Theorem 8.4.12 there is a set $T \subseteq V(C)$ such that $\mathcal{C}'_{C-T}$ is not matchable to T in C'_T. By Corollary 8.4.9, this means that $\mathcal{C}'_{C-T}$ has a subset $\mathcal{C}$ that is not matchable in C'_T to the set $T' \subseteq T$ of its neighbours, while T' is matchable to $\mathcal{C}$; let M' be such a matching. Then $(S, M) < (S \cup T', M \cup M')$, contradicting the maximality of (S, M).

Of the second statement, only the backward implication is non-trivial. Our assumptions now are that $\mathcal{C}'_{G-S}$ is matchable to S in G'_S and vice versa (by the choice of S), so Proposition 8.4.6 yields that G'_S has a 1-factor. This defines a matching of S in G that picks one vertex x_C from every component $C \in \mathcal{C}'_{G-S}$ and leaves the other components of $G - S$ untouched. Adding to this matching a 1-factor of $C - x_C$ for every $C \in \mathcal{C}'_{G-S}$ and a 1-factor of every other component of $G - S$, we obtain the desired 1-factor of G. □

Infinite matching theory may seem rather mature and complete as it stands, but there are still fascinating unsolved problems in the Erdős-Menger spirit concerning related discrete structures, such as posets or hypergraphs. We conclude with one about graphs.

Call an infinite graph G *perfect* if every induced subgraph $H \subseteq G$ has a complete subgraph K of order $\chi(H)$, and *strongly perfect* if K can always be chosen so that it meets every colour class of some $\chi(H)$-colouring of H. (Exercise 59 gives an example of a perfect graph that is not strongly perfect.) Call G *weakly perfect* if the chromatic number of every induced subgraph $H \subseteq G$ is at most the supremum of the orders of its complete subgraphs.

strongly perfect

weakly perfect

Conjecture. (Aharoni & Korman 1993)
Every weakly perfect graph without infinite independent sets of vertices is strongly perfect.

8.5 Graphs with ends: the topological viewpoint

In this section we shall develop a deeper understanding of the global structure of infinite graphs, especially locally finite ones, that can be attained only by studying their ends. This structure is intrinsically topological, but no more than the most basic concepts of point-set topology will be needed.

Our first goal will be to make precise the intuitive idea that the ends of a graph are the 'points at infinity' to which its rays converge. To do so, we shall define a topological space $|G|$ associated with a graph $G = (V, E, \Omega)$ and its ends.[8] By considering topological versions of

$|G|$

V, E, Ω

[8] The notation of $|G|$ comes from topology and clashes with our notation for the order of G. But there is little danger of confusion, so we keep both.

paths, cycles and spanning trees in this space, we shall then be able to extend to infinite graphs some parts of finite graph theory that would not otherwise have infinite counterparts (see the notes for more examples). Thus, the ends of an infinite graph turn out to be more than a curious new phenomenon: they form an integral part of the picture, without which it cannot be properly understood.

To build the space $|G|$ formally, we start with the set $V \cup \Omega$. For every edge $e = uv$ we add a set $\mathring{e} = (u,v)$ of continuum many points, making these sets $\mathring{e}$ disjoint from each other and from $V \cup \Omega$. We then choose for each e some fixed bijection between $\mathring{e}$ and the real interval $(0,1)$, and extend this bijection to one between $[u,v] := \{u\} \cup \mathring{e} \cup \{v\}$ and $[0,1]$. This bijection defines a metric on $[u,v]$; we call $[u,v]$ a *topological edge* with *inner points* $x \in \mathring{e}$. Given any $F \subseteq E$ we write $\mathring{F} := \bigcup \{\, \mathring{e} \mid e \in F \,\}$. When we speak of a 'graph' $H \subseteq G$, we shall often also mean its corresponding point set $V(H) \cup \mathring{E}(H)$.

(u,v)

$[u,v]$

$\mathring{F}$

Having thus defined the point set of $|G|$, let us choose a basis of open sets to define its topology. For every edge uv, declare as open all subsets of (u,v) that correspond, by our fixed bijection between (u,v) and $(0,1)$, to an open set in $(0,1)$. For every vertex u and $\epsilon > 0$, declare as open the 'open star around u of radius ϵ', that is, the set of all points on edges $[u,v]$ at distance less than ϵ from u, measured individually for each edge in its metric inherited from $[0,1]$. Finally, for every end ω and every finite set $S \subseteq V$, there is a unique component $C(S,\omega)$ of $G - S$ that contains rays from ω. Let $\Omega(S,\omega) := \{\, \omega' \in \Omega \mid C(S,\omega') = C(S,\omega) \,\}$. For every $\epsilon > 0$, write $\mathring{E}_\epsilon(S,\omega)$ for the set of all inner points of S–$C(S,\omega)$ edges at distance less than ϵ from their endpoint in $C(S,\omega)$. Then declare as open all sets of the form

$C(S,\omega)$

$\hat{C}_\epsilon(S,\omega)$

$$\hat{C}_\epsilon(S,\omega) := C(S,\omega) \cup \Omega(S,\omega) \cup \mathring{E}_\epsilon(S,\omega)\,.$$

This completes the definition of $|G|$, whose open sets are the unions of the sets we explicitly chose as open above.

closure $\overline{X}$

The *closure* of a set $X \subseteq |G|$ will be denoted by $\overline{X}$. For example, $\overline{V} = V \cup \Omega$ (because every neighbourhood of an end contains a vertex), and the closure of a ray is obtained by adding its end. More generally, the closure of the set of teeth of a comb contains a unique end, the end of its spine. Conversely, if $U \subseteq V$ and $R \in \omega \in \Omega \cap \overline{U}$, there is a comb with spine R and teeth in U (Exercise 61). In particular, the closure of the subgraph $C(S,\omega)$ considered above is the set $C(S,\omega) \cup \Omega(S,\omega)$.

$\overline{H}$

Given a subgraph $H = (U,F)$ of G, we write $\overline{H} := \overline{U} \cup \mathring{F}$ for its closure in $|G|$. Note that the ends in $\overline{H}$ are ends of G, not ends of H; in particular, $\overline{H}$ may well contain ends that have no ray in H. A subspace X of $|G|$ of the form $X = \overline{H}$ is a *standard* subspace. We denote F as $E(X)$, and call X *spanned by* H. When $F \subseteq E$ is a given set of edges

standard subspace

$E(X)$

and U is the set of their endvertices, we abbreviate $\overline{(U,F)}$ to $\overline{F}$, and also say that X is *spanned by* F. $\overline{F}$

By definition, $|G|$ is always Hausdorff; indeed one can show that it is normal. When G is connected and locally finite, then $|G|$ is compact:[9]

Proposition 8.5.1. *If G is connected and locally finite, then $|G|$ is a compact Hausdorff space.*

Proof. Let $\mathcal{O}$ be an open cover of $|G|$; we show that $\mathcal{O}$ has a finite subcover. Pick a vertex $v_0 \in G$, write D_n for the (finite) set of vertices at distance n from v_0, and put $S_n := D_0 \cup \ldots \cup D_{n-1}$. For every $v \in D_n$, let $C(v)$ denote the component of $G - S_n$ containing v, and let $\hat{C}(v)$ be its closure together with all inner points of $C(v)$–S_n edges. Then $G[S_n]$ and these $\hat{C}(v)$ together partition $|G|$. (8.1.2) $\hat{C}(v)$

We wish to prove that, for some n, each of the sets $\hat{C}(v)$ with $v \in D_n$ is contained in some $O(v) \in \mathcal{O}$. For then we can take a finite subcover of $\mathcal{O}$ for $G[S_n]$ (which is compact, being a finite union of edges and vertices), and add to it these finitely many sets $O(v)$ to obtain the desired finite subcover for $|G|$.

Suppose there is no such n. Then for each n the set V_n of vertices $v \in D_n$ such that no set from $\mathcal{O}$ contains $\hat{C}(v)$ is non-empty. Moreover, for every neighbour $u \in D_{n-1}$ of $v \in V_n$ we have $C(v) \subseteq C(u)$ because $S_{n-1} \subseteq S_n$, and hence $u \in V_{n-1}$; let $f(v)$ be such a vertex u. By the infinity lemma (8.1.2) there is a ray $R = v_0 v_1 \ldots$ with $v_n \in V_n$ for all n. Let ω be its end, and let $O \in \mathcal{O}$ contain ω. Since O is open, it contains a basic open neighbourhood of ω: there exist a finite set $S \subseteq V$ and $\epsilon > 0$ such that $\hat{C}_\epsilon(S,\omega) \subseteq O$. Now choose n large enough that S_n contains S and all its neighbours. Then $C(v_n)$ lies inside a component of $G - S$. As $C(v_n)$ contains the ray $v_n R \in \omega$, this component must be $C(S,\omega)$. Thus

$$\hat{C}(v_n) \subseteq \hat{C}_\epsilon(S,\omega) \subseteq O \in \mathcal{O},$$

contradicting the fact that $v_n \in V_n$. □

If G has a vertex of infinite degree then $|G|$ cannot be compact. (Why not?) But $\Omega \subseteq |G|$ can be compact; see Exercise 66 for when it is.

What else can we say about the space $|G|$ in general? For example, is it metrizable? Using a normal spanning tree T of G, it is indeed not difficult to define a metric on $|G|$ that induces its topology. But not every connected graph has a normal spanning tree, and it is not easy to determine in graph-theoretical terms which graphs do. Surprisingly, though, it is possible to deduce the existence of a normal spanning tree from that of a defining metric on $|G|$. Thus whenever $|G|$ is metrizable, a metric can be made visible in a natural and structural way.

