\cap N(v^+)|-1\big) \underset{(2)}{\leqslant} \|V^+,V\| - |V|$$

(a contradiction); for the last inequality note that, by (2), every $v^+ \in V^+$ has all its common neighbours with u on C, and hence in V. □

t-tough

Let us return to the question of whether an assumption that no small separator leaves many components can guarantee a Hamilton cycle. A graph G is called *t-tough*, where $t > 0$ is any real number, if for every separator S of G the graph $G - S$ has at most $|S|/t$ components. Clearly, hamiltonian graphs must be 1-tough – so what about the converse?

Unfortunately, it is not difficult to find even small graphs that are 1-tough but have no Hamilton cycle (Exercise 6), so toughness does not provide a characterization of hamiltonian graphs in the spirit of Menger's theorem or Tutte's 1-factor theorem. However, a famous conjecture asserts that t-toughness for some t will force hamiltonicity:

Toughness Conjecture. (Chvátal 1973)
There exists $t > 0$ such that every t-tough graph has a Hamilton cycle.

The toughness conjecture was long expected to hold with t as small as $t = 2$. This was disproved after many years, but the general conjecture remains open. See the exercises for how the conjecture ties in with the results given in the remainder of this chapter.

It may come as a surprise to learn that hamiltonicity is also related to the four colour problem. As we noted in Chapter 6.6, the four colour theorem is equivalent to the non-existence of a planar snark, i.e. to the assertion that every bridgeless planar cubic graph has a 4-flow. It is easily checked that 'bridgeless' can be replaced with '3-connected' in this assertion, and that every hamiltonian graph has a 4-flow (Ex. 16, Ch. 6). For a proof of the four colour theorem, therefore, it would suffice to show that every 3-connected planar cubic graph has a Hamilton cycle!

Unfortunately, this is not the case: the first counterexample was found by Tutte in 1946. Ten years later, Tutte proved the following deep theorem as a best possible weakening:

Theorem 10.1.4. (Tutte 1956)
Every 4-connected planar graph has a Hamilton cycle.

Although, at first glance, it appears that the study of Hamilton cycles is a part of graph theory that cannot possibly extend to infinite graphs, there is a fascinating conjecture that does just that. Recall that a *circle* in an infinite graph G is a homeomorphic copy of the unit circle S^1 in the topological space $|G|$ formed by G and its ends (see Chapter 8.6). A *Hamilton circle* of G is a circle that contains every vertex of G. *Hamilton circle*

Conjecture. (Bruhn 2003)
Every locally finite 4-connected planar graph has a Hamilton circle.

10.2 Hamilton cycles and degree sequences

Historically, Dirac's theorem formed the point of departure for the discovery of a series of weaker and weaker degree conditions, all sufficient for hamiltonicity. The development culminated in a single theorem that encompasses all the earlier results: the theorem we shall prove in this section.

If G is a graph with n vertices and degrees $d_1 \leqslant \ldots \leqslant d_n$, then the n-tuple $(d_1, \ldots, d_n)$ is called the *degree sequence* of G. Note that this sequence is unique, even though G has several vertex enumerations giving rise to its degree sequence. Let us call an arbitrary integer sequence $(a_1, \ldots, a_n)$ *hamiltonian* if every graph with n vertices and a degree sequence pointwise greater than $(a_1, \ldots, a_n)$ is hamiltonian. (A sequence $(d_1, \ldots, d_n)$ is *pointwise greater* than $(a_1, \ldots, a_n)$ if $d_i \geqslant a_i$ for all i.) *degree sequence* *hamiltonian sequence* *pointwise greater*

The following theorem characterizes all hamiltonian sequences:

Theorem 10.2.1. (Chvátal 1972)
An integer sequence $(a_1, \dots, a_n)$ *such that* $0 \leqslant a_1 \leqslant \dots \leqslant a_n < n$ *and* $n \geqslant 3$ *is hamiltonian if and only if the following holds for every* $i < n/2$*:*

$$a_i \leqslant i \Rightarrow a_{n-i} \geqslant n - i\,.$$

$(a_1, \dots, a_n)$ *Proof.* Let $(a_1, \dots, a_n)$ be an arbitrary integer sequence such that $0 \leqslant a_1 \leqslant \dots \leqslant a_n < n$ and $n \geqslant 3$. We first assume that this sequence satisfies the condition of the theorem and prove that it is hamiltonian.

Suppose not. Then there exists a graph whose degree sequence
$(d_1, \dots, d_n)$ $(d_1, \dots, d_n)$ satisfies

$$d_i \geqslant a_i \qquad \text{for all } i \tag{1}$$

$G = (V, E)$ but which has no Hamilton cycle. Let $G = (V, E)$ be such a graph, chosen with the maximum number of edges.

By (1), our assumptions for $(a_1, \dots, a_n)$ transfer to the degree sequence $(d_1, \dots, d_n)$ of G; thus,

$$d_i \leqslant i \Rightarrow d_{n-i} \geqslant n - i \qquad \text{for all } i < n/2. \tag{2}$$

x, y Let x, y be distinct and non-adjacent vertices in G, with $d(x) \leqslant d(y)$ and $d(x) + d(y)$ as large as possible. One easily checks that the degree sequence of $G + xy$ is pointwise greater than $(d_1, \dots, d_n)$, and hence than $(a_1, \dots, a_n)$. Hence, by the maximality of G, the new edge xy lies on a
$x_1, \dots, x_n$ Hamilton cycle H of $G + xy$. Then $H - xy$ is a Hamilton path $x_1, \dots, x_n$ in G, with $x_1 = x$ and $x_n = y$ say.

As in the proof of Dirac's theorem, we now consider the index sets

$$I := \{\, i \mid xx_{i+1} \in E \,\} \quad \text{and} \quad J := \{\, j \mid x_j y \in E \,\}\,.$$

Then $I \cup J \subseteq \{1, \dots, n-1\}$, and $I \cap J = \emptyset$ because G has no Hamilton cycle. Hence

$$d(x) + d(y) = |I| + |J| < n\,, \tag{3}$$

h so $h := d(x) < n/2$ by the choice of x.

Since $x_i y \notin E$ for all $i \in I$, all these x_i were candidates for the choice of x (together with y). Our choice of $\{x, y\}$ with $d(x) + d(y)$ maximum thus implies that $d(x_i) \leqslant d(x)$ for all $i \in I$. Hence G has at least $|I| = h$ vertices of degree at most h, so $d_h \leqslant h$. By (2), this implies that $d_{n-h} \geqslant n - h$, i.e. the $h+1$ vertices with the degrees $d_{n-h}, \dots, d_n$ all have degree at least $n - h$. Since $d(x) = h$, one of these vertices,
z z say, is not adjacent to x. Since

$$d(x) + d(z) \geqslant h + (n - h) = n\,,$$

this contradicts the choice of x and y by (3).

Let us now show that, conversely, for every sequence $(a_1, \ldots, a_n)$ as in the theorem, but with

$$a_h \leqslant h \quad \text{and} \quad a_{n-h} \leqslant n-h-1$$

for some $h < n/2$, there exists a graph that has a pointwise greater degree sequence than $(a_1, \ldots, a_n)$ but no Hamilton cycle. As the sequence h

$$(\underbrace{h, \ldots, h}_{h \text{ times}}, \underbrace{n-h-1, \ldots, n-h-1}_{n-2h \text{ times}}, \underbrace{n-1, \ldots, n-1}_{h \text{ times}})$$

is pointwise greater than $(a_1, \ldots, a_n)$, it suffices to find a graph with this degree sequence that has no Hamilton cycle.

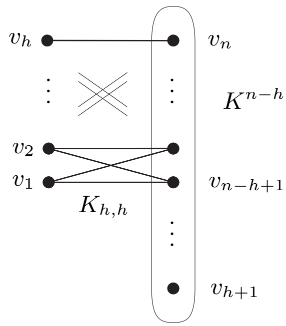

Fig. 10.2.1. Any cycle containing $v_1, \ldots, v_h$ misses v_{h+1}

Figure 10.2.1 shows such a graph G, with vertices $v_1, \ldots, v_n$ and the edge set

$$\{\, v_i v_j \mid i, j > h \,\} \cup \{\, v_i v_j \mid i \leqslant h;\ j > n-h \,\};$$

it is the union of a K^{n-h} on the vertices $v_{h+1}, \ldots, v_n$ and a $K_{h,h}$ with partition sets $\{v_1, \ldots, v_h\}$ and $\{v_{n-h+1}, \ldots, v_n\}$. Deleting these latter h vertices leaves more than h components. Hence G is not 1-tough, and thus not hamiltonian. $\square$

By applying Theorem 10.2.1 to graphs of the form $G * K^1$, one can easily prove the following adaptation of the theorem to Hamilton paths. Let an integer sequence be called *path-hamiltonian* if every graph with a pointwise greater degree sequence has a Hamilton path.