[9] Topologists call $|G|$ the *Freudenthal compactification* of G.

Theorem 8.5.2. *For a connected graph G, the space $|G|$ is metrizable if and only if G has a normal spanning tree.*

The proof of Theorem 8.5.2 is indicated in Exercises 33 and 68.

Our next aim is to review, or newly define, some topological notions of paths and connectedness, of cycles, and of spanning trees. By substituting these topological notions with respect to $|G|$ for the corresponding graph-theoretical notions with respect to G, one can extend to locally finite graphs a number of theorems about paths, cycles and spanning trees in finite graphs whose ordinary infinite versions are false. We shall do this, as a case in point, for the tree-packing theorem (2.4.1) of Nash-Williams and Tutte.

X

connected

Let X be an arbitrary Hausdorff space. (Later, this will be a subspace of $|G|$.) X is *(topologically) connected* if it is not a union of two disjoint non-empty open subsets.[10] Note that continuous images of connected spaces are connected. For example, since the real interval $[0,1]$ is connected,[11] so are its continuous images in X.

arc

A *homeomorphic* image of $[0,1]$ in X is an *arc* in X; it *links* the images of 0 and 1, which are its *endpoints*. Every finite path in G defines an arc in $|G|$ in an obvious way. Similarly, every ray defines an arc linking its starting vertex to its end, and a double ray in G forms an arc in $|G|$ together with the two ends of its tails if these ends are distinct.

end degrees in subspaces

Consider an end ω in a standard subspace X of $|G|$, and $k \in \mathbb{N} \cup \{\infty\}$. If k is the maximum number of arcs in X that have ω as their common endpoint and are otherwise disjoint, then k is the *(topological) vertex-degree* of ω *in* X. The *(topological) edge-degree* of ω *in* X is defined analogously, using edge-disjoint arcs. Similarly to Theorem 8.2.5 one can show that these maxima are always attained, so every end of G that lies in X has a topological vertex- and edge-degree there. For $X = |G|$ and G locally finite, the (topological) end degrees in X coincide with the combinatorial end degrees defined earlier.

Unlike ordinary paths, arcs in $|G|$ can jump across an infinite cut without containing an edge from it—but only if the cut is infinite:

Lemma 8.5.3. (Jumping Arc Lemma)
Let G be connected and locally finite, and let $F \subseteq E(G)$ be a cut with sides V_1, V_2.

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \smallsetminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*

(ii) *If F is infinite, then $\overline{V_1} \cap \overline{V_2} \neq \emptyset$, and there may be such an arc.*

---

[10] These subsets would be complements of each other, and hence also be closed. Note that 'open' and 'closed' means open and closed *in* X: when X is a subspace of $|G|$ with the subspace topology, the two sets need not be open or closed in $|G|$.

[11] This takes a few lines to prove—can you prove it?

Proof. (i) Suppose that F is finite. Let S be the set of vertices incident with edges in F. Then S is finite and separates V_1 from V_2, so for every $\omega \in \Omega$ the connected graph $C(S, \omega)$ misses either V_1 or V_2. But then so does every basic open set of the form $\hat{C}_\epsilon(S, \omega)$. Therefore no end ω lies in the closure of both V_1 and V_2. (8.2.2)

As $|G| \smallsetminus \mathring{F} = \overline{G[V_1]} \cup \overline{G[V_2]}$ and this union is disjoint, no connected subset of $|G| \smallsetminus \mathring{F}$ can meet both V_1 and V_2. Since arcs are continuous images of $[0, 1]$ and hence connected, there is no V_1–V_2 arc in $|G| \smallsetminus \mathring{F}$.

(ii) Suppose now that F is infinite. Since G is locally finite, the set U of endvertices of F in V_1 is also infinite. By the star-comb lemma (8.2.2), there is a comb in G with teeth in U; let ω be the end of its spine. Then every basic open neighbourhood $\hat{C}_\epsilon(S, \omega)$ of ω meets $U \subseteq V_1$ infinitely and hence also meets V_2, giving $\omega \in \overline{V_1} \cap \overline{V_2}$.

To obtain a V_1–V_2 arc in $|G| \smallsetminus \mathring{F}$, all we need now is an arc in $\overline{G[V_1]}$ and another in $\overline{G[V_2]}$, both ending in ω. Such arcs exist, for example, if the graphs $G[V_i]$ are connected: we can then pick a sequence of vertices in V_i converging to ω, and apply the star-comb lemma in $G[V_i]$ to obtain a comb whose spine is a ray in $G[V_i]$ converging to ω. Concatenating these two rays yields the desired jumping arc. □

To some extent, arcs in $|G|$ assume the role that paths play in finite graphs. So arcs are important—but how do we find them? It is not always possible to construct arcs as explicitly as in the proof of Lemma 8.5.3 (ii). Figure 8.5.1, for example, shows an arc that goes through continuum many ends; such arcs cannot be constructed greedily by following a ray into its end and emerging from that end on another ray, and repeating this finitely often.

There are two basic methods to obtain an arc between two given points, say two vertices x and y. One is to use compactness to obtain as a limit of finite x–y paths a *topologial* x–y path, a continuous map $\pi\colon [0, 1] \to |G|$ sending 0 to x and 1 to y. A theorem from general topology then tells us that this path can be made injective, i.e., that the image of π contains an x–y arc.[12] Another method is to prove that the subspace in which we wish to find an x–y arc is topologically connected, and use this fact to deduce that it contains the desired arc. Our next two lemmas show how to implement this approach in practice.

Being linked by an arc is an equivalence relation on the points of our Hausdorff space X: every x–y arc A has a first point p on any y–z arc A' (because A' is closed), and the obvious segments Ap and pA' together form an x–z arc in X. The corresponding equivalence classes are the *arc-components* of X. If X has only one arc-component, then X is *arc-connected*.

arc-component

arc-connected

[12] This approach is explained in a long hint for Exercise 89.

Since $[0,1]$ is connected, arc-connectedness implies connectedness. The converse implication is false in general, even for spaces $X \subseteq |G|$ with G locally finite. But it holds in an important special case:

Lemma 8.5.4. *If G is locally finite, then every connected standard subspace of $|G|$ is arc-connected.*

The proof of Lemma 8.5.4 is not that easy. A proof borrowing a lemma from general topology is indicated in Exercise 69.

The lemma implies that the arc-components of standard subspaces of $|G|$ are closed. Indeed, such an arc-component A is connected, so its closure $\overline{A}$ is connected (and standard). Hence $\overline{A}$ is arc-connected by the lemma, giving $A = \overline{A}$ by definition of A.

Connected standard subspaces of $|G|$ containing two given points are much easier to construct than an arc between two points. This has to do with the fact that they can be described in purely graph-theoretical terms, with reference only to G itself rather than to $|G|$. The description can be viewed as a topological analogue of the fact that a subgraph H of G is connected if and only if it contains an edge from every cut of G that separates two of its vertices:

Lemma 8.5.5. *If G is locally finite, then a standard subspace of $|G|$ is connected (equivalently: arc-connected) if and only if it contains an edge from every finite cut of G of which it meets both sides.*

D *Proof.* Let $X = \overline{(U,D)}$ be a standard subspace of $|G|$. If X is not connected, we can partition it into disjoint non-empty open and closed subsets O_1 and O_2. As X is standard, these O_i are closed in $|G|$, and $U_i := O_i \cap U \neq \emptyset$. Let $\mathcal{P}$ be a maximal set of edge-disjoint U_1–U_2 paths in G, and put

$$F := \bigcup \{ E(P) \mid P \in \mathcal{P} \}.$$

Then $D \cap F = \emptyset$, and no component of $G - F$ meets both U_1 and U_2. Extending $\{U_1, U_2\}$ to a partition of V in such a way that each component of $G - F$ has all its vertices in one class, we obtain a cut $F' \subseteq F$ of G of which X meets both sides. As $D \cap F = \emptyset$, it thus suffices to show that F is finite.

If F is infinite, then so is $\mathcal{P}$. As G is locally finite, the vertices of each $P \in \mathcal{P}$ are incident with only finitely many edges of G. We can thus inductively find an infinite subset of $\mathcal{P}$ consisting of paths that are not only edge-disjoint but disjoint. The endvertices in U_1 of these paths have a limit point ω in $|G|$, which is also a limit point of their endvertices in U_2. Since both O_1 and O_2 are closed in $|G|$, we thus have $\omega \in O_1 \cap O_2$, contradicting the choice of the O_i. This completes the backward implication of the lemma.

For the forward implication, suppose that G has a finite cut $F = E(V_1, V_2)$ such that X meets both V_1 and V_2 but has no edge in F. Then

$$X \subseteq |G| \setminus \mathring{F} = \overline{G[V_1]} \cup \overline{G[V_2]}\,,$$

and this union is disjoint by Lemma 8.5.3 (i). The induced partition of X into non-empty closed subsets of X shows that X is not connected. □

A *circle* in a topological space is a homeomorphic image of the unit circle $S^1 \subseteq \mathbb{R}^2$. For example, if G is the 2-way infinite ladder shown in Figure 8.1.3, and we delete all its rungs (the vertical edges), what remains is a disjoint union of two double rays; its closure in $|G|$, obtained by adding the two ends of G, is a circle. Similarly, the double ray 'round the outside' of the 1-way ladder forms a circle together with the unique end of that ladder. *circle*

It is not hard to show that no arc in $|G|$ can consist entirely of ends. This implies that every circle in $|G|$ is a standard subspace; the set of edges spanning it will be called its *circuit*. *circuit*

A more adventurous example of a circle is shown in Figure 8.5.1. Let G be the graph obtained from the binary tree T_2 by joining for every finite 0–1 sequence ℓ the vertices $\ell 01$ and $\ell 10$ by a new edge e_ℓ. Together with all the (uncountably many) ends of G, the double rays $D_\ell \ni e_\ell$ shown in the figure form an arc A in $|G|$, whose union with the bottom double ray D is a circle in $|G|$ (Exercise 75). Note that no two of the double rays in A are consecutive: between any two there lies a third (cf. Exercise 76).

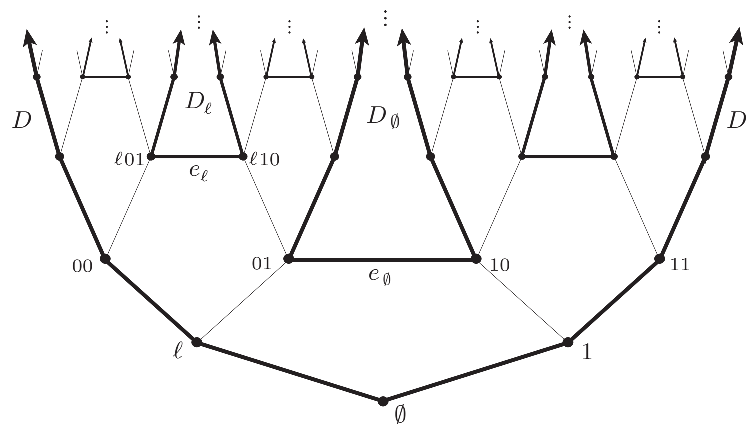


Fig. 8.5.1. The Wild Circle

A *topological spanning tree* of G is an arc-connected standard subspace T of $|G|$ that contains every vertex but contains no circle. Since *topological spanning tree*

standard subspaces are closed, T also contains every end. With respect to the addition or deletion of edges, it is both minimally arc-connected and maximally 'acirclic'. As with ordinary trees, one can show that every two points of T are joined by only one arc in T. Thus, adding a new edge e to T creates a unique circle in $T \cup e$; its edges form the *fundamental circuit* C_e of e with respect to T. Similarly, for every edge $f \in E(T)$ the space $T \smallsetminus \mathring{f}$ has exactly two arc-components; the set of edges between these is the *fundamental cut* D_f of T. As in finite graphs, we have $e \in D_f$ if and only if $f \in C_e$, for all $f \in E(T)$ and $e \in E \smallsetminus E(T)$. Since the two arc-components of $T \smallsetminus \mathring{f}$ are closed but disjoint, Lemma 8.5.3 (ii) implies that D_f is finite if G is locally finite.

fundamental circuit

fundamental cut

One might expect that the closure $\overline{T}$ of an ordinary spanning tree T of G is always a topological spanning tree of $|G|$. However, this can fail in two ways: if T has a vertex of infinite degree then $\overline{T}$ may fail to be arc-connected (Exercise 74), although it will be topologically connected, because T is; if T is locally finite, then $\overline{T}$ will be arc-connected but may contain a circle (Figure 8.5.2). On the other hand, a subgraph whose closure is a topological spanning tree may well be disconnected: the vertical rays in the $\mathbb{N} \times \mathbb{N}$ grid, for example, form a topological spanning tree of the grid (together with its unique end).

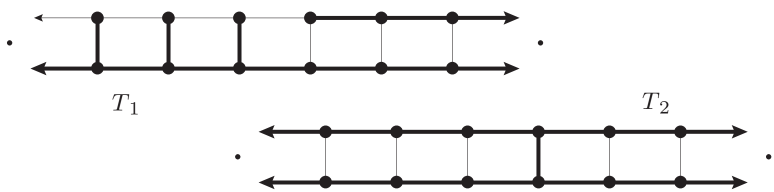


Fig. 8.5.2. $\overline{T}_1$ is a topological spanning tree, but $\overline{T}_2$ contains three circles

In general, there seems to be no canonical way to construct topological spanning trees, and it is unknown whether every connected graph has one. Countable connected graphs, however, do have topolo-
(8.2.4) gical spanning trees by Theorem 8.2.4:

Lemma 8.5.6. *The closure of any normal spanning tree is a topological spanning tree.*

(1.5.5)
(8.2.3)

Proof. Let T be a normal spanning tree of G. By Lemma 8.2.3, every end ω of G contains a normal ray R of T. Then $R \cup \{\omega\}$ is an arc linking ω to the root of T, so $\overline{T}$ is arc-connected.

It remains to check that $\overline{T}$ contains no circle. Suppose it does, and let A be the u–v arc obtained from that circle by deleting the inner
f points of an edge $f = uv$ it contains. Clearly, $f \in T$. Assume that $u < v$ in the tree-order of T, let T_u and T_v denote the components of $T - f$ containing u and v, and notice that $V(T_v)$ is the up-closure $\lfloor v \rfloor$ of v in T.

Now let $S := \lceil u \rceil$. By Lemma 1.5.5 (ii), $\lfloor v \rfloor$ is the vertex set of a component C of $G - S$. Thus, $V(C) = V(T_v)$ and $V(G - C) = V(T_u)$, so the set $E(C, S)$ of edges between these sets meets $E(T)$ precisely in f. Thus, $\overline{C}$ and $\overline{G - C}$ partition $|G| \smallsetminus \mathring{E}(C, S) \supseteq A$ into two open sets both meeting A. This contradicts the fact that A is topologically connected. $\square$

As an application of our new concepts, let us now extend the tree-packing theorem (2.4.1) of Nash-Williams and Tutte to locally finite graphs. Its naive extension, with ordinary spanning trees, fails. Indeed, for every $k \in \mathbb{N}$ one can construct a $2k$-edge-connected locally finite graph that is left disconnected by the deletion of the edges in any one finite circuit (Exercise 14). Such a graph will have at least $k(\ell - 1)$ edges across any vertex partition into ℓ sets, but it cannot have more than two edge-disjoint spanning trees: adding an edge of one of these to another creates a (finite) fundamental circuit there, whose deletion would not disconnect any third spanning tree.

As soon as we generalize spanning trees to topological spanning trees, however, Theorem 2.4.1 does extend:[13]

Theorem 8.5.7. *The following statements are equivalent for all $k \in \mathbb{N}$ and locally finite multigraphs G:* G

(i) *G has k edge-disjoint topological spanning trees.*

(ii) *For every finite partition of $V(G)$, into ℓ sets say, G has at least $k\,(\ell - 1)$ cross-edges.*

We begin our proof of Theorem 8.5.7 with a compactness extension of the finite theorem, which will give us a slightly weaker statement at the limit.

Lemma 8.5.8. *If for every finite partition of $V(G)$, into ℓ sets say, G has at least $k\,(\ell - 1)$ cross-edges, then G has k edge-disjoint spanning subgraphs whose closures in $|G|$ are topologically connected.*

Proof. Pick an enumeration $v_0, v_1, \ldots$ of $V(G)$. For every $n \in \mathbb{N}$ let G_n be the finite multigraph obtained from G by contracting every component of $G - \{v_0, \ldots, v_n\}$ to a vertex, deleting any loops but no parallel edges that arise in the contraction. Then $G[v_0, \ldots, v_n]$ is an induced submultigraph of G_n. Let $\mathcal{V}_n$ denote the set of all k-tuples $(H_n^1, \ldots, H_n^k)$ of edge-disjoint connected spanning subgraphs of G_n. (2.4.1) (8.1.2)

Since every partition P of $V(G_n)$ induces a partition of $V(G)$, since G has enough cross-edges for that partition, and since all these cross-edges are also cross-edges of P, Theorem 2.4.1 implies that $\mathcal{V}_n \neq \emptyset$. As every $(H_n^1, \ldots, H_n^k) \in \mathcal{V}_n$ induces an element $(H_{n-1}^1, \ldots, H_{n-1}^k)$ of $\mathcal{V}_{n-1}$,

13 Note that all our definitions extend naturally to multigraphs.

the infinity lemma (8.1.2), yields a sequence $(H_n^1, \dots, H_n^k)_{n\in\mathbb{N}}$ of k-tuples, one from each $\mathcal{V}_n$, with a limit $(H^1, \dots, H^k)$ defined by the nested unions

$$H^i := \bigcup_{n\in\mathbb{N}} H_n^i[v_0, \dots, v_n]\,.$$

These H^i are edge-disjoint for distinct i (because the H_n^i are), but they need not be connected. To show that they have connected closures, it suffices by Lemma 8.5.5 to show that each of them has an edge in every finite cut F of G. Given F, choose n large enough that all the edges of F lie in $G[v_0, \dots, v_n]$. Then F is also a cut of G_n. Now consider the k-tuple $(H_n^1, \dots, H_n^k)$ which the infinity lemma picked from $\mathcal{V}_n$. Each of these H_n^i is a connected spanning subgraph of G_n , so it contains an edge from F. But H_n^i agrees with H^i on $\{v_0, \dots, v_n\}$, so H^i too contains this edge from F. □

Lemma 8.5.9. *Every connected standard subspace of $|G|$ that contains $V(G)$ also contains a topological spanning tree of G.*

Proof. Let X be a connected standard subspace of $|G|$ containing $V(G)$. Then G too must be connected, so it is countable. Let $e_0, e_1, \dots$ be an enumeration of $E(X)$, and consider these edges in turn. Starting with $X_0 := X$, define $X_{n+1} := X_n \smallsetminus \mathring{e}_n$ if this keeps X_{n+1} connected; if not, put $X_{n+1} := X_n$. Finally, let $T := \bigcap_{n\in\mathbb{N}} X_n$.