Corollary 10.2.2. *An integer sequence $(a_1, \ldots, a_n)$ such that $n \geqslant 2$ and $0 \leqslant a_1 \leqslant \ldots \leqslant a_n < n$ is path-hamiltonian if and only if every $i \leqslant n/2$ is such that $a_i < i \Rightarrow a_{n+1-i} \geqslant n-i$.* $\square$

10.3 Hamilton cycles in the square of a graph

G^d

Given a graph G and a positive integer d, we denote by G^d the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most d in G. Clearly, $G = G^1 \subseteq G^2 \subseteq \ldots$ Our goal in this section is to prove the following fundamental result:

Theorem 10.3.1. (Fleischner 1974)
If G is a 2-connected graph, then G^2 has a Hamilton cycle.

The proof of Theorem 10.3.1 will go roughly as follows. We start by finding a cycle C in G. Using induction, we shall cover the remaining vertices by C-paths in G^2. The first and last edges of those paths will be edges of G, like those of C. By deleting some of these edges and doubling others, we turn the union of C and all the C-paths into a multigraph with even degrees, and find an Euler tour in it. This Euler tour W will pass some vertices more than once, but all edges in such multiple passes will be edges of G. For all but one of the passes through a given vertex we can therefore try to replace its two G-edges by an edge of G^2 (Fig. 10.3.1), hoping to turn our Euler tour into a Hamilton cycle of G^2. The main difficulty will be to ensure that these lifts of passes are compatible, i.e., that we do not attempt to lift an edge at both its ends.

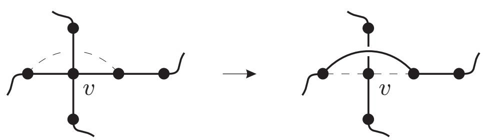

Fig. 10.3.1. Reducing the degree of v in W by lifting a pass

Lemma 10.3.2. *For every 2-connected graph G and $x \in V(G)$, there is a cycle $C \subseteq G$ that contains x as well as a vertex $y \neq x$ with $N_G(y) \subseteq V(C)$.*

Proof. If G has a Hamilton cycle, there is nothing more to show. If not, let $C' \subseteq G$ be any cycle containing x; such a cycle exists, since G is 2-connected. Let D be a component of $G - C'$. Assume that C' and D are chosen so that $|D|$ is minimum. Since G is 2-connected, D has at least two neighbours on C'. Then C' contains a path P between two such neighbours u and v, whose interior $\mathring{P}$ does not contain x and has no neighbour in D (Fig. 10.3.2).

Replacing P in C' by a u–v path through D, we obtain a cycle C that contains x and a vertex $y \in D$. If y had a neighbour z in $G - C$, then z would lie in a component $D' \subsetneq D$ of $G - C$, contradicting the choice of C' and D. Hence all the neighbours of y lie on C, and C satisfies the assertion of the lemma. $\square$

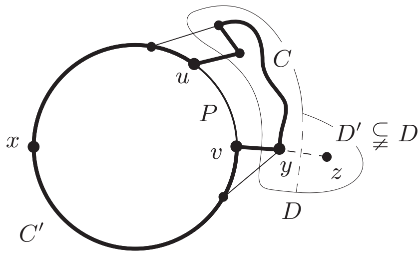

Fig. 10.3.2. The proof of Lemma 10.3.2

For our proof of Theorem 10.3.1 we need some more definitions. Let G be a multigraph, and W a walk in G. A *pass* of W *through* a vertex x is a subwalk of the form $uexfv$, where e and f are edges. (We also count $uexfv$ as a pass of W if $W = xfv \dots uex$.) By *lifting* this pass we mean replacing it in W by a new u–v edge if $u \neq v$, or by the single vertex u if $u = v$. A *multipath* is a multigraph obtained from a path by replacing some of its edges by double edges. Given $C \subseteq G$, we define a C-*trail* to be either a C-path or a cycle meeting C in exactly one vertex. *pass* *lift* *multipath* *C-trail*