Since T is closed and contains $V(G)$, it is still a standard subspace. And T has an edge in every finite cut of G, because X does and its last edge in that cut will never be deleted. So T is arc-connected, by Lemma 8.5.5. But T contains no circle: that would contain an edge, which should have got deleted since deleting an edge from a circle cannot destroy connectedness. □

Proof of Theorem 8.5.7. The implication (ii)→(i) follows from our two lemmas. For (i)→(ii), let G have edge-disjoint topological spanning trees $T_1, \dots, T_k$, and consider a partition P of $V(G)$ into ℓ sets. If there are infinitely many cross-edges, there is nothing to show; so we assume there are only finitely many. For each $i \in \{1, \dots, k\}$, let T_i' be the multigraph of order ℓ which the edges of T_i induce on P.

To establish that G has at least $k(\ell-1)$ cross-edges, we show that the graphs T_i' are connected. If not, then some T_i' has a vertex partition crossed by no edge of T_i. This partition induces a cut of G that contains no edge of T_i. By our assumption that G has only finitely many cross-edges, this cut is finite. By Lemma 8.5.5, this contradicts the connectedness of T_i. □

As a more comprehensive application of our new theory, we show in the remainder of this section how the notion of cycle space from finite graph theory extends to locally finite graphs by making use of infinite circuits and topological spanning trees. All the applications of finite cycle spaces covered in this book can be shown to extend to this new infinite cycle space, while their naive extensions based on finite circuits can be shown to fail.

Call a family $(D_i)_{i \in I}$ of subsets of $E(G)$ *thin* if no edge lies in D_i for infinitely many i. Let the *sum* $\sum_{i \in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i. Now define the *(topological) cycle space* $\mathcal{C}(G)$ of G as the subspace of its edge space $\mathcal{E}(G)$ consisting of all sums of (thin families of) circuits. (Note that $\mathcal{C}(G)$ is closed under addition: just combine the two thin families into one.) Clearly, this definition of $\mathcal{C}(G)$ agrees with that from Chapter 1.9 when G is finite.

thin

sum

topological cycle space

We say that a given set $\mathcal{Z}$ of circuits *generates* $\mathcal{C}(G)$ if every element of $\mathcal{C}(G)$ is a sum of (a possibly infinite thin family of) elements of $\mathcal{Z}$. For example, the cycle space of the ladder in Figure 8.1.3 can be generated by all its squares (the 4-element circuits), or by the infinite circuit consisting of all horizontal edges and all squares but one. Similarly, the 'wild circuit' of Figure 8.5.1 is the sum of all the finite face boundaries of that graph, which thus generate it.

generates

The following theorem summarizes how the properties of the cycle spaces of finite graphs, familiar from Chapter 1.9, extend to locally finite graphs with topological cycle spaces. There are similar extensions of the properties of the cut space, and of its duality with the cycle space; see Exercises 92, 88 and the notes.

Theorem 8.5.10. (Diestel & Kühn 2004; Berger & Bruhn 2009) *Let $G = (V, E, \Omega)$ be a locally finite connected graph.*

(i) *The fundamental circuits of any topological spanning tree of G generate $\mathcal{C}(G)$.*

(ii) *$\mathcal{C}(G)$ consists of those subsets of E that meet every finite cut in an even number of edges.*

(iii) *Every element of $\mathcal{C}(G)$ is a disjoint sum of circuits.*

(iv) *A set $D \subseteq E$ lies in $\mathcal{C}(G)$ if and only if the degree of every vertex in (V, D), and the edge-degree of every end in $\overline{(V, D)}$, are even.*[14]

[14] This statement is not yet well defined: since ends in subspaces even of locally finite graphs can have infinite degree, we have to agree first when the infinite degree of an end is deemed to be 'even'. Such a division of 'infinite' into 'even' and 'odd' has indeed been found for the proof of (iv), but it is not that simple. See the notes.

While Theorem 8.5.10 (iv) is too difficult to prove here, (i) and (ii) will be easy. We shall also prove (iii), which is much more interesting now than for finite graphs.

Proof of Theorem 8.5.10 (i)–(ii). We prove both assertions simultaneously by showing that the following three statements are equivalent, given any topological spanning tree T of G and edge set $D \subseteq E$:

$T,\ D$

- $D \in \mathcal{C}(G)$;
- D meets every finite cut F of G in an even number of edges;
- D is a sum of fundamental circuits.

The third of these assertions implies the first by definition of $\mathcal{C}(G)$. Let us prove the second from the first. By assumption, D is a sum of a thin family of circuits. Only finitely many of these can meet F, so it suffices to show that every circuit meets F evenly. This follows from Lemma 8.5.3 (i): given a circle C in $|G|$, the segments of C between any (consecutive) edges it has in F are arcs whose vertices all lie on the same side of the cut F. These sides alternate as we follow C round. Therefore, there is an even number of such arcs, and hence of edges that C has in F.

It remains to prove the third assertion from the second. Write C_e for the fundamental circuit of an edge $e \notin E(T)$, and D_f for the fundamental cut of an edge $f \in E(T)$. Recall that, since G is locally finite, these D_f are finite cuts, so the second statement applies to them. We show that

$$D = \sum_{e \in D \smallsetminus E(T)} C_e\,. \tag{$*$}$$

(Since $f \in C_e \Leftrightarrow e \in D_f$ and fundamental cuts are finite, the C_e in this sum form a thin family, so the sum is well defined.)

To prove $(*)$, we consider the edges of G separately. An edge $e \notin T$ clearly lies in D if and only if it lies in the sum in $(*)$, since C_e is the unique fundamental circuit containing e. Now consider an edge $f \in T$. Since $f \in D_f$ and $D \cap D_f$ is even by assumption, f lies in D if and only if an odd number of edges $e \neq f$ from D_f lie in D, or equivalently (since $D_f \cap E(T) = \{f\}$), if and only if an odd number of edges $e \notin E(T)$ from D lie in D_f. As $e \in D_f \Leftrightarrow f \in C_e$ for such e, this is the case if and only if f lies in the sum in $(*)$. □

Recall that, in the proof of Theorem 8.5.10 (iii) for finite graphs, we could simply construct the disjoint circuits greedily: we would 'follow the (remaining) edges round' until a circuit was found, delete it, and repeat. For infinite G, however, it is no longer straightforward to isolate a single circuit C from a given element D of $\mathcal{C}(G)$. For example, without using our knowledge that the edge set D of the wild circle in the graph G of Figure 8.5.1 is a circuit, we can see at once that it must lie in $\mathcal{C}(G)$: it is the sum of all the finite circuits bounding a face. Our proof of (iii) must therefore be able to decompose D into disjoint circuits. Since D itself is the only circuit contained in D, the proof thus has to reconstruct the complicated wild circle just from the—seemingly much weaker—information that $D \in \mathcal{C}(G)$. And it has to do so 'generically', without appealing to the special structure of this graph G.

Our proof of (iii) will run as follows. In order to collect all the edges of some given $D \in \mathcal{C}(G)$ into a decomposition of D into circuits, we shall have to be able to find a circuit $C \subseteq D$ through a given edge $e \in D$. This amounts to finding an arc A in $\overline{D}$ between the endvertices of e. Using compactness, we shall construct a limit of arcs A_n between these vertices in finite minors G_n of G. This limit will not itself be an arc, but it will be connected. As connected standard subspaces of $|G|$ are arc-connected (Lemma 8.5.4), we shall then be able to find A inside this limit.

Proof of Theorem 8.5.10 (iii). Let $D \in \mathcal{C}(G)$ be given, and enumerate (1.9.1)
its edges. We inductively construct a sequence of disjoint circuits $C \subseteq D$ each containing the smallest edge in our enumeration of D that is not yet contained in the circuits constructed before. Then all these circuits will form the desired partition of D.

Suppose then that we have already constructed finitely many disjoint circuits all contained in D. Deleting these edges from D leaves a set D' of edges that is again in $\mathcal{C}(G)$; let e be its smallest edge, in our D', e
enumeration of D. We shall find an arc A between the endvertices of e in the standard subspace that $D' \setminus \{e\}$ spans in $|G|$. Then $A \cup e$ will be the circle defining our next circuit.

Enumerate the vertices of G as $v_0, v_1, \ldots$, with $e = v_0v_1$. For each $e = v_0v_1$
$n \in \mathbb{N}$ let G_n be the multigraph obtained from G by contracting every component of $G - \{v_0, \ldots, v_n\}$ to a vertex, deleting any loops but keeping parallel edges that arise in the contraction. Note that $G[v_0, \ldots, v_n]$ is G_n
an induced submultigraph of G_n, and that both $V(G_n)$ and $E(G_n)$ are finite.