(1.2.1) (1.8.1)

Proof of Theorem 10.3.1. Let $G = (V, E)$ be a 2-connected graph. We prove the following stronger assertion by induction on $|G|$: $G = (V, E)$

For every vertex $x \in V$ there is a Hamilton cycle in G^2 whose edges at x lie in E. x

If G is hamiltonian, there is nothing more to show. If not, let C and y be as provided by Lemma 10.3.2. For $i = 1, 2$ let $r_i, s_i \in V(C)$ and $g_i, h_i \in E(C)$ be such that C, y

$$C = x g_1 r_1 \dots s_1 h_1 y h_2 s_2 \dots r_2 g_2 x\,;$$ $r_i, s_i;\ g_i, h_i$

see Figure 10.3.3. (These vertices and edges need not all be distinct.)

Our first aim is to construct for every component D of $G - C$ a set of C-trails in $G^2 + \overline{E}$, where $\overline{E}$ will be a set of additional edges parallel to edges of G. Every vertex of D will lie on exactly one such trail, and every edge of such a trail that is incident with a vertex of C will lie in E or in $\overline{E}$.

If D consists of a single vertex u, we pick any C-trail in G containing u, and let E_D be the set of its two edges. If $|D| > 1$, let $\tilde{D}$ be the (2-connected) graph obtained from G by contracting $G - D$ to a vertex $\tilde{x}$. Applying the induction hypothesis to $\tilde{D}$, we obtain a Hamilton cycle $\tilde{H}$ of $\tilde{D}^2$ whose edges at $\tilde{x}$ lie in $E(\tilde{D})$. Write $\tilde{E}$ for the set of those edges of $\tilde{H}$ that are not edges of G^2; these include its two edges at $\tilde{x}$. Replacing the edges from $\tilde{E}$ by edges of G or new edges $\overline{e} \in \overline{E}$, we shall turn $E(\tilde{H})$ into the edge set of a union of C-trails.

Consider an edge $uv \in \tilde{E}$, with $u \in D$. Then either $v = \tilde{x}$, or u and v have distance at most 2 in $\tilde{D}$ but not in G, and are hence neighbours of $\tilde{x}$ in $\tilde{D}$. In either case, G contains a u–C edge. Let E_D be obtained from $E(\tilde{H}) \smallsetminus \tilde{E}$ by adding at every vertex $u \in D$ as many u–C edges from E as u has incident edges in $\tilde{E}$; if u has two incident edges in $\tilde{E}$ but in G sends only one edge e to C, we add both e and a new edge

$\overline{e}$ $\overline{e}$ parallel to e. Then every vertex of D has the same degree (two) in (V, E_D) as in $\tilde{H}$, so E_D is the edge set of a union of C-trails. Let

G_0
$$G_0 := \Big(V, E(C) \cup \bigcup_D E_D\Big)$$

be the union of C and all these C-trails, for all components D of $G - C$ together.

Our next aim is to turn G_0 into an Eulerian multigraph by doubling some edges of C. Since G_0 is connected, it will suffice to do this in such a way that all degrees become even (Theorem 1.8.1).[2] The vertices of G_0 outside C already have degree 2. To make the degrees even also at the vertices of C we consider these in reverse order, starting with x and ending with r_1. Let u be the vertex currently considered, and let v be

$\overline{e}$ the vertex to be considered next. Add a new edge $\overline{e}$ parallel to $e = uv$ if and only if u has odd degree in the multigraph obtained from G_0 so far. When finally $u = r_1$ is considered, every other vertex has even degree, so r_1 must have even degree too (Proposition 1.2.1), and no edge parallel to g_1 will be added.

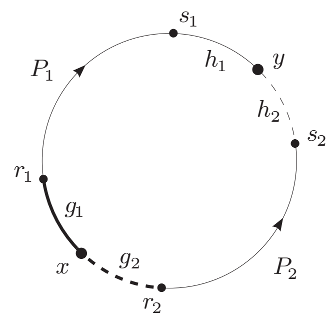

Fig. 10.3.3. Broken edges may not exist in G_1; bold edges are known to be single (if they exist)

From the Eulerian multigraph thus obtained we may now delete one or two double edges, as follows (Fig. 10.3.3). If g_2 has a parallel edge $\overline{g}_2$, we delete both g_2 and $\overline{g}_2$. If h_2 has a parallel edge $\overline{h}_2$, we delete h_2 and $\overline{h}_2$ unless this (together with the deletion of g_2 and $\overline{g}_2$) would

[2] To deduce the multigraph version of Theorem 1.8.1, subdivide every edge once to obtain a simple graph.

disconnect our multigraph. We write G_1 for the Eulerian multigraph thus obtained, $\overline{E} = E(G_1) \smallsetminus E(G^2)$ for the set of all its new parallel edges, and $C_1 := G_1[V(C)]$.

G_1 $\overline{E}$ C_1

Let us note two properties of G_1, which follow from its construction and the definition of y:

$$\textit{The edges of } G_1 \textit{ at vertices of } C \textit{ all lie in } E \cup \overline{E}. \tag{1}$$

$$N_{G_1}(y) \subseteq \{s_1, s_2\}; \textit{ thus, } y \textit{ has degree 2 or 4 in } G_1. \tag{2}$$

Let $P_1 = x_0^1 \dots x_{\ell_1}^1$ be the (maximal) x–y multipath in C_1 containing g_1, and let $P_2 = x_0^2 \dots x_{\ell_2}^2$ be the multipath consisting of the other edges of C_1. Unless P_2 is empty, we think of it as running from $x_0^2 \in \{x, r_2\}$ to $x_{\ell_2}^2 \in \{y, s_2\}$. We write e_j^i for the x_{j-1}^i–x_j^i edge of P_i in $E(C)$, and $\overline{e}_j^i$ for its possible parallel edge in $\overline{E}$ $(i = 1, 2)$.

ℓ_i, x_j^i P_i $e_j^i, \overline{e}_j^i$

Our plan is to find an Euler tour W_1 of G_1 that can be transformed into a Hamilton cycle of G^2. In order to endow W_1 more easily with the required properties, we shall not define it directly. Instead, we shall derive W_1 from an Euler tour W_2 of a related multigraph G_2, which we define next.

For $i = 1, 2$ and every $j = 1, \dots, \ell_i - 1$ such that $\overline{e}_{j+1}^i \in G_1$, we delete e_j^i and $\overline{e}_{j+1}^i$ from G_1 and add a new edge f_j^i joining x_{j-1}^i to x_{j+1}^i; we shall say that f_j^i *represents* the path $x_{j-1}^i e_j^i x_j^i \overline{e}_{j+1}^i x_{j+1}^i \subseteq P_i$ (Fig. 10.3.4). Note that every such replacement leaves the current multigraph connected, and it preserves the parity of all degrees. Hence, the multigraph G_2 obtained from G_1 by all these replacements is Eulerian.

f_j^i G_2

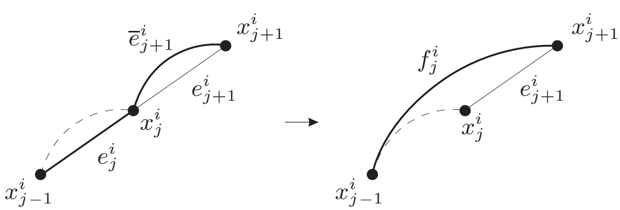

Fig. 10.3.4. Replacing e_j^i and $\overline{e}_{j+1}^i$ by a new edge f_j^i

Pick an Euler tour W_2 of G_2. To transform W_2 into an Euler tour W_1 of G_1, replace every edge in $E(W_2) \smallsetminus E(G_1)$ by the path it represents.