We may think of $E(G_n)$ as a subset of $E(G)$. Then the cuts of G_n are also cuts of G. By (ii), D' meets these evenly; in particular, every vertex of G_n is incident with an even number of edges in D'. Hence $D' \cap E(G_n) \in \mathcal{C}(G_n)$, by Proposition 1.9.1. For each n, pick a circuit $C_n \subseteq D' \cap E(G_n)$ through e; note that $e \in E(G_n)$ even for $n = 0$, since e is incident with v_0. Let $A_n := C_n \setminus \{e\}$, and define

$$\mathcal{V}_n := \{A_m \cap E(G_n) \mid m \geqslant n\}.$$

Since the A_m define connected subgraphs of G_m, and G_n arises from G_m by contracting but not deleting edges (other than loops), each of the edge sets $A_m \cap E(G_n) \in \mathcal{V}_n$ defines a connected subgraph of G_n.

We now apply the infinity lemma to the auxiliary graph with vertex set $\bigcup_n \mathcal{V}_n$ obtained by joining, for all $m \geqslant n > 0$, the vertex $A_m \cap E(G_n) \in \mathcal{V}_n$ to the vertex $A_m \cap E(G_{n-1}) \in \mathcal{V}_{n-1}$. The lemma yields a nested sequence $D_0 \subseteq D_1 \subseteq \ldots$ of edge sets $D_n \in \mathcal{V}_n$; let $D'' := \bigcup_n D_n$
D'', X and $X := \overline{D''}$. Clearly $e \notin D''$, but X contains the endvertices of e. It thus suffices to show that X is arc-connected: then $X \cup e$ contains a circle through e, and $D' \supseteq D'' \cup \{e\}$ contains our desired circuit.

To show that X is arc-connected, it suffices by Lemma 8.5.5 to show that D'' has an edge in every finite cut F of G such that X has vertices u, v on different sides of F. Choose n large enough that $G[v_0, \ldots, v_n]$ contains u, v and all the edges from F. Then every component of $G - \{v_0, \ldots, v_n\}$ that was contracted to a vertex of G_n has all its vertices on the same side of F. Hence F is also a cut of G_n, with u and v on different sides. Since D_n defines a connected subgraph in G_n that contains u and v, we thus have $\emptyset \neq D_n \cap F = D'' \cap F$ as desired. □

Corollary 8.5.11. *$\mathcal{C}(G)$ is generated by finite circuits, and is closed under infinite (thin) sums.*

(8.2.4) *Proof.* By Theorem 8.2.4, G has a normal spanning tree, T say. By Lemma 8.5.6, its closure $\overline{T}$ in $|G|$ is a topological spanning tree. The fundamental circuits of $\overline{T}$ coincide with those of T, and are therefore finite. By Theorem 8.5.10 (i), they generate $\mathcal{C}(G)$.

Let $\sum_{i \in I} D_i$ be a sum of elements of $\mathcal{C}(G)$. By Theorem 8.5.10 (iii), each D_i is a disjoint union of circuits. Together, these form a thin family, whose sum equals $\sum_{i \in I} D_i$ and lies in $\mathcal{C}(G)$. □

8.6 Recursive structures

In this section we introduce another tool that is commonly used in infinite graph theory: to define a class of graphs recursively, so as to be able later to prove assertions about these graphs by (transfinite) induction. Rather than attempting a systematic treatment of this technique we give two examples; more can be found in the exercises.

Our first example is very simple: it describes the structure of a tree by recursively pruning away leaves and isolated ends. Let T be any tree, equipped with a root and the corresponding tree-order on its vertices. We recursively label the vertices of T by ordinals, as follows. Given an ordinal α, assume that we have decided for every $\beta < \alpha$ which of the vertices of T to label β, and let T_α be the subgraph of T induced by the vertices that are still unlabelled. Assign label α to every vertex t of T_α whose up-closure $\lfloor t \rfloor_{T_\alpha} = \lfloor t \rfloor_T \cap T_\alpha$ in T_α is a chain. The recursion terminates at the first α not used to label any vertex; for this α we put $T_\alpha =: T^*$.

For each α, the vertices labelled α form an up-set in T_α: if $\lfloor t \rfloor_{T_\alpha}$ is a chain, then so is $\lfloor t' \rfloor_{T_\alpha}$ for every $t' \in \lfloor t \rfloor_{T_\alpha}$. Every T_α, therefore, is a down-set in T (induction on α) and hence connected. Thus, T_α is a tree, and the set of vertices labelled α induces in T_α a disjoint union of paths.

recursively prunable

Let us call T *recursively prunable* if every vertex of T gets labelled in this way, i.e., if $T^* = \emptyset$. We may then be able to prove assertions about T, or about graphs containing T as a normal spanning tree, by dealing in turn with those chains as they get deleted. The following proposition shows that the recursively prunable trees form a natural class also in structural terms:

Proposition 8.6.1. *A rooted tree is recursively prunable if and only if it contains no subdivision of the infinite binary tree T_2 as a subgraph.*

Proof. Let T be any rooted tree. Suppose first that T is not recursively prunable, i.e. that $T^* \neq \emptyset$. Since no vertex of T^* gets labelled when the recursion terminates, every $t \in T^*$ has two incomparable vertices of T^* above it. As T^* is connected, it is now easy to find a subdivision of T_2 in T^* inductively, along the levels of T_2.

Conversely, suppose that T contains a subdivision T' of T_2. We shall see in a moment that T' can be chosen 'upwards' in T, that is, in such a way that the tree-order which T induces on its vertices agrees with its own tree-order as induced by T_2. If this is the case, then every vertex of T' has two incomparable vertices of T' above it (in both orders). Hence there can be no minimal ordinal α such that a vertex of T' is labelled α. Thus all of T' remains unlabelled, and $\emptyset \neq T' \subseteq T^*$ as desired.

It remains to show that T' can indeed be chosen in this way. Let T' be any subdivision of T_2 in T, and let u be minimal in the tree-order of T among the vertices of T'. Induction on the levels of the tree $\lfloor u \rfloor_{T'}$ shows that $\leqslant_{T'}$ and $\leqslant_T$ agree on $\lfloor u \rfloor_{T'}$: any upper neighbour in T' of a vertex $t \in \lfloor u \rfloor_{T'}$ must lie above t also in T, since the unique lower neighbour of t in T is either not in T' (if $t = u$), or by induction it is the unique lower neighbour of t also in T'. Pick any branch vertex v of T' in $\lfloor u \rfloor_{T'}$. Then $\lfloor v \rfloor_{T'}$ is the desired subdivision of T_2 in T. □

The charm of the recursive pruning discussed above lies in the fact that it removes the 'messy bits' of a given tree in an automated sort of way: we do not have to know where they are, but if our given tree contains a 'clean' ever-branching subtree, then the recursion will reveal it.

And there is another way of viewing it. We might think of rooted paths (paths with a first vertex, which we take to be the root) as particularly basic objects, and call them *rooted trees of rank* 0. We could then define rooted trees of higher ordinal rank inductively, taking as the rooted trees of rank α those that do not have any rank $\beta < \alpha$ but in which it is possible to delete a path starting at the root so as to leave components that each have some rank $< \alpha$ when taken with the induced tree-order. Then the rooted trees of rank $\leqslant \alpha$ will be precisely those that are recursively prunable with labels not exceeding α (Exercise 100).

We now apply the same idea to graphs that are not trees. Let us assign *rank* 0 to all the finite graphs. Given an ordinal $\alpha > 0$, we assign *rank* α to every graph G that does not already have a rank $\beta < \alpha$ and which has a finite set U of vertices such that every component of $G - U$ has some rank $< \alpha$.

rank

When disjoint graphs G_i have ranks $\alpha_i < \alpha$, their union clearly has a rank of at most α; if the union is finite, it has rank $\max_i \alpha_i$. Induction on α shows that subgraphs of graphs of rank α also have a rank of at most α. Conversely, joining finitely many new vertices to a graph (no matter how) will not change its rank.

Not every graph has a rank. Indeed, the ray cannot have a rank, since deleting finitely many of its vertices always leaves a component that is also a ray. As subgraphs of graphs with a rank also have a rank, this means that only rayless graphs can have a rank. But all these do:

Lemma 8.6.2. *A graph has a rank if and only if it is rayless.*

Proof. Consider a graph G that has no rank. Then one of its components, C_0 say, has no rank; let v_0 be a vertex in C_0. Now $C_0 - v_0$ has a component C_1 that has no rank; let v_1 be a neighbour of v_0 in C_1. Continuing inductively, we find a ray $v_0 v_1 \ldots$ in G. $\square$

Because of Lemma 8.6.2, we call the ranking defined above the *ranking of rayless graphs*. As an application of this ranking, we now prove the unfriendly partition conjecture from Section 8.1 for rayless graphs.

Theorem 8.6.3. *Every countable rayless graph G has an unfriendly partition.*

Proof. To help with our formal notation, we shall think of a partition of a set V as a map $\pi\colon V \to \{0,1\}$. We apply induction on the rank of G. When this is zero then G is finite, and an unfriendly partition can be obtained by maximizing the number of edges across the partition. Suppose now that G has rank $\alpha > 0$, and assume the theorem as true for graphs of smaller rank.

Let U be a finite set of vertices in G such that each of the components $C_0, C_1, \ldots$ of $G - U$ has rank $< \alpha$. Partition U into the set U_0 of vertices that have finite degree in G, the set U_1 of vertices that have infinitely many neighbours in some C_n, and the set U_2 of vertices that have infinite degree but only finitely many neighbours in each C_n.

For every $n \in \mathbb{N}$ let $G_n := G[U \cup V(C_0) \cup \ldots \cup V(C_n)]$. This is a graph of some rank $\alpha_n < \alpha$, so by induction it has an unfriendly partition π_n. Each of these π_n induces a partition of U. Let π_U be a partition of U induced by π_n for infinitely many n, say for $n_0 < n_1 < \ldots$.

Choose n_0 large enough that G_{n_0} contains all the neighbours of vertices in U_0, and the other n_i large enough that every vertex in U_2 has more neighbours in $G_{n_i} - G_{n_{i-1}}$ than in $G_{n_{i-1}}$, for all $i > 0$. Let π be the partition of G defined by letting $\pi(v) := \pi_{n_i}(v)$ for all $v \in G_{n_i} - G_{n_{i-1}}$ and all i, where $G_{n_{-1}} := \emptyset$. Note that $\pi|_U = \pi_{n_0}|_U = \pi_U$.

Let us show that π is unfriendly. We have to check that every vertex is *happy with* π, i.e., that it has at least as many neighbours in the opposite class under π as in its own.[15] To see that a vertex $v \in G - U$ is happy with π, let i be minimal such that $v \in G_{n_i}$ and recall that v was happy with π_{n_i}. As both v and its neighbours in G lie in $U \cup V(G_{n_i} - G_{n_{i-1}})$, and π agrees with π_{n_i} on this set, v is happy also with π. Vertices in U_0 are happy with π, because they were happy with π_{n_0}, and π agrees with π_{n_0} on U_0 and all its neighbours. Vertices in U_1 are also happy. Indeed, every $u \in U_1$ has infinitely many neighbours in some C_n, and hence in some $G_{n_i} - G_{n_{i-1}}$; choose i minimal. Then u has infinitely many opposite neighbours in $G_{n_i} - G_{n_{i-1}}$ under π_{n_i}. Since π_{n_i} agrees with π on both U and $G_{n_i} - G_{n_{i-1}}$, our vertex u has infinitely many opposite neighbours also under π. Vertices in U_2, finally, are happy with every π_{n_i}. By our choice of n_i, at least one of their opposite neighbours under π_{n_i} must lie in $G_{n_i} - G_{n_{i-1}}$. Since π_{n_i} agrees with π on both U_2 and $G_{n_i} - G_{n_{i-1}}$, this gives every $u \in U_2$ at least one opposite neighbour under π in every $G_{n_i} - G_{n_{i-1}}$. Hence u has infinitely many opposite neighbours under π, which clearly makes it happy. □

Exercises

1.$^-$ Show that a connected graph is countable if all its vertices have countable degrees.

2.$^-$ Given countably many sequences $\sigma^i = s_1^i, s_2^i, \dots$ $(i \in \mathbb{N})$ of natural numbers, find one sequence $\sigma = s_1, s_2, \dots$ that beats every σ^i eventually, i.e. such that for every i there exists an $n(i)$ such that $s_n > s_n^i$ for all $n \geqslant n(i)$.

3. Can a countable set have uncountably many subsets whose intersections have finitely bounded size?

4.$^-$ Let T be an infinite rooted tree. Show that every ray in T has an increasing tail, that is, a tail whose sequence of vertices increases in the tree-order associated with T and its root.

5.$^-$ Let G be an infinite graph and $A, B \subseteq V(G)$. Show that if no finite set of vertices separates A from B in G, then G contains an infinite set of disjoint A–B paths.

[15] It is only by tradition that such partitions are called ‘unfriendly’; our vertices love them.

6.$^-$ In Proposition 8.1.1, the existence of a spanning tree was proved using Zorn's lemma 'from below', to find a maximal acyclic subgraph. For finite graphs, one can also use induction 'from above', to find a minimal spanning connected subgraph. What happens if we apply Zorn's lemma 'from above' to find such a subgraph?

For the next two exercises it may help to consider the cycle space of the given graph, defined as for finite graphs in Chapter 1.9.

7.$^-$ Show that if a graph has a spanning tree with infinitely many chords then all its spanning trees have infinitely many chords.

8. Show that if a graph contains infinitely many distinct cycles then it contains infinitely many edge-disjoint cycles.

9. Let G be a countable infinitely connected graph. Show that G has, for every $k \in \mathbb{N}$, an infinitely connected spanning subgraph of girth at least k.

10. Construct, for any given $k \in \mathbb{N}$, a planar k-connected graph. Can you construct one whose girth is also at least k? Can you construct an infinitely connected planar graph?

11. Theorem 8.1.3 implies that there exists an $\mathbb{N} \to \mathbb{N}$ function f_χ such that, for every $k \in \mathbb{N}$, every infinite graph of chromatic number at least $f_\chi(k)$ has a finite subgraph of chromatic number at least k. (E.g., let f_χ be the identity on $\mathbb{N}$.) Find similar functions f_δ and f_κ for the minimum degree and connectivity, or show that no such functions exist.

12.$^+$ Show that, given $k \in \mathbb{N}$ and an edge e in a graph G, there are only finitely many bonds in G that consist of exactly k edges and contain e.

13.$^-$ Extend Theorem 2.4.4 to infinite graphs.

14.$^+$ For every $k \in \mathbb{N}$, construct a k-connected locally finite graph such that the deletion of the edge set of any cycle disconnects that graph. Deduce that the tree-packing theorem (2.4.1) of Nash-Williams and Tutte fails for infinite graphs.

(Hint. Start with a k-connected finite graph G_0. If G_0 has a cycle C such that deleting $E(C)$ does not disconnect G_0, graft some more copies of G_0 on to $E(C)$ to give C that property. Continue inductively.)

15. Give a proof of Theorem 8.1.3 for countable graphs that is based on the fact that, in this case, the topological space X defined in the third proof of the theorem is sequentially compact. (Thus, every infinite sequence of points in X has a convergent subsequence: there is an $x \in X$ such that every neighbourhood of x contains a tail of the subsequence.)

16. Show that the restriction to countable sets X of the compactness principle in Appendix A is equivalent to the infinity lemma.

17. In the text, the unfriendly partition conjecture is proved for locally finite graphs, using the infinity lemma.

(i) Give an alternative proof using the compactness principle from Appendix A.

(ii) The proof in the text, by the infinity lemma, required a modification of the statement. Is this still necessary? Which step in the proof using the compactness principle reflects the requirement in the infinity lemma that every admissible partial solution must induce an admissible solution on a smaller substructure? Where is the local finiteness used?

18. (i) Prove the unfriendly partition conjecture for countable graphs with all degrees infinite.

(ii) Can you adapt the proof to cover also those countable graphs that have finitely many vertices of finite degree?

19. Rephrase Gallai's partition theorem of Exercise 39 (i), Chapter 1, in terms of degrees, and extend the equivalent version to locally finite graphs.

20. Prove Theorem 8.4.8 for locally finite graphs. Does your proof extend to arbitrary countable graphs?

21. Extend the marriage theorem to locally finite graphs, but show that it fails for countable graphs with infinite degrees.

22.$^+$ Show that a locally finite graph G has a 1-factor if and only if, for every finite set $S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ odd (finite) components. Find a counterexample that is not locally finite.

23.$^+$ Extend Kuratowski's theorem to countable graphs.

24.$^-$ A vertex $v \in G$ is said to *dominate* an end ω of G if any of the following three assertions holds; show that they are equivalent.

(i) For some ray $R \in \omega$ there is an infinite v–$(R - v)$ fan in G.

(ii) For every ray $R \in \omega$ there is an infinite v–$(R - v)$ fan in G.

(iii) No finite subset of $V(G - v)$ separates v from a ray in ω.

25. Show that a graph G contains a $TK^{\aleph_0}$ if and only if some end of G is dominated by infinitely many vertices.

26. Construct a countable graph with uncountably many thick ends.

27. Show that a locally finite connected vertex-transitive graph has exactly 0, 1, 2 or infinitely many ends.

28.$^+$ Show that the automorphisms of a graph $G = (V, E)$ act naturally on its ends, i.e., that every automorphism $\sigma\colon V \to V$ can be extended to a map $\sigma\colon \Omega(G) \to \Omega(G)$ such that $\sigma(R) \in \sigma(\omega)$ whenever R is a ray in an end ω. Prove that, if G is connected, every automorphism σ of G fixes a finite set of vertices or an end. If σ fixes no finite set of vertices, can it fix more than one end? More than two?

29.$^-$ Show that a locally finite spanning tree of a graph G contains a ray from every end of G.

30. A ray in a graph *follows* another ray if the two have infinitely many vertices in common. Show that if T is a normal spanning tree of G then every ray of G follows a unique normal ray of T.

31. Show that the following assertions are equivalent for connected countable graphs G.

    (i) G has a locally finite spanning tree.

    (ii) G has a locally finite normal spanning tree.

    (iii) Every normal spanning tree of G is locally finite.

    (iv) For no finite separator $X \subseteq V(G)$ does $G - X$ have infinitely many components.

32. Use the previous exercise to show that every (countable) planar 3-connected graph has a locally finite spanning tree.

33. Let G be a connected graph. Call a set $U \subseteq V(G)$ *dispersed* if every ray in G can be separated from U by a finite set of vertices. (In the topology of Section 8.5, these are precisely the closed subsets of $V(G)$.)

    (i) Prove Jung's theorem that G has a normal spanning tree if and only if $V(G)$ is a countable union of dispersed sets.

    (ii) Deduce that if G has a normal spanning tree then so does every connected minor of G.