W_2 W_1

By (2), there are either one or two passes of W_1 through y. If $d(y) = 4$ then G_1 is connected but $G_1 - \{h_2, \overline{h}_2\}$ is not, by definition of G_1. Hence the proper y–y subwalk W of W_1 starting or ending with the edge h_2 must end or start with the edge $\overline{h}_2$: otherwise it would contain an s_2–y walk in $G_1 - \{h_2, \overline{h}_2\}$, and deleting h_2 and $\overline{h}_2$ from G_1 could not affect its connectedness. Therefore W_1 has no pass through y containing both h_2 and $\overline{h}_2$: this would close the subwalk W and thus imply $W = W_1$, contrary to the definition of W. Reversing W if necessary, we may thus assume:

If W_1 contains two passes through y, then one of these contains h_1 and h_2. (3)

Q

Our plan is to transform W_1 into a Hamilton cycle of G^2 by lifting, at every vertex $v \in C$, all but one of the passes of W_1 through v. We begin by marking the one pass at each vertex that we shall not lift. At x we mark an arbitrary pass Q of W_1. At y we mark the pass containing h_1. For every $v \in V(C) \setminus \{x, y\}$ there is a unique pair (i, j) such that $v = x^i_j$. If $j \geqslant 1$, we mark the pass through $v = x^i_j$ that contains e^i_j. If $j = 0$ (which can happen only if $v = x^2_0 = r_2$), we mark the pass through v that contains $\overline{e}^2_1$ if $\overline{e}^2_1 \in G_1$; otherwise we mark an arbitrary pass through v. At every vertex $v \notin C$ we mark the unique pass of W_1 through v. Thus:

At every vertex $v \in G$ we marked exactly one pass through v. (4)

To avoid conflicts when we later lift the unmarked passes, we need that no edge of W_1 is left unmarked at both its ends:

For every edge $e = uv$ in W_1 we marked at least one of the two passes of W_1 containing e (one through u, the other through v). If $u = x$, we marked the pass through v. (5)

This is clear for edges not in C_1. For every edge $e \in C_1$, there is a unique pair (i, j) such that $e = e^i_j$ or $e = \overline{e}^i_j$; then $j \geqslant 1$. If $e = e^i_j$, we marked the pass of W_1 through x^i_j that contains e; for $e = h_2$ this follows from (3). If $e = \overline{e}^i_j$, we marked the pass through x^i_{j-1} containing e. Indeed, note first that e is not incident with x: recall that $\overline{g}_1$ never existed, and if $\overline{g}_2$ existed it was deleted in the definition of G_1. Hence unless P_2 starts at r_2 and $e = \overline{e}^2_1$, an edge f^i_{j-1} was defined to represent the path $x^i_{j-2} e^i_{j-1} x^i_{j-1} \overline{e}^i_j x^i_j$. Since W_2 contained f^i_{j-1}, this path is a pass in W_1. We marked this pass, because it is a pass through x^i_{j-1} containing e^i_{j-1}. Finally, if P_2 starts at r_2 and $e = \overline{e}^2_1$, we marked the pass through r_2 containing e explicitly. This completes the proof of (5).

By (1), all unmarked passes lift to edges of G^2. As different unmarked passes never share an edge (5), lifting them all at once turns W_1 into one closed walk $\overline{H}$ in $G^2 + \overline{E}$ (which inherits the cyclic ordering of its edges from W_1). By (4), $\overline{H}$ still contains all the vertices v of G: if uv is an edge of a pass marked at v, then $\overline{H}$ contains either the edge uv or the lift wv of a pass $weufv$. Also by (4), $\overline{H}$ traverses every vertex only once. In particular, $\overline{H}$ cannot contain a pair of parallel edges. We can therefore replace every edge $\overline{e}$ in $\overline{H}$ by its parallel edge $e \in E$, to obtain a Hamilton cycle H of G^2. Since we marked Q, and by (5) no edge of Q was lifted at its other end, $\overline{H}$ contains the edges of Q. By (1), these lie in $E \cup \overline{E}$. Hence the edges of H at x lie in E, as desired. □

Fleischner's theorem has a natural extension to infinite graphs, which is much harder to prove:

Theorem 10.3.3. (Georgakopoulos 2009)
The square of every 2-connected locally finite graph contains a Hamilton circle.

We close the chapter with a far-reaching conjecture generalizing Dirac's theorem:

Conjecture. (Seymour 1974)
Let G be a graph of order $n \geqslant 3$, and let k be a positive integer. If G has minimum degree

$$\delta(G) \geqslant \frac{k}{k+1}\, n\,,$$

then G has a Hamilton cycle H such that $H^k \subseteq G$.