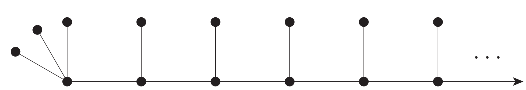
34. (i) Prove that if a given end of a graph contains k disjoint rays for every $k \in \mathbb{N}$ then it contains infinitely many disjoint rays.

    (ii) Prove that if a given end of a graph contains k edge-disjoint rays for every $k \in \mathbb{N}$ then it contains infinitely many edge-disjoint rays.

35.$^{+}$ Prove that if a graph contains k disjoint double rays for every $k \in \mathbb{N}$ then it contains infinitely many disjoint double rays.

36. Show that, in the ubiquity conjecture, the host graphs G considered can be assumed to be locally finite too.

37. Show that the modified comb below is not ubiquitous with respect to the subgraph relation. Does it become ubiquitous if we delete its 3-star on the left?



38. Imitate the proof of Theorem 8.2.6 to find a function $f: \mathbb{N} \to \mathbb{N}$ such that whenever an end ω of a graph G contains $f(k)$ disjoint rays there is a $k \times \mathbb{N}$ grid in G whose rays all belong to ω.

39. Show that there is no universal locally finite connected graph for the subgraph relation.

40. Construct a universal locally finite connected graph for the minor relation. Is there one for the topological minor relation?

41.$^-$ Show that each of the following operations performed on the Rado graph R leaves a graph isomorphic to R:

(i) taking the complement, i.e. changing all edges into non-edges and vice versa;

(ii) deleting finitely many vertices;

(iii) changing finitely many edges into non-edges or vice versa;

(iv) changing all the edges between a finite vertex set $X \subseteq V(R)$ and its complement $V(R) \setminus X$ into non-edges, and vice versa.

42.$^-$ Prove that the Rado graph is homogeneous.

43. Show that a homogeneous countable graph is determined uniquely, up to isomorphism, by the class of (the isomorphism types of) its finite subgraphs.

44. Recall that subgraphs $H_1, H_2, \ldots$ of a graph G are said to *partition* G if their edge sets form a partition of $E(G)$. Show that the Rado graph can be partitioned into any given countable set of countable locally finite graphs, as long as each of them contains at least one edge.

45.$^-$ A linear order is called *dense* if between any two elements there lies a third.

(i) Find, or construct, a countable dense linear order that has neither a maximal nor a minimal element.

(ii) Show that this order is unique, i.e. that every two such orders are order-isomorphic. (Definition?)

(iii) Show that this ordering is universal among the countable linear orders. Is it homogeneous? (Supply appropriate definitions.)

46. Given a bijection f between $\mathbb{N}$ and $[\mathbb{N}]^{<\omega}$, let G_f be the graph on $\mathbb{N}$ in which $u, v \in \mathbb{N}$ are adjacent if $u \in f(v)$ or vice versa. Prove that all such graphs G_f are isomorphic.

47. (for set theorists) Show that, given any countable model of set theory, the graph whose vertices are the sets and in which two sets are adjacent if and only if one contains the other as an element, is the Rado graph.

48. Let G be a locally finite graph. Let us say that a finite set S of vertices *separates* two ends ω and ω' if $C(S, \omega) \neq C(S, \omega')$. Use Proposition 8.4.1 to show that if ω can be separated from ω' by $k \in \mathbb{N}$ but no fewer vertices, then G contains k disjoint double rays each with one tail in ω and one in ω'. Is the same true for all graphs that are not locally finite?

49.$^+$ Prove the following more structural version of Exercise 34 (i). Let ω be an end of a countable graph G. Show that either G contains a $TK^{\aleph_0}$ with all its rays in ω, or there are disjoint finite sets $S_0, S_1, S_2, \ldots$ such that $|S_1| \leqslant |S_2| \leqslant \ldots$ and, with $C_i := C(S_0 \cup S_i, \omega)$, we have for all $i < j$ that $C_i \supseteq C_j$ and $G_i := G[S_i \cup C_i]$ contains $|S_i|$ disjoint S_i–S_{i+1} paths.

50.$^-$ Given sets A, B of vertices in a graph G, show that either G contains infinitely many edge-disjoint A–B paths or there is a finite set of edges separating A from B in G.

51. Construct an example of a small limit of large waves. Can you find a locally finite one?

52.$^+$ Prove Theorem 8.4.2 for trees.

53.$^+$ Prove Pym's theorem (8.4.7).

54. (i)$^-$ Prove the naive extension of Dilworth's theorem to arbitrary infinite posets P: if P has no antichain of order $k \in \mathbb{N}$, then P can be partitioned into fewer than k chains. (A proof for countable P will do.)

(ii)$^-$ Find a poset that has no infinite antichain and no partition into finitely many chains.

(iii) For posets without infinite chains, deduce from Theorem 8.4.8 the following Erdős-Menger-type extension of Dilworth's theorem: every such poset has a partition $\mathcal{C}$ into chains such that some antichain meets all the chains in $\mathcal{C}$.

55. Let G be a countable graph in which for every partial matching there is an augmenting path. Let M be any matching. Is there a sequence, possibly transfinite, of augmenting paths (each for the then current matching) that turns M into a 1-factor?

56. Find an uncountable graph in which every partial matching admits an augmenting path but which has no 1-factor.

57. Show that every locally finite factor-critical graph is finite.

58.$^-$ Let G be a countable graph whose finite subgraphs are all perfect. Show that G is weakly perfect but not necessarily perfect.

59.$^+$ Let G be the incomparability graph of the binary tree. (Thus, $V(G) = V(T_2)$, and two vertices are adjacent if and only if they are incomparable in the tree-order of T_2.) Show that G is perfect but not strongly perfect.

60. Let G be a countable connected graph with vertices $v_0, v_1, \dots$. For every $n \in \mathbb{N}$ write $S_n := \{v_0, \dots, v_{n-1}\}$. Prove the following statements:

(i) For every end ω of G there is a unique sequence $C_0 \supseteq C_1 \supseteq \dots$ of components C_n of $G - S_n$ such that $C_n = C(S_n, \omega)$ for all n.

(ii) For every infinite sequence $C_0 \supseteq C_1 \supseteq \dots$ of components C_n of $G - S_n$ there exists a unique end ω such that $C_n = C(S_n, \omega)$ for all n.

61. Let G be a graph, $U \subseteq V(G)$, and $R \in \omega \in \Omega(G)$. Show that G contains a comb with spine R and teeth in U if and only if $\omega \in \overline{U}$.

The *end space* of a graph G is the subspace $\Omega(G)$ of $|G|$.

62. Above every horizontal edge of the plane graph shown in Figure 8.5.1 add infinitely many horizontal edges in the plane, so as to turn every pair of rays whose associated 0–1 sequences define the same rational number into a ladder. Prove or disprove that the end space of the resulting graph is homeomorphic to $[0,1]$.

63. A compact metric space is a *Cantor set* if the singletons are its only connected subsets and every point is an accumulation point.

    (i) Characterize the trees whose end space is a Cantor set.

    (ii) Show that the end space of a connected locally finite graph is a subset of a Cantor set.

64. (i) Show that if $G = IH$ with finite branch sets, then the end spaces of G and H are homeomorphic.

    (ii) Let T_n denote the n-ary tree, the rooted tree in which every vertex has exactly n successors. Show that all these trees have homeomorphic end spaces.

65. Give an independent proof of Proposition 8.5.1 using sequential compactness and the infinity lemma.

66.$^+$ Let G be a connected countable graph that is not locally finite. Show that $|G|$ is not compact, but that $\Omega(G)$ is compact if and only if for every finite set $S \subseteq V(G)$ only finitely many components of $G - S$ contain a ray.

67. Given graphs $H \subseteq G$, let $\eta\colon \Omega(H) \to \Omega(G)$ assign to every end of H the unique end of G containing it as a subset (of rays). For the following questions, assume that H is connected and $V(H) = V(G)$.

    (i) Show that η need not be injective. Must it be surjective?

    (ii) Investigate how η relates the subspace $\Omega(H)$ of $|H|$ to its image in $|G|$. Is η always continuous? Is it open? Do the answers to these questions change if η is known to be injective?

    (iii) A spanning tree is called *end-faithful* if η is bijective, and *topologically end-faithful* if η is a homeomorphism. Show that every connected countable graph has a topologically end-faithful spanning tree.

68.$^+$ Let G be a connected graph. Assuming that G has a normal spanning tree, define a metric on $|G|$ that induces its usual topology. Conversely, use Jung's theorem of Exercise 33 to show that if $V \cup \Omega \subseteq |G|$ is metrizable then G has a normal spanning tree.

A topological space X is *locally connected* if for every $x \in X$ and every neighbourhood U of x there is an open connected neighbourhood $U' \subseteq U$ of x. A *continuum* is a compact, connected Hausdorff space. By a theorem of general topology, every locally connected metric continuum is arc-connected.

69.$^+$ Show that, for G connected and locally finite, every connected standard subspace of $|G|$ is locally connected. Using the theorem cited above, deduce Lemma 8.5.4.

70.$^+$ (for topologists) In a locally compact, connected, and locally connected Hausdorff space X, consider sequences $U_1 \supseteq U_2 \supseteq \dots$ of open, non-empty, connected subsets with compact frontiers such that $\bigcap_{i\in\mathbb{N}} \overline{U_i} = \emptyset$. Call such a sequence *equivalent* to another such sequence if every set of one sequence contains some set of the other sequence and vice versa. Note that this is indeed an equivalence relation, and call its classes the *Freudenthal ends* of X. Now add these to the space X, and define a natural topology on the extended space $\hat{X}$ that makes it homeomorphic to $|X|$ if X is a graph, by a homeomorphism that is the identity on X.