For $k = 1$, this is precisely Dirac's theorem. The conjecture was proved for large enough n (depending on k) by Komlós, Sárközy and Szemerédi (1998).

Exercises

1. An oriented complete graph is called a *tournament*. Show that every tournament contains a (directed) Hamilton path.

2. Show that every uniquely 3-edge-colourable cubic graph is hamiltonian. ('Unique' means that all 3-edge-colourings induce the same edge partition.)

3. Given an even positive integer k, construct for every $n \geqslant k$ a k-regular graph of order $2n+1$.

4. Prove or disprove the following strengthening of Proposition 10.1.2: 'Every k-connected graph G with $|G| \geqslant 3$ and $\chi(G) \geqslant |G|/k$ has a Hamilton cycle.'

5. Let G be a graph, and $H := L(G)$ its line graph.
   (i) Show that H is hamiltonian if G has a spanning Eulerian subgraph.
   (ii)$^+$ Deduce that H is hamiltonian if G is 4-edge-connected.

6. (i)$^-$ Show that hamiltonian graphs are 1-tough.
   (ii) Find a graph that is 1-tough but not hamiltonian.

7. Prove the toughness conjecture for planar graphs. Does it hold with $t = 2$, or even with some $t < 2$?

8.$^-$ Find a hamiltonian graph whose degree sequence is not hamiltonian.

9.$^-$ Let G be a graph with fewer than i vertices of degree at most i, for every $i < |G|/2$. Use Chvátal's theorem to show that G is hamiltonian. (Thus in particular, Chvátal's theorem implies Dirac's theorem.)

10. Prove that the square G^2 of a k-connected graph G is k-tough. Use this to deduce Fleischner's theorem for graphs satisfying the toughness conjecture with $t = 2$.

11. Show that Exercise 6 (i) has the following weak converse: for every non-hamiltonian graph G there exists a graph G' that has a pointwise greater degree-sequence than G but is not 1-tough.

12. (i) Show that, unlike the graphs satisfying Dirac's condition of $\delta \geqslant n/2$, graphs satisfying the degree condition of Theorem 10.1.3 can be sparse: there exists an integer d for which there are arbitrarily large graphs of average degree at most d that satisfy the condition.

    (ii) Show that there is no integer d that bounds the average degrees of arbitrarily large graphs satisfying Chvátal's degree condition.

13. Find a connected graph G whose square G^2 has no Hamilton cycle.

14.$^-$ Deduce from the proof of Fleischner's theorem that the square of a 2-connected graph contains a Hamilton path between any two vertices.

15. Show by induction on $|G|$ that the third power G^3 of any connected graph G of order at least 3 contains a Hamilton cycle.

16.$^+$ Let G be a graph in which every vertex has odd degree. Show that every edge of G lies on an even number of Hamilton cycles.

    (Hint. Let $xy \in E(G)$ be given. The Hamilton cycles through xy correspond to the Hamilton paths in $G - xy$ from x to y. Consider the set $\mathcal{H}$ of *all* Hamilton paths in $G - xy$ starting at x, and show that an even number of these end in y. To show this, define a graph on $\mathcal{H}$ so that the desired assertion follows from Proposition 1.2.1.)

Notes

The problem of finding a Hamilton cycle in a graph has the same kind of origin as its Euler tour counterpart and the four colour problem: all three problems come from mathematical puzzles older than graph theory itself. What began as a game invented by W.R. Hamilton in 1857 – in which 'Hamilton cycles' had to be found on the graph of the dodecahedron – re-emerged over a hundred years later as a combinatorial optimization problem of prime importance: the *travelling salesman problem*. Here, a salesman has to visit a number of customers, and his problem is to arrange these in a suitable circular route. (For reasons not included in the mathematical brief, the route has to be such that after visiting a customer the salesman does not pass through that town again.) Much of the motivation for considering Hamilton cycles comes from variations of this algorithmic problem.

The lack of a good characterization of hamiltonicity also has to do with an algorithmic problem: deciding whether or not a given graph is hamiltonian is NP-hard (indeed, this was one of the early prototypes of an NP-complete decision problem), while the existence of a good characterization would place it in NP $\cap$ co-NP, which is widely believed to equal P. Thus, unless P = NP, no good characterization of hamiltonicity exists. See the introduction to Chapter 12.7, or the end of the notes for Chapter 12, for more.

The 'proof' of the four colour theorem indicated at the end of Section 10.1, which is based on the (false) premise that every 3-connected cubic planar graph is hamiltonian, is usually attributed to the Scottish mathematician P.G. Tait. Following Kempe's flawed proof of 1879 (see the notes for Chapter 5), it seems that Tait believed to be in possession of at least one 'new proof of Kempe's theorem'. However, when he addressed the Edinburgh Mathematical Society on this subject in 1883, he seems to have been aware that he could not *really* prove the above statement about Hamilton cycles. His account in P.G. Tait, Listing's topologie, *Phil. Mag.* **17** (1884), 30–46, makes some entertaining reading.

A shorter proof of Tutte's theorem that 4-connected planar graphs are hamiltonian has been given by C. Thomassen, A theorem on paths in planar graphs, *J. Graph Theory* **7** (1983), 169–176. Tutte's counterexample to Tait's assumption that even 3-connectedness suffices (at least for cubic graphs) is shown in Bollobás, and in J.A. Bondy & U.S.R. Murty, *Graph Theory with Applications*, Macmillan 1976 (where Tait's attempted proof is discussed in some detail).

Bruhn's conjecture generalizing Tutte's theorem to infinite graphs was first stated in R. Diestel, The cycle space of an infinite graph, *Comb. Probab. Comput.* **14** (2005), 59–79. As the notion of a Hamilton circle is relatively recent, earlier generalizations of Hamilton cycle theorems asked for spanning double rays. Now a ray can pass through a finite separator only finitely often, so a necessary condition for the existence of a spanning ray or double ray is that the graph has at most one or two ends, respectively. Confirming a long-standing conjecture of Nash-Williams, X. Yu, Infinite paths in planar graphs I–V, *J. Graph Theory* (2004–08), proved that a 4-connected planar graph with at most two ends contains a spanning double ray. N. Dean, R. Thomas and X Yu, Spanning paths in infinite planar graphs, *J. Graph Theory* **23** (1996), 163–174, proved Nash-Williams's conjecture that a one-ended 4-connected planar graph has a spanning ray.

Proposition 10.1.2 is due to Chvátal and Erdős (1972). Theorem 10.1.3 was found much later: by A.S. Asratian and N.K. Khachatrian, Some localization theorems on hamiltonian circuits, *J. Comb. Theory, Ser. B* **49** (1990), 287–294. Since its hamiltonicity condition is local, the theorem might generalize to Hamilton circles in locally finite graphs, similarly to Fleischner's theorem. This appears to be a hard problem.

The toughness invariant and conjecture were proposed by V. Chvátal, Tough graphs and hamiltonian circuits, *Discrete Math.* **5** (1973), 215–228. If true with $t = 2$, the conjecture would have implied Fleischner's thereom; see Exercise 10. However, it was disproved for $t = 2$ by D. Bauer, H.J. Broersma & H.J. Veldman, Not every 2-tough graph is hamiltonian, *Discrete Appl. Math.* **99** (2000), 317–321. Theorem 10.2.1 is due to V. Chvátal, On Hamilton's ideals, *J. Comb. Theory, Ser. B* **12** (1972), 163–168.

The extension of Fleischner's theorem to locally finite graphs, Theorem 10.3.3, was proved by A. Georgakopoulos, Infinite Hamilton cycles in squares of locally finite graphs, *Adv. Math.* **220** (2009), 670–705. Our short proof of Fleischner's theorem is a windfall of that proof.

Seymour's conjecture is from P.D. Seymour, Problem 3, in (T.P. McDonough and V.C. Mavron, eds.) *Combinatorics*, Cambridge University Press 1974. Its proof for large n is due to J. Komlós, G.N. Sárközy & E. Szemerédi, Proof of the Seymour conjecture for large graphs, *Ann. Comb.* **2** (1998), 43–60.

Finally, let us mention Thomassen's conjecture (1986) that every 4-connected line graph is hamiltonian. T. Kaiser, Hamilton cycles in 5-connected line graphs, *Eur. J. Comb.* **33** (2012), 924–947, arXiv:1009.3754, proved that 5-connected line graphs of minimum degree at least 6 are hamiltonian.