71. Let G be a locally finite graph, and X a standard subspace of $|G|$ spanned by a set of at least two edges. Show that X is a circle if and only if, for every two distinct edges $e, e' \in E(X)$, the subspace $X \setminus \mathring{e}$ is connected but $X \setminus (\mathring{e} \cup \mathring{e}')$ is disconnected.

72. Does every infinite locally finite 2-connected graph contain an infinite circuit? Does it contain an infinite bond?

73. Show that the union of all the edges contained in an arc or circle C in $|G|$ is dense in C.

74. Let T be a spanning tree of a graph G. Note that $\overline{T}$ is a connected subset of $|G|$. Without using Lemma 8.5.4, show that if T is locally finite then $\overline{T}$ is arc-connected. Find an example where $\overline{T}$ is not arc-connected.

75.$^+$ Prove that the circle shown in Figure 8.5.1 is really a circle, by exhibiting a homeomorphism with S^1.

76.$^+$ Every arc induces on its points a linear ordering inherited from $[0,1]$. Call an arc in $|G|$ *wild* if it induces on some subset of its vertices the ordering of the rationals. Show that every arc containing uncountably many ends is wild.

77. Find a graph G with a connected standard subspace of $|G|$ that is the closure of a disjoint union of circles.

78. Without using Theorem 8.5.10 show that, for G locally finite, a closed standard subspace C of $|G|$ is a circle in $|G|$ if and only if C is connected, every vertex in C is incident with exactly two edges in C, and every end in C has vertex-degree 2 (equivalently: edge-degree 2) in C.

79. Let T be a locally finite tree. Construct a continuous map $\sigma\colon [0,1] \to |T|$ that maps 0 and 1 to the root and traverses every edge exactly twice, once in each direction. (Formally: define σ so that every inner point of an edge is the image of exactly two points in $[0,1]$.)

(Hint. Define σ as a limit of similar maps σ_n for finite subtrees T_n.)

80. Let G be a connected locally finite graph. Show that the following assertions are equivalent for a spanning subgraph T of G:

(i) $\overline{T}$ is a topological spanning tree of $|G|$;

(ii) T is edge-maximal such that $\overline{T}$ contains no circle;

(iii) T is edge-minimal with $\overline{T}$ arc-connected.

81. Observe that a topological spanning tree need not be homeomorphic to a tree. Is it homeomorphic to the space $|T|$ for a suitable tree T?

82. Show that connected graphs with only one end have topological spanning trees.

83. To show that Theorem 3.2.6 does not generalize to infinite graphs with the 'finitary' cycle space as defined in Chapter 1.9, construct a 3-connected locally finite planar graph with a separating cycle that is not a finite sum of non-separating induced cycles. Can you find an example where even infinite sums of finite non-separating induced cycles do not generate all separating cycles?

84.$^-$ In a locally finite connected graph G let F be a set of edges not containing a circuit. Show that F can be extended to the edge set of a topological spanning tree of G.

85. Extend Exercise 37 of Chapter 1 to characterizations of the bonds, and of the finite bonds, in a locally finite connected graph.

86.$^-$ As a converse to Theorem 8.5.10 (i), show that the fundamental circuits of an ordinary spanning tree T of a locally finite graph G do not generate $\mathcal{C}(G)$ unless $\overline{T}$ is a topological spanning tree.

87.$^-$ Prove that the edge set of a countable graph G can be partitioned into finite circuits if G has no odd cut. Where does your argument break down if G is uncountable?

88. Explain why Theorem 8.5.10 (iii) is needed in the proof that $\mathcal{C}(G)$ is closed under infinite sums (Corollary 8.5.11): can't we just combine the constituent sums of circuits for the D_i (from our assumption that $D_i \in \mathcal{C}(G)$) into one big family? If not, can you prove the same using statement (i) of Theorem 8.5.10 rather than (iii)?

89.$^+$ Let G be a locally finite graph, $x, y \in G$ two vertices, and $P_0, P_1, \ldots$ an infinite sequence of x–y paths in G. Show that $|G|$ contains an x–y arc A each of whose edges lies eventually on every path in some fixed subsequence of the P_n.

(Hint. Exercise 79 provides some practice in an easier setting. Remember that ends can be specified as in Exercise 60.)

90. Apply Exercise 89 to give a more direct proof of Theorem 8.5.10 (iii).

For the next four exercises, let G be a locally finite connected graph. Let $\mathcal{C} = \mathcal{C}(G)$, and define the *cut space* $\mathcal{C}^* = \mathcal{C}^*(G)$ of G as in Chapter 1.9. Note that cuts may now be infinite. Define 'generate' for cuts as for circuits, allowing sums over infinite thin families of cuts. Given a set $\mathcal{F} \subseteq \mathcal{E}(G)$, write $\mathcal{F}^\perp := \{ D \in \mathcal{E}(G) \mid \forall F \in \mathcal{F} : |D \cap F| \in 2\mathbb{N} \}$ and $\mathcal{F}_{\text{fin}} := \{ F \in \mathcal{F} : |F| < \infty \}$.

91. (i)$^-$ Show that $\mathcal{C}^*$ is a subspace of $\mathcal{E}(G)$ generated by finite cuts.

(ii) Show that every cut is a disjoint union of bonds.

(iii)$^+$ Show that the fundamental cuts of any ordinary spanning tree of G generate $\mathcal{C}^*$, but that those of a topological spanning tree need not.

(iv)$^+$ Show that $\mathcal{C}^*$ is closed under infinite (thin) sums.

92. (i)$^-$ Find in this book a proof, or sketch of a proof, for each of the following two statements: $\mathcal{C} = (\mathcal{C}^*{}_{\text{fin}})^\perp$ and $\mathcal{C}^* = (\mathcal{C}_{\text{fin}})^\perp$.
(ii)$^+$ Show that $\mathcal{C}^{*\perp} = \mathcal{C}_{\text{fin}}$ and, if G is 2-connected, $\mathcal{C}^\perp = \mathcal{C}^*{}_{\text{fin}}$.

93. Write $\mathcal{D}$ for the set of circuits in G, and $\mathcal{B}$ for the set of bonds.
(i)$^-$ Show that $(\mathcal{D}_{\text{fin}})^\perp = (\mathcal{C}_{\text{fin}})^\perp$ and $(\mathcal{B}_{\text{fin}})^\perp = (\mathcal{C}^*{}_{\text{fin}})^\perp$.
(ii)$^+$ Find 2-connected graphs for which $\mathcal{D}^\perp \supsetneq \mathcal{C}^\perp$ and $\mathcal{B}^\perp \supsetneq \mathcal{C}^{*\perp}$, respectively.

94. Extending Gallai's partition theorem of Exercise 39 (ii), Chapter 1, show that $E(G)$ can be partitioned into a set $C \in \mathcal{C}$ and a set $D \in \mathcal{C}^*$. (This strengthens Exercise 19.)

95.$^+$ Let $G = (V, E)$ be a connected locally finite graph and H an abelian group. Let the group $\mathcal{C}_H$ of *H-circulations* on $|G|$ consist of the maps $\psi\colon \vec{E} \to H$ that satisfy (F1) and $\psi(X, Y) = 0$ for any finite cut $E(X, Y)$ of G. (See Chapter 6.1 for notation.) Extend Exercise 8 of Chapter 6 to $\mathcal{C}_H$, with $\mathcal{E}_H$ and $\mathcal{D}_H$ as defined there.

96.$^+$ Let G be a locally finite graph, and X a connected standard subspace of $|G|$. Call a continuous map $\sigma\colon S^1 \to X$ a *topological Euler tour* of X if it traverses every edge in $E(X)$ exactly once. (Formally: every inner point of an edge in $E(X)$ must be the image of exactly one point in S^1.) Use compactness to show that X admits a topological Euler tour if and only if $E(X) \in \mathcal{C}(G)$.
(Hint. Exercise 79 provides some practice in an easier setting.)

97.$^+$ An *open Euler tour* in an infinite connected graph G is a 2-way infinite walk $\ldots e_{-1} v_0 e_0 \ldots$ that contains every edge of G exactly once. Show that G contains an open Euler tour if and only if G is countable, every vertex has even or infinite degree, and any finite cut $F = E(V_1, V_2)$ with both V_1 and V_2 infinite is odd.

98. Show that a countable tree has uncountably many ends if and only if it contains a subdivision of the binary tree T_2. Deduce that a countable connected graph has either countably many or continuum many ends.

99.$^+$ Show that the vertices of any infinite connected locally finite graph can be enumerated in such a way that every vertex is adjacent to some later vertex.

100. Show that a tree is recursively prunable with labels $\leqslant \alpha$ if and only if it has rank $\leqslant \alpha$ in the 'ranking of rooted' trees defined in the second paragraph after the proof of Proposition 8.6.1.

101. Construct a countable tree that has rank ω in the ranking of rayless graphs. Can you find one such tree that contains all the others? Or one that is contained in them all?

102. A graph $G = (V, E)$ is called *bounded* if for every vertex labelling $\ell\colon V \to \mathbb{N}$ there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ that exceeds the labelling along any ray in G eventually. (Formally: for every ray $v_1 v_2 \ldots$ in G there exists an n_0 such that $f(n) > \ell(v_n)$ for every $n > n_0$.) Prove the following assertions: