ON THE ARITHMETIC SELF-INTERSECTION NUMBER OF THE DUALIZING SHEAF FOR FERMAT CURVES OF PRIME EXPONENT

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Abstract. In this article we improve the upper bound for the arithmetic self-intersection number of the dualizing sheaf of the minimal regular model for the Fermat curves F_p of prime exponent, given by the second author in [Kü2].

Contents

0. Introduction

The main motivation of Arakelov to develop an arithmetic intersection theory was the idea of proving the Mordell conjecture by mimicking the proof in the function field case done by Parshin [Pa1]. Let E be a number field. A central step in this program relies on suitable upper bounds for the arithmetic self-intersection number $\overline{\omega}_{\mathrm{Ar}}{}^2$, where $\overline{\omega}_{\mathrm{Ar}}$ is the dualizing sheaf $\omega_{\mathcal{X}} = \omega_{\mathcal{X}/\mathcal{O}_E} \otimes f^*\omega_{\mathcal{O}_E/\mathbb{Z}}$ equipped with the Arakelov metric (see [Ar], p.1177, [MB1], p.75), of an arithmetic surface $\mathcal{X} \to \operatorname{Spec}\mathcal{O}_E$ that varies in certain complete families (cf. [Pa2], [MB2], or Vojta's appendix in [La]). However finding such bounds turned out to be an intricate problem. The best results obtained so far give asymptotics or upper bounds for $\overline{\omega}_{\mathrm{Ar}}{}^2$ on regular models for certain discrete families of curves as modular curves (see [AU], [MU], [JK1] and [Kü2]) and Fermat curves (see [Kü2]). Bounds for these curves have been asked for since the beginning of Arakelov theory (see e.g. [La], p. 130 or [MB2], 8.2). In this article we improve the upper bound of $\overline{\omega}_{\mathrm{Ar}}{}^2$ for Fermat curves F_p of prime exponent. Our calculations rely on a careful analysis of the cusps behaviour above the prime p. This

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allows us to compute exactly the "algebraic contributions" of a formula for $\overline{\omega}_{\mathrm{Ar}}{}^2$ in [Kü2]. We also take into account the difference between the minimal regular model $\mathfrak{F}_p^{min}$ and the regular model $\mathfrak{F}_p$ constructed in [Mc], i.e. the minimal desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p + y^p = z^p$ with prime exponent p. This leads to the following result.

Theorem 0.1. *Let $\pi : \mathfrak{F}_p^{min} \to \operatorname{Spec} \mathbb{Z}[\zeta_p]$ be the minimal regular model of the Fermat curve $F_p : x^p + y^p = z^p$ of prime exponent and genus g. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric satisfies*

$$\overline{\omega}^2_{\mathfrak{F}_p^{min},Ar} \leq (2g-2)\left(\log|\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_p):\mathbb{Q}]\left(\kappa_1 \log p + \kappa_2\right) + \frac{3p^2-14p+15}{p(p-3)}\log p\right),$$

where $\kappa_1, \kappa_2 \in \mathbb{R}_+^*$ are positive constants independent of p.

Proof: See Theorem 8.4. □

It is a well known fact that $\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}} = (-1)^{\frac{p-1}{2}} p^{p-2}$ and $[\mathbb{Q}(\zeta_p):\mathbb{Q}] = p-1$ and so Theorem 0.1 yields

$$\overline{\omega}^2_{\mathfrak{F}_p^{min},\mathrm{Ar}} \leq (2g-2)\left((p-1)\left(\kappa_1 \log p + \kappa_2\right) + \frac{2p^2-p-5}{p}\log p\right).$$

In comparison to previous results in [Kü2] our explicit calculation of the algebraic contributions reduces the maximal possible growth of $\overline{\omega}^2_{\mathfrak{F}_p^{min},\mathrm{Ar}}$ as a function in p by a factor $g(F_p)p^6$. In the forthcoming thesis of the first named author the more general case of Fermat curves with squarefree exponents will be considered.

1. Intersection theory for arithmetic surfaces

We start by reminding some notation used in the context of Arakelov Theory. Most of it will be very similar to the notation used in [So].

Definition 1.1. An *arithmetic surface* $\mathcal{X}$ is a regular integral scheme of dimension 2 together with a projective flat morphism $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$, where $\mathcal{O}_E$ is the ring of integers of a number field E. Moreover we assume that the *generic fiber* $X_E = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} E$ of f is geometrically irreducible, i.e. $\mathcal{X}$ is a regular model for X_E over $\operatorname{Spec} \mathcal{O}_E$. We denote the complex valued points $\mathcal{X}(\mathbb{C})$ by $\mathcal{X}_\infty$; this is a compact, 1-dimensional, complex manifold, which may have several connected components. Actually we have the decomposition

$$\mathcal{X}_\infty = \coprod_{\sigma: E \hookrightarrow \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C}),$$

where $\mathcal{X}_\sigma(\mathbb{C})$ denotes the set of complex valued points of the curve $\mathcal{X}_\sigma = \mathcal{X} \times_{\operatorname{Spec} E,\sigma} \operatorname{Spec} \mathbb{C}$ coming from the embedding $\sigma : E \hookrightarrow \mathbb{C}$. For each $s \in \operatorname{Spec} \mathcal{O}_E$ we define the fibre above s as $\mathcal{X}_s := \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} k(s)$. We have $\mathcal{X}_{(0)} = X_E$. Any point $s \neq (0)$ will be called a *closed point* and the corresponding fibre $\mathcal{X}_s$ a *special fibre.*

Remark 1.2. Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ be an arithmetic surface in the sense of Definition 1.1. Due to the fact that $\operatorname{Spec} \mathcal{O}_E$ is Noetherian and that f is of finite type it follows that $\mathcal{X}$ is Noetherian as well.

Definition 1.3. We denote by $Z^1(\mathcal{X})$ the *group of Weil divisors of* $\mathcal{X}$, by $\mathrm{Cl}(\mathcal{X})$ the *divisor class group of* $\mathcal{X}$ i.e. the group of Weil divisors divided by the subgroup of principal divisors $R^1(\mathcal{X})$, and by $\mathrm{Pic}(\mathcal{X})$ the *Picard group of* $\mathcal{X}$.

Remark 1.4. Since $\mathcal{X}$ is a regular Noetherian integral scheme, the divisor class group $\mathrm{Cl}(\mathcal{X})$ of $\mathcal{X}$ is isomorphic to the Picard group $\mathrm{Pic}(\mathcal{X})$ (see [Li2], p.257: Corollary 1.19 and p.271: Proposition 2.16). Let us denote by g' the canonical surjection $g' : Z^1(\mathcal{X}) \to \mathrm{Cl}(\mathcal{X})$ and by g the isomorphism $g : \mathrm{Cl}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$. For any divisor $\mathcal{D} \in Z^1(\mathcal{X})$ we denote the corresponding invertible sheaf $(g \circ g')(\mathcal{D})$ by $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$.

Definition 1.5. We set $\mathrm{Cl}(\mathcal{X})_{\mathbb{Q}} = \mathrm{Cl}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Obviously $\mathrm{Cl}(\mathcal{X})_{\mathbb{Q}}$ is a group again. The difference is that we are now allowed to work with divisors with rational coefficients. We will use $Z^1(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$ for the analog construction for the group of Weil divisors and the Picard group. The morphisms g', g of Remark 1.4 extend to morphisms $g'_{\mathbb{Q}} := g' \otimes \mathrm{id}_{\mathbb{Q}}, g_{\mathbb{Q}} := g \otimes \mathrm{id}_{\mathbb{Q}}$ of the groups $Z^1(\mathcal{X})_{\mathbb{Q}}, \mathrm{Cl}(\mathcal{X})_{\mathbb{Q}}$ and $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$. Again, for $\mathcal{D} \in Z^1(\mathcal{X})_{\mathbb{Q}}$ we will denote by $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ its image with respect to $g_{\mathbb{Q}} \circ {g'}_{\mathbb{Q}}$ in $\mathrm{Pic}(\mathcal{X})_{\mathbb{Q}}$.

Lemma 1.6. *Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ be an arithmetic surface and $s \in \operatorname{Spec} \mathcal{O}_E$ a closed point. Then*

$$\mathcal{X}_s = \frac{1}{m} \operatorname{div}(h)$$

in $Z^1(\mathcal{X})_{\mathbb{Q}}$, where $\mathcal{X}_s = f^* s$, $h \in K(\mathcal{X})$ and $m \in \mathbb{Z}$.

Proof: We know that the divisor class group $\mathrm{Cl}(\operatorname{Spec} \mathcal{O}_E)$ is finite and so we can find a positive integer m and a rational function $g \in K(\operatorname{Spec} \mathcal{O}_E)$ with the property that $m \cdot s = \operatorname{div}(g)$. Since $\mathcal{X}$ is regular it follows that $f^* s = \mathcal{X}_s$ (see [Li2], p.351: Lemma 3.9) and so $f^*(m \cdot s) = m \cdot \mathcal{X}_s = \operatorname{div}(h)$ for a $h \in K(\mathcal{X})$. Now, in $Z^1(\mathcal{X})_{\mathbb{Q}}$ we may divide this equation by m and the lemma is proven. $\square$

Definition 1.7. Let $\mathcal{D}$, $\mathcal{E}$ be effective divisors without common component, $x \in \mathcal{X}$ a closed point and f, g local equations of $\mathcal{D}$, $\mathcal{E}$ in the local ring $\mathcal{O}_{\mathcal{X},x}$. Then we define the *intersection number* $i_x(\mathcal{D}, \mathcal{E})$ in x as the length of $\mathcal{O}_{\mathcal{X},x}/(f,g)$ as a $\mathcal{O}_{\mathcal{X},x}$-module. The symbol $i_x(\mathcal{D}, \mathcal{E})$ is bilinear and so we may extend the intersection number to all pairs of divisors of $\mathcal{X}$ that have no common component (just write $\mathcal{D}$ as $\mathcal{D}_+ - \mathcal{D}_-$ with $\mathcal{D}_+$ and $\mathcal{D}_-$ effective and then define $i_x(\mathcal{D}, \mathcal{E}) := i_x(\mathcal{D}_+, \mathcal{E}) - i_x(\mathcal{D}_-, \mathcal{E})$). Now let $s \in \operatorname{Spec} \mathcal{O}_E$ be a closed point. The *intersection number of $\mathcal{D}$ and $\mathcal{E}$ above s* is then defined as

$$i_s(\mathcal{D}, \mathcal{E}) := \sum_{x \in \mathcal{X}_s} i_x(\mathcal{D}, \mathcal{E})[k(x) : k(s)],$$

where x runs through the closed points of $\mathcal{X}_s$ and $k(x)$, $k(s)$ denote the residue class field of x, s respectively. If it is clear from the context which intersection number we compute (above which s), we simply write $\mathcal{D} \cdot \mathcal{E}$.

Definition 1.8. Let $s \in \operatorname{Spec} \mathcal{O}_E$ be a closed point and $\mathcal{E}$ a vertical divisor contained in the special fiber $\mathcal{X}_s$. According to the moving lemma (see e.g. [Li2], p.379: Corollary 1.10) there exists a principal divisor (f) so that $\mathcal{D} := \mathcal{E} + (f)$ and $\mathcal{E}$ have no common component. Since $(f) \cdot \mathcal{E} = 0$ (see. e.g. [La], p.58: Theorem 3.1.) we may define the *self-intersection* of $\mathcal{E}$ as

$$\mathcal{E}^2 := \mathcal{D} \cdot \mathcal{E}\,.$$

Remark 1.9. Another possible way to define $\mathcal{E}^2$ can be done via cohomological methods (see e.g. [De]).

2. Canonical divisors on an arithmetic surface

Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ be an arithmetic surface in the sense of Definition 1.1. As f is a local complete intersection (see [Li2], p.232: Example 3.18.), we can define the canonical sheaf $\omega_{\mathcal{X}/\operatorname{Spec} \mathcal{O}_E}$ of $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ (see e.g. [Li2], p.239: Definition 4.7.).

Remark 2.1. Since the scheme $\operatorname{Spec} \mathcal{O}_E$ is a locally Noetherian scheme and f is a flat projective local complete intersection of relative dimension 1, the canonical sheaf is isomorphic to the 1-dualizing sheaf (see [Li2], p.247: Theorem 4.32.).

Definition 2.2. We call any divisor $\mathcal{K}$ of $\mathcal{X}$ with $\mathcal{O}_{\mathcal{X}}(\mathcal{K}) \cong \omega_{\mathcal{X}/\operatorname{Spec} \mathcal{O}_E}$ a *canonical divisor*. This divisor exists because of Remark 1.4.

Remark 2.3. By abuse of language we call a divisor $\mathcal{K} \in Z^1(\mathcal{X})_{\mathbb{Q}}$ with $\mathcal{O}_{\mathcal{X}}(\mathcal{K}) = \omega_{\mathcal{X}/\operatorname{Spec} \mathcal{O}_E}$ in $\operatorname{Pic}(\mathcal{X})_{\mathbb{Q}}$ a canonical divisor as well.

Remark 2.4. Let $s \in \operatorname{Spec} \mathcal{O}_E$ be a closed or the generic point. For each fibre $\mathcal{X}_s \to \operatorname{Spec} k(s)$ we get a canonical sheaf $\omega_{\mathcal{X}_s/\operatorname{Spec} k(s)}$. We have the relation $\omega_{\mathcal{X}_s/\operatorname{Spec} k(s)} \cong \omega_{\mathcal{X}/\operatorname{Spec} \mathcal{O}_E}|_{\mathcal{X}_s}$ (see [Li2], p.239: Theorem 4.9). If s is the generic point we can define a canonical divisor K of $X := \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_E} \operatorname{Spec} E$ in the same way we did with the arithmetic surface. Similar to the relation between the canonical sheaves we get $\mathcal{K}|_X \cong K$.

Now let $\mathcal{E}$ be a vertical divisor contained in a special fiber $\mathcal{X}_s$ and $\mathcal{K}$ a canonical divisor on $\mathcal{X}$. Since any other canonical divisor is rationally equivalent to $\mathcal{K}$ the intersection number $\mathcal{K} \cdot \mathcal{E}$ depends uniquely on $\omega_{\mathcal{X}/\mathcal{O}_{\operatorname{Spec} E}}$ and not on the choice of a representative $\mathcal{K}$. We have the following important theorem:

Theorem 2.5 (Adjunction formula)**.** *Let $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ be an arithmetic surface, $s \in \operatorname{Spec} \mathcal{O}_E$ a closed point and $\mathcal{E}$ a vertical divisor contained in the special fiber $\mathcal{X}_s$. Then we have*

$$2p_a(\mathcal{E}) - 2 = \mathcal{E}^2 + \mathcal{K} \cdot \mathcal{E}\,, \tag{2.1}$$

where $p_a(\mathcal{E})$ is the arithmetic genus of $\mathcal{E}$.

Proof: See [Li1] Theorem 3.2. □

Later on it will be important to construct the canonical divisor explicitly. The following proposition will help us with that.

Proposition 2.6. *Let $\mathcal{C} \in Z^1(\mathcal{X})_{\mathbb{Q}}$ be a divisor on $\mathcal{X}$ which satisfies the adjunction formula* (2.1) *and whose restriction to the generic fibre X is a canonical divisor of X. Then $\mathcal{C}$ is a canonical divisor on $\mathcal{X}$.*

Proof: Let $\mathcal{K}$ be a canonical divisor on $\mathcal{X}$ (we already know that it exists). We want to show that $\mathcal{K} \sim \mathcal{C}$ and so that $\mathcal{C}$ is a canonical divisor as well. We denote the horizontal part of the divisors by $\mathcal{K}_h$ and $\mathcal{C}_h$. Since the restriction to the generic fibre of both divisors is a canonical divisor of X we have $\mathcal{K}|_X = \mathcal{K}_h|_X \sim \mathcal{C}_h|_X = \mathcal{C}|_X$ and so there exists a rational element $g \in K(X)$, which yields $\mathcal{K}|_X - \operatorname{div}(g) = \mathcal{C}|_X$. Because we have $K(X) \cong K(\mathcal{X})$, we can interpret g as an element of $K(\mathcal{X})$ and so obtain a principal divisor whose restriction to X is $\operatorname{div}(g)$. We denote this principal divisor by $\operatorname{div}(g)$ as well. If we now set $\mathcal{C}' := \mathcal{C} + \operatorname{div}(g)$ we get a divisor with the properties that $\mathcal{C}' \sim \mathcal{C}$ and $\mathcal{C}'_h = \mathcal{K}_h$. Since we are just interested in $\mathcal{C}$ up to rational equivalence we may assume from now on that the horizontal part of $\mathcal{C}$ is the same as the one of $\mathcal{K}$.
Let $s \in \operatorname{Spec} \mathcal{O}_E$ be a closed point and $\mathcal{X}_s$ the fibre above it. We denote by $\mathcal{K}_s$ and $\mathcal{C}_s$ the vertical divisor of $\mathcal{K}$ and $\mathcal{C}$ which have support in $\mathcal{X}_s$. Since $\mathcal{K}$ and $\mathcal{C}$ fulfill the adjunction formula and have the same horizontal part we have

$$0 = (\mathcal{K}_s - \mathcal{C}_s) \cdot (\mathcal{K} - \mathcal{C}) = (\mathcal{K}_s - \mathcal{C}_s) \cdot (\mathcal{K}_s - \mathcal{C}_s)\,.$$

and so $\mathcal{K}_s - \mathcal{C}_s = q\mathcal{X}_s$, where q is a rational number (see [La], p.61: Proposition 3.5.). Now, according to Lemma 1.6, we find $m \in \mathbb{Z}$ and $h \in K(\mathcal{X})$ so that $\mathcal{K}_s - \mathcal{C}_s = q\mathcal{X}_s = \frac{q}{m}\operatorname{div}(h)$ and so we have $\mathcal{K}_s = \mathcal{C}_s$ in $\operatorname{Cl}(\mathcal{X})_{\mathbb{Q}}$. If we set $\mathcal{C}' := \mathcal{C} + \frac{q}{m}\operatorname{div}(h)$ we have just changed the components of $\mathcal{C}$ with support in $\mathcal{X}_s$. Again, we have $\mathcal{C}' \sim \mathcal{C}$ and now $\mathcal{K}_h + \mathcal{K}_s = \mathcal{C}'_h + \mathcal{C}'_s$. Continuing successively with the other closed points of $\operatorname{Spec} \mathcal{O}_E$ we arrive at a divisor $\mathcal{C}''$ with $\mathcal{C}'' = \mathcal{K}$ and $\mathcal{C}'' \sim \mathcal{C}$ as we claimed at the beginning. □

Remark 2.7. The Proposition 2.6 uses the fact that in $Z^1(\mathcal{X})_{\mathbb{Q}}$ the special fibres are divisors coming from functions (see Lemma 1.6). In other words, the canonical divisor in the sense of Remark 2.3 is only defined up to rational multiples of principal divisors and therefore in particular defined only up to special fibres (in $Z^1(\mathcal{X})_{\mathbb{Q}}$).

3. Arithmetic intersection numbers for hermitian line bundles

Definition 3.1. A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, h)$ is a line bundle $\mathcal{L}$ on $\mathcal{X}$ together with a smooth, hermitian metric h on the induced holomorphic line bundle $\mathcal{L}_\infty = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{C}$ on $\mathcal{X}_\infty$. We denote the norm associated with h by $||\cdot||$. Two hermitian line bundles $\overline{\mathcal{L}}$, $\overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_{\mathcal{X}}, |\cdot|)\,,$$

where $|\cdot|$ denotes the usual absolute value. The *arithmetic Picard group* $\widehat{\operatorname{Pic}}(\mathcal{X})$ is the group of isomorphy classes of hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$, the group structure being given by the tensor product.

Definition 3.2. Let $\overline{\mathcal{L}}$, $\overline{\mathcal{M}}$ be two hermitian line bundles on $\mathcal{X}$ and l, m non-trivial, global sections, whose induced divisors $\mathrm{div}(l)$ and $\mathrm{div}(m)$ on $\mathcal{X}$ have no horizontal component in common. Then we define the *intersection number at the finite places* $(l.m)_{\mathrm{fin}}$ of l and m by the formula

$$\begin{aligned}(l.m)_{\mathrm{fin}} \quad &:= \quad \sum_{x\in\mathcal{X}} \log \sharp \left(\mathcal{O}_{\mathcal{X},x}/(l_x, m_x)\right) = \sum_{x\in\mathcal{X}} i_x(\mathrm{div}(l), \mathrm{div}(m)) \log |k(x)| \\ &= \quad \sum_{s\in\mathrm{Spec}\,\mathcal{O}_E} \left(\sum_{x\in\mathcal{X}_s} i_x(\mathrm{div}(l), \mathrm{div}(m))[k(x):k(s)]\right) \log |k(s)| \ ,\end{aligned}$$

where l_x and m_x are local equations of l and m at the point $x \in \mathcal{X}$; the sum runs through the closed points x of $\mathcal{X}$.
The sections l and m induce global sections on $\mathcal{L}_\infty$ and $\mathcal{M}_\infty$, which we denote by abuse of notation again by l and m. We assume that the associated divisors $\mathrm{div}(l)$ and $\mathrm{div}(m)$ on $\mathcal{X}_\infty$ have no points in common. Writing $\mathrm{div}(l) = \sum_\alpha p_\alpha P_\alpha$ with $p_\alpha \in \mathbb{Z}$ and $P_\alpha \in \mathcal{X}_\infty$, we set

$$(\log ||m||)[\mathrm{div}(l)] := \sum_\alpha p_\alpha \log ||m(P_\alpha)|| \,,$$

where $||\cdot||$ is the norm which is associated to the metric of $\mathcal{M}_\infty$. The *intersection number at the infinite places* $(l.m)_\infty$ of l and m is now given by the formula

$$(l.m)_\infty := -(\log ||m||)[\mathrm{div}(l)] - \int_{\mathcal{X}_\infty} \log ||l|| \cdot c_1(\overline{\mathcal{M}}) \,, \tag{3.1}$$

where the first Chern form $c_1(\overline{\mathcal{M}}) \in H^{1,1}(\mathcal{X}_\infty, \mathbb{R})$ of $\overline{\mathcal{M}}$ is given, away from the divisor $\mathrm{div}(m)$ on $\mathcal{X}_\infty$, by

$$c_1(\overline{\mathcal{M}}) = \mathrm{dd^c}(-\log ||m(\cdot)||^2) \,;$$

the integral in (3.1) has to be understood as integrating with respect to the extension of $c_1(\overline{\mathcal{M}})$ to all of $\mathcal{X}_\infty$. We define the *arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ of $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + (l.m)_\infty \,. \tag{3.2}$$

For general $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ we can choose line bundles $\mathcal{L}_i$ and $\mathcal{M}_j$ $(i, j = 1, 2)$ for which non-trivial global sections exist, such that $\mathcal{L}_i$ has disjoint global sections with $\mathcal{M}_j$ for $i, j = 1, 2$ and

$$\mathcal{L} \cong \mathcal{L}_1 \otimes \mathcal{L}_2^{\otimes -1}, \mathcal{M} \cong \mathcal{M}_1 \otimes \mathcal{M}_2^{\otimes -1} \,. \tag{3.3}$$

We provide $\mathcal{L}_{i\infty}$ and $\mathcal{M}_{j_\infty}$ with metrics in such a way that the equivalences in (3.3) are isometries. Then we define $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ by linearity. The *arithmetic self-intersection number* of $\overline{\mathcal{L}}$ is given by $\overline{\mathcal{L}}.\overline{\mathcal{L}}$.

Theorem 3.3 (Arakelov, Deligne et al.)**.** *Formula* (3.2) *induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \to \mathbb{R} \,.$$

Proof: See for example [So]. □

Remark 3.4. Theorem 3.3 is a generalisation, essentially due to Deligne, of the arithmetic intersection pairing, invented by Arakelov, where only hermitian line bundle, whose Chern forms are multiples of a fixed volume form, are considered.

If the genus of $\mathcal{X}$ is greater than one, then for each σ we have on $\mathcal{X}_\sigma(\mathbb{C})$ the *canonical volume form*

$$\nu_{\mathrm{can}}^{\sigma}(z) = \frac{i}{2g} \sum_j |f_j^\sigma|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $f_1^\sigma(z)\mathrm{d}z$, ... $f_g^\sigma(z)\mathrm{d}z$ is an orthonormal basis of $H^0(\mathcal{X}_\sigma(\mathbb{C}), \Omega^1)$ equipped with the natural scalar product. We write ν_{can} for the induced volume form on $\mathcal{X}_\infty$ and for ease of notation we set

$$\overline{\mathcal{O}}(D) = \overline{\mathcal{O}}(D)_{\nu_{\mathrm{can}}}.$$

Here the norm of the section 1_D of $\mathcal{O}(D)$ is given by $\|1_D\| = g(D, \cdot)$ where g is the canonical green function (see e.g. [La]).
Due to Arakelov is the observation that there is a unique metric $\|\cdot\|_{\mathrm{Ar}}$ on $\omega_{\mathcal{X}}$ such that for all sections P of $\mathcal{X}$ it holds the adjunction formula

$$\overline{\omega}_{\mathrm{Ar}}.\overline{\mathcal{O}}(P) + \overline{\mathcal{O}}(P)^2 = \log|\Delta_{E|\mathbb{Q}}|, \tag{3.4}$$

where $\overline{\omega}_{\mathrm{Ar}} = (\omega_{\mathcal{X}}, \|\cdot\|_{\mathrm{Ar}})$. Moreover $\overline{\omega}_{\mathrm{Ar}}$ is a ν_{can}-admissible line bundle (see [Ar], p.1189 ff.).

Remark 3.5. In Remark 2.7 we saw that the canonical divisor is in particular only defined up to rational multiples of the special fibres. Because of formula (3.4) this indeterminacy will be deleted by the norm of the section.

Convention 3.6. *Analog to the first and second section we will allow rational coefficients for* $\widehat{\mathrm{Pic}}(\mathcal{X})$*. The corresponding group will be denoted by* $\widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$*. Furthermore, we will extend the arithmetic intersection numbers to this group. Unless otherwise specified, we will always assume to work with rational coefficients.*

Assumption 3.7. *Let* $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_E$ *be an arithmetic surface and write* Y *for its generic fiber. We fix* $Q, P_1, ..., P_r \in Y(E)$ *such that* $Y \setminus \{Q, P_1, ..., P_r\}$ *is hyperbolic. Then we consider any arithmetic surface* $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_E$ *equipped with a dominant morphism of arithmetic surfaces* $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ *such that the induced morphism* $\boldsymbol{\beta} : X \to Y$ *of algebraic curves defined over* E *is unramified above* $Y(E) \setminus \{Q, P_1, ..., P_r\}$*. Let* $g \geq 2$ *be the genus of* X *and* $d = \deg(\boldsymbol{\beta})$*. We write* $\boldsymbol{\beta}^* Q = \sum b_j S_j$ *and the points* S_j *will be called labeled. Set* $b_{\max} = \max_j\{b_j\}$*. Divisors on* X *with support in the labeled points are called labeled. Finally, a prime* $\mathfrak{p}$ *is said to be bad if the fiber of* $\mathcal{X}$ *above* $\mathfrak{p}$ *is reducible*[1].

[1] note that a prime of bad reduction need not be a bad prime

Theorem 3.8. *Let $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{Y}$ be a morphism of arithmetic surfaces as in Assumption 3.7. Assume that all labeled points are E-rational points and that all labeled divisors of degree zero are torsion, then the arithmetic self-intersection number of the dualizing sheaf on $\mathcal{X}$ satisfies the inequality*

$$\overline{\omega}^2_{Ar} \leq (2g-2)\left(\log|\Delta_{E|\mathbb{Q}}|^2 + [E:\mathbb{Q}]\,(\kappa_1 \log b_{\max} + \kappa_2) + \sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p})\right), \tag{3.5}$$

where $\kappa_1, \kappa_2 \in \mathbb{R}^*_+$ are positive constants that dependent only on Y and the points $Q, P_1, ..., P_r$. The coefficients $a_{\mathfrak{p}} \in \mathbb{Q}$ are determined by certain local intersection numbers (see formula (3.6) *below).*

Proof: See [Kü2] Theorem I. The method of proof uses classical Arakelov theory, as well as generalized arithmetic intersection theory (see [Kü1]), which allows to use a refinement of a result of Jorgenson and Kramer [JK2]. □

Definition 3.9. To keep the notation simple, we write $\mathcal{S}_j$ for the Zariski closure in $\mathcal{X}$ of a labeled point S_j. Let $\mathcal{K}$ be a canonical divisor of $\mathcal{X}$, then for each labeled point S_j we can find a divisor $\mathcal{F}_j$ such that

$$\left(\mathcal{S}_j + \mathcal{F}_j - \frac{1}{2g-2}\mathcal{K}\right) \cdot \mathcal{C} = 0$$

for all vertical irreducible components $\mathcal{C}$ of $\mathcal{X}$. Similarly we find for each labeled point S_j a divisor $\mathcal{G}_j$ such that also for all $\mathcal{C}$ as before

$$\left(\mathcal{S}_j + \mathcal{G}_j - \frac{1}{d}\boldsymbol{\beta}^*\overline{Q}\right) \cdot \mathcal{C} = 0\,.$$

Notice that we can choose $\mathcal{F}_j$ and $\mathcal{G}_j$ to have support in the fiber above the bad primes (Lemma 1.6). The rational numbers $a_{\mathfrak{p}}$ in Theorem 3.8 are determined by the following arithmetic intersection numbers of trivially metrised hermitian line bundles

$$\sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}) = -\frac{2g}{d}\sum_j b_j\,\overline{\mathcal{O}}(\mathcal{G}_j)^2 + \frac{2g-2}{d}\sum_j b_j\,\overline{\mathcal{O}}(\mathcal{F}_j)^2. \tag{3.6}$$

Remark 3.10. Since the divisors $\mathcal{G}_j$ and $\mathcal{F}_j$ are vertical the hermitian line bundles $\overline{\mathcal{O}}(\mathcal{G}_j)$ and $\overline{\mathcal{O}}(\mathcal{F}_j)$ have a trivial metric. In order to indicate this circumstance we will write $\mathcal{O}(\mathcal{G}_j)$ and $\mathcal{O}(\mathcal{F}_j)$ instead of $\overline{\mathcal{O}}(\mathcal{G}_j)$ and $\overline{\mathcal{O}}(\mathcal{F}_j)$. The intersection number at the infinite places of $\mathcal{O}(\mathcal{G}_j)^2$ and $\mathcal{O}(\mathcal{F}_j)^2$ is zero, and so the computation of (3.6) becomes a pure algebraic problem.

4. Fermat curves and their natural Belyi uniformization

For the rest of this article we will consider the Fermat curve

$$F_p : X^p + Y^p = Z^p,$$

where $p > 3$ is prime number, together with the natural morphism

$$\beta : F_p \to \mathbb{P}^1 \tag{4.1}$$

given by $(x : y : z) \mapsto (x^p : y^p)$. Since the morphism β is defined over $\mathbb{Q}$, it is defined over any number field. It is a Galois covering of degree p^2 and, since there are only the three branch points $P_x = (0 : 1), P_y = (1 : 0)$ and $P_z = (1 : -1)$, it is a Belyi morphism. All the ramification orders equal p. In [MR] Murty and Ramakrishnan give the associated Belyi uniformisation $F_p(\mathbb{C}) \setminus \boldsymbol{\beta}^{-1}\{P_x, P_y, P_z\} \cong \Gamma_P \setminus \mathbb{H}$. The subgroup Γ_P of $\Gamma(2)$ is given by $\Gamma_p = \ker \psi$ where $\psi : \Gamma(2) \to \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ maps the generators of $\Gamma(2)$ to the elements $(1, 0)$ and $(0, 1)$.
The ramification points of $\boldsymbol{\beta}$ are defined over $\mathbb{Q}(\zeta_p)$. A ramification point that maps to P_x is of the form $(0 : \zeta_p^i : 1)$ and we denote it by S_x; this abuse of notation will be justified by the Lemma 6.2 below, which shows that the properties of S_x, relevant for our considerations, do not depent on the exponent i. Similar we denote by S_y (resp. S_z) a preimage of P_y (resp. P_z), they are of the form $(\zeta_p^i : 0 : 1)$ (resp. $(\zeta_p^i : -1 : 0)$). A ramification point will also be called a *cusp*. Divisors with support in the cusps having degree zero are called *cuspidal divisor*.

Proposition 4.1. *Let F_p a Fermat curve and $\beta : F_p \to \mathbb{P}^1$ the morphism in* (4.1).

(i) *The group of cuspidal divisors generate a torsion subgroup of* $\mathrm{Cl}(F_p)$.
(ii) *Let $S \in F_p(\mathbb{Q}(\zeta_p))$ be a cusp, then $(2g - 2)S$ is a canonical divisor.*

Proof: The first statement follows from [Ro], p. 101: Theorem 1. So only the second statement is left. By the Hurwitz formula there exists a canonical divisor with support in the cusps. Then by (i) the claim follows. □

5. A regular model and the minimal model for F_p

In this section we are going to sketch the construction done by McCallum [Mc] of a regular model and the minimal model of the curve $F_p : x^p + y^p = z^p$ over $S = \operatorname{Spec} R$, where $R = \mathbb{Z}_p[\zeta_p]$ denotes the ring of integers of the field $\mathbb{Q}_p(\zeta_p)$ and ζ_p a primitive p-th root of unity. In order to simplify our computations we may consider the curve

$$C_p : x^p + y^p = 1 \tag{5.1}$$

in $\mathbb{A}^2_S$ because the model, we are starting with, is just the normalization of the projective completion of C_p. Let $(\pi) := (1 - \zeta_p)$ be the prime ideal which is lying above (p); in fact since p is totally ramified in $\mathbb{Q}_p(\zeta_p)$ we have $p = u\pi^{p-1}$ with an element $u \in \mathbb{Z}_p[\zeta_p]^*$. Reduction modulo p gives us a p-tuple line which is non-regular. Moving this line to the x-axis, or in other words setting

$$X = x \quad \text{and} \quad Y = y + x - 1\,, \tag{5.2}$$

equation (5.1) becomes

$$-u\pi^{p-1}\phi(X, -Y - 1) + u\pi^{p-1}\phi(Y) + Y^p = 0\,,$$

where
$$\phi(X,Y) := \frac{(X+Y)^p - X^p - Y^p}{p}$$
and $\phi(X) := \phi(X,1)$. Now, by blowing up the line $\pi = Y = 0$, one obtains a model which is covered by the two affine open sets: we introduce new variables a and b. Setting $b = \frac{\pi}{Y}$, we have $U_1 = \operatorname{Spec}(R[X,Y,b]/(bY-\pi, F_1(X,Y)))$ where
$$F_1(X,Y) = -ub^{p-1}\phi(X,-Y-1) + ub^{p-1}\phi(Y) + Y\,;$$
setting $a = \frac{Y}{\pi}$ the second affine open set is $U_2 = \operatorname{Spec}(R[X,Y,a]/(a\pi - Y, F_2(X,Y)))$ where
$$F_2(X,Y) = -u\phi(X,-Y-1) + u\phi(Y) + \pi a^p\,.$$
The geometric special fibre $U_1 \times_S \operatorname{Spec}\overline{k(\pi)} \cup U_2 \times_S \operatorname{Spec}\overline{k(\pi)}$ of this model consists of a component L (which is located just in U_1 and associated to the ideal $(\overline{Y}, \overline{b})$ in $R[X,Y,b]/(bY-\pi, F_1(X,Y))$ and components $L_x, L_y, L_{\alpha_1}, \ldots, L_{\alpha_r}, L_{\beta_1}, \ldots, L_{\beta_s}$ which intersect L and correspond to the different roots of the polynomial
$$\phi(X,-1) = -X(X-1)\prod_{i=1}^{r}(X-\alpha_i)^2\prod_{j=1}^{s}(X-\beta_j)\,;$$
we have $\alpha \in k(\pi)$, $\alpha \neq 0, 1$ and $\beta \notin k(\pi)$. The L_{α_i} appear with multiplicity 2 whereas all other components with multiplicity 1. There is also a line L_z crossing the point at infinity on L, which we cannot see in this affine model. There are just singularities left on the double lines L_{α_i}. Blowing up these singularities we achieve new components $L_{\alpha_{i,j}}$ crossing L_{α_i}. All components have genus 0. For later applications we define the index set

(5.3) $$I := \{x, y, z, \beta_i, \alpha_j, \alpha_{j,k}, \ldots\}\,.$$

Let us denote the model we achived by $\mathfrak{F}_p$. The scheme $\mathfrak{F}_p$ is a regular model and its geometric special fibre $\mathfrak{F}_p \times_{\operatorname{Spec} R} \operatorname{Spec}\overline{k(\pi)}$ corresponding to (π) has the configuration as in figure 1 where all components of the fibre have genus 0 and the pair (n,m) indicates the multiplicity n and the self-intersection m of the component ([Mc], Theorem 3.).

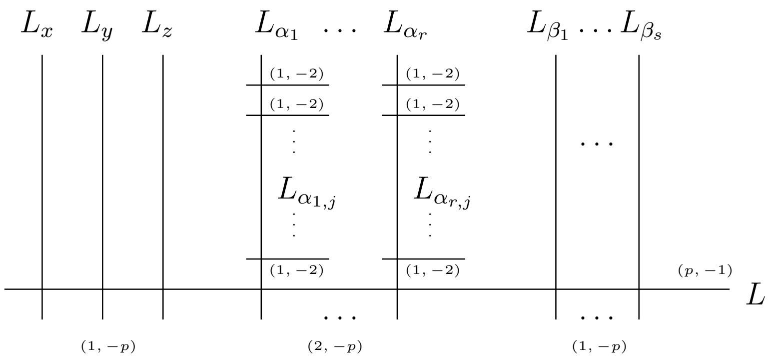

FIGURE 1. The configuration of the geometric special fibre $\mathfrak{F}_p \times_{\operatorname{Spec} R} \operatorname{Spec}\overline{k(\pi)}$.

Remark 5.1. If we now blow down the curve L (which is the only one with self-intersection -1), we get the minimal regular model $\mathfrak{F}_p^{min}$ (see [Ch], p.315: Theorem 3.1).

Remark 5.2. A regular model over $\mathbb{Z}[\zeta_p]$ can be obtained by glueing the model $\mathfrak{F}_p$ over S and the smooth model of F_p over $\operatorname{Spec}\mathbb{Z}[\zeta_p] \setminus \{(\pi)\}$. We will denote this model as well by $\mathfrak{F}_p$. According to Remark 5.1 there is just one prime of bad reduction, namely (π) (cf. [Li2], p.462: Proposition 1.21. (b)). This is the only bad prime of the scheme $\mathfrak{F}_p$.

Theorem 5.3. *The morphism $\beta : F_p \to \mathbb{P}^1$ extends to a morphism of arithmetic surfaces*

$$\beta : \mathfrak{F}_p \longrightarrow \mathbb{P}^1_{\mathbb{Z}[\zeta_p]}.$$

Moreover, the assumptions of Theorem 3.8 are fulfilled, if we choose P_x, P_y or P_z as the distinguished point Q in Assumption 3.7.

Proof: The morphism $\beta : F_p \to \mathbb{P}^1$ obviously extends to a morphism of models $\beta : \mathfrak{F}_p^0 \to \mathbb{P}^1_{\mathbb{Z}[\zeta_p]}$, where $\mathfrak{F}_p^0 = \operatorname{Proj}\mathbb{Z}[\zeta_p][X,Y,Z]/(X^p + Y^p - Z^p)$. Since we where just performing a sequence of blow-ups in order to obtain $\mathfrak{F}_p$ from $\mathfrak{F}_p^0$, it also extends to a morphism $\beta : \mathfrak{F}_p \to \mathbb{P}^1_{\mathbb{Z}[\zeta_p]}$. Now $\mathbb{P}^1 \setminus \{P_x, P_y, P_z\}$ is hyperbolic and since $\beta : F_p \to \mathbb{P}^1$ has only the three branch points P_x, P_y and P_z it is unramified above $\mathbb{P}^1 \setminus \{P_x, P_y, P_z\}$. Furthermore, since β is non-constant its extension is a dominant morphism as in Assumption 3.7. Finally it follows with Proposition 4.1 (a), that, if we choose any of the points P_x, P_y or P_z as the point Q, the labeled divisor of degree zero are torsion. □

Convention 5.4. *We make for the rest of this work the convention that $Q = P_x$.*

Remark 5.5. Because of symmetry we could have chosen $Q = P_y$ or $Q = P_z$ in Convention 5.4 as well. Then, some of the following computations in this work would have to be done with respect to this choice.

The rest of this paper is devoted to calculate the quantities $a_{\mathfrak{p}}$ in Theorem 3.8.

6. Extensions of cusps and canonical divisors on $\mathfrak{F}_p$

Definition 6.1. If we take the Zariski-closure of a cusp S_x in $\mathfrak{F}_p$, we get a horizontal divisor, which we denote by $\mathcal{S}_x$. Again, similar for y and z.

For any two divisors $\mathcal{D}$ and $\mathcal{E}$ of $\mathfrak{F}_p$ we say that $\mathcal{D}$ *intersects* $\mathcal{E}$, if $\operatorname{supp}\mathcal{D} \cap \operatorname{supp}\mathcal{E} \neq 0$.

Proposition 6.2. *Let $\mathcal{S}$ and $\mathcal{S}'$ be horizontal divisors of $\mathfrak{F}_p$ coming from different cusps S and S' on F_p. Then the following properties are true:*

(i) *$\mathcal{S}$ does not intersect $\mathcal{S}'$.*
(ii) *If $\mathcal{S} = \mathcal{S}_x$ (resp. $\mathcal{S}_y, \mathcal{S}_z$), then $\mathcal{S}$ only intersects the component L_x (resp. L_y, L_z) in the special fiber $\mathfrak{F}_p \times_{\operatorname{Spec}\mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\pi)$ (see figure 2).*

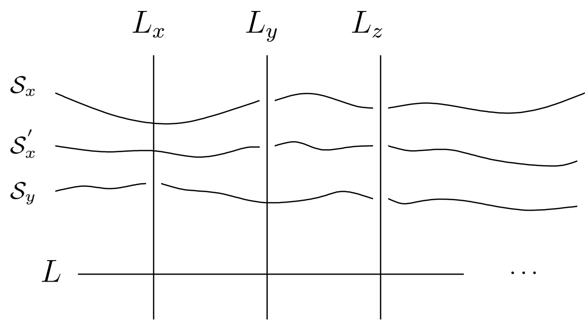

FIGURE 2. The divisors $\mathcal{S}_x, \mathcal{S}_x^{'}$ and $\mathcal{S}_y$, where $\mathcal{S}_x^{'}$ is coming from another cusp of the form $(0 : \zeta_p^j : 1)$.

Proof: For the proof, we need to work with the explicit description of the regular model $\mathfrak{F}_p$. So if we talk about a cusp in the following, we will mean a point of the form $(0 : \zeta_p^i - 1 : 1)$ ($(\zeta_p^i : \zeta_p^i - 1 : 1)$ resp.) which is just S_x (S_y resp.) after the transformation (5.2). For any element in the ring $\mathbb{Z}[\zeta_p][X, Y, b]$ ($\mathbb{Z}[\zeta_p][X, Y, a]$ resp.) we will denote by a bar the corresponding element in the ring $\mathbb{Z}[\zeta_p][X, Y, b]/(bY - \pi, F_1(X, Y))$ ($\mathbb{Z}[\zeta_p][X, Y, a]/(a\pi - Y, F_2(X, Y))$ resp.).
Now let $\mathcal{S}, \mathcal{S}^{'}$ be two horizontal divisors on $\mathfrak{F}_p$ associated with cusps $S, S^{'}$ and let $Q \in \operatorname{supp} \mathcal{S} \cap \operatorname{supp} \mathcal{S}^{'}$ be a point. We will denote by m the maximal ideal corresponding to Q. If the cusps lie above different branch points, for example $S = (0 : \zeta_p^i - 1 : 1)$ and $S^{'} = (\zeta_p^j : \zeta_p^j - 1 : 1)$, we have $\overline{X}, \overline{X} - \overline{\zeta_p^j} \in m$. But then $\overline{\zeta_p^j} \in m$ which is impossible since $\overline{\zeta_p^j}$ is a unit. So let S and $S^{'}$ lie above the same branch point. Without loss of generality we may assume $S = (\zeta_p^i : \zeta_p^i - 1 : 1)$ and $S^{'} = (\zeta_p^j : \zeta_p^j - 1 : 1)$. It is a basic result from number theory that $(\zeta_p^i - 1)/\pi$ is a unit in $\mathbb{Z}[\zeta_p]$ if $i \not\equiv 0 \mod p$. We will denote this unit by ϵ_i. If Q is a point in the fibre $\mathfrak{F}_p \times_{\operatorname{Spec} \mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\mathfrak{q})$, where $\mathfrak{q} \in \operatorname{Spec} \mathbb{Z}[\zeta_p]$, then $\overline{\mathfrak{q}} \subseteq m$. On the other hand since $\overline{X} - \overline{\zeta_p^i}, \overline{X} - \overline{\zeta_p^j} \in m$ we have $\zeta_p^i - \zeta_p^j = \zeta_p^i(1 - \zeta_p^{j-i}) = \zeta_p^i \epsilon_{j-i}\pi$ and so $(\overline{\pi}) \subseteq m$. Now if $\mathfrak{q}$ is different from (π) and so in particular coprime to (π) we have $\overline{1} \in m$ which gives us a contradiction again. It follows that the only possibility for Q to be in a special fibre is to be in the fibre of bad reduction $\mathfrak{F}_p \times_{\operatorname{Spec} \mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\pi)$. Now since S and $S^{'}$ are $\mathbb{Q}(\zeta_p)$-rational points $\mathcal{S}$ and $\mathcal{S}^{'}$ are reduced to single points P and $P^{'}$ in this fibre. A direct computation shows that

$$M = \left(\overline{X} - \overline{\zeta_p^i}, \overline{\pi}, \overline{a} - \overline{\epsilon_i}\right)$$

and

$$M^{'} = \left(\overline{X} - \overline{\zeta_p^j}, \overline{\pi}, \overline{a} - \overline{\epsilon_j}\right)$$

are the ideals corresponding to these points. If we take a look at the affine open set U_2, described in the previous section, we can easily verify that M and $M^{'}$ are indeed maximal ideals and that $\mathcal{S}$ and $\mathcal{S}'$ are reduced to these points in the fibre of bad reduction since

$$\overline{\pi}\,(\overline{a}-\overline{\epsilon_i}) = \overline{Y} - \overline{\zeta_p^i} + \overline{1}$$

and $\overline{\pi}\,(\overline{a}-\overline{\epsilon_j}) = \overline{Y} - \overline{\zeta_p^j} + \overline{1}$. Now if $P = P' = Q$ we have

$$\epsilon_i - \epsilon_j = \frac{\zeta_p^i - 1}{\pi} - \frac{\zeta_p^j - 1}{\pi} = \frac{\zeta_p^i - \zeta_p^j}{\pi} = \frac{\zeta_p^i(1-\zeta_p^{j-i})}{\pi} = \zeta_p^i \epsilon_{j-i}\,.$$

and so $\overline{\zeta_p^i \epsilon_{j-i}} \in m$. But since $\zeta_p^i \epsilon_{j-i} \in \mathbb{Z}[\zeta_p]^*$, this gives us a contradiction and we have completed the proof of (i).
Now let $S = (0 : \zeta_p^i - 1 : 1)$, so S is S_x after the transformation (5.2). Again $\mathcal{S} \cap \mathfrak{F}_p \times_{\operatorname{Spec}\mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\pi)$ is reduced to a single point P. Let M be the corresponding maximal ideal, so $M = (\overline{X}, \overline{\pi}, \overline{a} - \overline{\epsilon_i})$. The irreducible component L_x corresponds (in U_2) to the prime ideal $I = (\overline{\pi}, \overline{X})$. Obviously $I \subset M$ and so P is just in the component L_x in the fibre of bad reduction (remember that the component L does not lie in U_2). Since $\mathcal{S}$ is only reduced to P it only intersects L_x. Similar computations for S_y and S_z yield (ii). □

Lemma 6.3. *Let $\mathfrak{F}_p \to \operatorname{Spec}\mathbb{Z}[\zeta_p]$ be the arithmetic surface constructed above. There exists a canonical divisor $\mathcal{C} \in Z^1(\mathfrak{F}_p)_{\mathbb{Q}} = Z^1(\mathfrak{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ on $\mathfrak{F}_p$ of the form*

$$\mathcal{C} = (2g-2)\mathcal{S} + \mathcal{V}\,,$$

where $\mathcal{S}$ is a horizontal divisor coming from a cusp, $g = g(F_p)$ is the genus of F_p and $\mathcal{V}$ denotes a vertical divisor having support in the special fibre $\mathfrak{F}_p \times_{\operatorname{Spec}\mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\pi)$.

Proof: It follows from Proposition 4.1 that

$$(2g-2)S$$

is a canonical divisor in $Z^1(F_p)_{\mathbb{Q}}$, where S is any cusp. If we now set

$$\mathcal{C}_0 := (2g-2)\mathcal{S} + \mathcal{V}_0\,,$$

where $\mathcal{S}$ is the Zariski closure of S and $\mathcal{V}_0$ is a sum of divisors, having support in the closed fibres, so that $\mathcal{C}_0$ fulfills the adjunction formula, then $\mathcal{C}_0$ is a canonical divisor of $\mathfrak{F}_p$ (see Proposition 2.6). Note that similar arguments, as in the proof of Proposition 2.6, assure that $\mathcal{V}_0$ exists. For all primes $\mathfrak{q} \in \operatorname{Spec}\mathbb{Z}[\zeta_p]$ not dividing p - in fact these are the primes of good reduction - the special fibre $\mathfrak{F}_p \times_{\operatorname{Spec}\mathbb{Z}[\zeta_p]} \operatorname{Spec} k(\mathfrak{q})$ is smooth and so it consists of a single irreducible component. Since the self-intersection of this fibre is zero (see [La]: p.61: Proposition 3.5.) we can add any multiple of it to $\mathcal{C}_0$ and the resulting divisor still fulfills the adjunction formula. Using this fact we can transform $\mathcal{C}_0$ into a divisor $\mathcal{C} = (2g-2)\mathcal{S} + \mathcal{V}$, where $\mathcal{V}$ is a vertical divisor having support in the special fibre over π. Again, by Proposition 2.6, this is a canonical divisor. □

Now we are ready to compute the canonical divisor for the model $\mathfrak{F}_p$. In the previous lemma we saw that such a divisor can be constructed with a horizontal divisor $\mathcal{S}$ coming from a cusp and vertical divisors having support in the fibre of bad reduction. Now let S_x be a cusp,

$$\mathcal{V}_x = \lambda_x L_x + \lambda_y L_y + \lambda_z L_z \tag{6.1}$$

and

$$\mathcal{V}_\Sigma = \sum_{i=1}^{r} \left(\sum_{j=1}^{p} \lambda_{\alpha_{i,j}} L_{\alpha_{i,j}} + \lambda_{\alpha_i} L_{\alpha_i} \right) + \sum_{i=1}^{s} \lambda_{\beta_i} L_{\beta_i}\,, \tag{6.2}$$

where

$$\lambda_x = \left(\frac{2g-p}{p}\right), \tag{6.3}$$

$$\lambda_y = \lambda_z = \lambda_{\beta_i} = \lambda_{\alpha_{j,k}} = -\left(\frac{p-2}{p}\right) \text{ for all } i = 1, \dots, s \text{ and } j = 1, \dots, r\,, \tag{6.4}$$

$$\lambda_{\alpha_j} = -2\left(\frac{p-2}{p}\right) \text{ for all } j = 1, \dots, r\,. \tag{6.5}$$

Then we claim that the divisor $\mathcal{C}_x$ given by

$$\mathcal{C}_x = (2g-2)\mathcal{S}_x + \mathcal{V}_x + \mathcal{V}_\Sigma \tag{6.6}$$

is a canonical divisor. Notice that L is not included in $\mathcal{C}_x$, since it is modulo the full fiber just a linear combination of the other components.

Lemma 6.4. *The divisor $\mathcal{C}_x$ in* (6.6) *is indeed a canonical divisor.*

Proof: From Lemma 6.3 we know that there exists a canonical divisor of the form (6.6) with (6.1) and (6.2) for some coefficients λ. The only thing we need to do is to show that for these λ is no other choice possible than the one we made in (6.3), (6.4) and (6.5). So the whole idea of the proof is the repeating use of the adjunction formula (see [Li2], p.390: Theorem 1.37) combined with the fact that the genus of the components of the special fibre is zero (see [Mc], p.59: Theorem 3) to approve the choice we made. We start with the observation

$$2\lambda_{\alpha_{i,j}} = \lambda_{\alpha_i}\,. \tag{6.7}$$

Indeed, according to the adjunction formula $L^2_{\alpha_{i,j}} + \mathcal{C}_x \cdot L_{\alpha_{i,j}} = 2g(L_{\alpha_{i,j}}) - 2$ and $L^2_{\alpha_{i,j}} = -2$ (see previous section), we have

$$0 = L_{\alpha_{i,j}} \cdot \mathcal{C}_x = L_{\alpha_{i,j}} \cdot \left(\sum_{l=1}^{p} \lambda_{\alpha_{i,j}} L_{\alpha_{i,j}} + \lambda_{\alpha_i} L_{\alpha_i} \right) = \lambda_{\alpha_{i,j}}(-2) + \lambda_{\alpha_i}\,.$$

Now using (6.7) and the formula for L_{α_i}, we get

$$p - 2 = L_{\alpha_i} \cdot \mathcal{C}_x = \sum_{j=1}^{p} \lambda_{\alpha_{i,j}} + \lambda_{\alpha_i}(-p) = \frac{p}{2}\lambda_{\alpha_i} - p\lambda_{\alpha_i} = -\frac{p}{2}\lambda_{\alpha_i}\,.$$

Similar computations yield λ_y, λ_z and the λ_{β_i}. Finally, one observes that

$$p-2 = \mathcal{C}_x \cdot L_x = (2g-2)\mathcal{S}_x \cdot L_x + \lambda_x L_x^2 = (2g-2) + \lambda_x(-p)$$

and with this we finish our proof. $\square$

With a view to this lemma we see that the vertical part of two divisors coming from cusps that lie over different branch points, say $\mathcal{C}_x$ and $\mathcal{C}_y$, just differs in the parts $\mathcal{V}_x$ and $\mathcal{V}_y$.

7. The algebraic contributions to $\overline{\omega}_{\mathrm{Ar}}{}^2$

We now calculate certain intersection numbers, which will be used later to complete the computations of the coefficient a_p.

Lemma 7.1. *For $\mathcal{V}_\Sigma$ given in* (6.2) *we have*

$$\mathcal{V}_\Sigma \cdot \mathcal{V}_\Sigma = (p-3)(-p)\left(\frac{p-2}{p}\right)^2 .$$

Proof: In all the computations in this proof we have to remember the coefficients we calculated in Lemma 6.4. If we write $\mathcal{V}_\Sigma = \mathcal{V}_{\Sigma_\alpha} + \mathcal{V}_{\Sigma_\beta}$, where $\mathcal{V}_{\Sigma_\alpha}$ denotes the part coming from the L_α and $\mathcal{V}_{\Sigma_\beta}$ the part coming from the L_β, we have

$$\mathcal{V}_\Sigma \cdot \mathcal{V}_\Sigma = \mathcal{V}_{\Sigma_\alpha} \cdot \mathcal{V}_{\Sigma_\alpha} + \mathcal{V}_{\Sigma_\beta} \cdot \mathcal{V}_{\Sigma_\beta} ,$$

since each of the components of $\mathcal{V}_{\Sigma_\alpha}$ does not intersect any component of $\mathcal{V}_{\Sigma_\beta}$ and vice versa. From figure 1 we see that each L_{β_i} just intersects itself and that the number of self-intersection is $-p$. Since there are s lines L_{β_i}, we have

$$\mathcal{V}_{\Sigma_\beta} \cdot \mathcal{V}_{\Sigma_\beta} = s(-p)\left(\frac{p-2}{p}\right)^2 .$$

Now let $\mathcal{C}$ be a canonical divisor. According to the adjunction formula, we have $\mathcal{C} \cdot L_{\alpha_{i,j}} = 0$ and, since each $L_{\alpha_{i,j}}$ just intersects the $\mathcal{V}_{\Sigma_\alpha}$ part of $\mathcal{C}$, the equation $0 = \mathcal{C} \cdot L_{\alpha_{i,j}} = \mathcal{V}_{\Sigma_\alpha} \cdot L_{\alpha_{i,j}}$. This yields

$$\mathcal{V}_{\Sigma_\alpha} \cdot \mathcal{V}_{\Sigma_\alpha} = \mathcal{V}_{\Sigma_\alpha} \cdot \sum_{i=1}^{r} \lambda_{\alpha_i} L_{\alpha_i} = \sum_{i=1}^{r} \lambda_{\alpha_i}\left(\mathcal{V}_{\Sigma_\alpha} \cdot L_{\alpha_i}\right) ,$$

where each addend is

$$\begin{aligned}
\lambda_{\alpha_i}\left(\mathcal{V}_{\Sigma_\alpha} \cdot L_{\alpha_i}\right) &= \lambda_{\alpha_i}\left(\left(\sum_{i=1}^{p} \lambda_{\alpha_{i,j}} L_{\alpha_{i,j}} + \lambda_{\alpha_i} L_{\alpha_i}\right) \cdot L_{\alpha_i}\right) \\
&= \lambda_{\alpha_i}\left(\frac{p}{2}\lambda_{\alpha_i} + \lambda_{\alpha_i}(-p)\right) \\
&= -\frac{p}{2}\lambda_{\alpha_i}^2 = 2(-p)\left(\frac{p-2}{p}\right)^2 .
\end{aligned}$$

Since there are r lines L_{α_i}, we have

$$\mathcal{V}_\Sigma \cdot \mathcal{V}_\Sigma = (2r+s)(-p)\left(\frac{p-2}{p}\right)^2 = (p-3)(-p)\left(\frac{p-2}{p}\right)^2$$

□

Lemma 7.2. *Let $\mathcal{V}_x$ be a vertical divisors as in* (6.1) *which belongs to a cusp. Then*

$$\mathcal{V}_x \cdot \mathcal{V}_x = (-p)\left(\frac{2g-p}{p}\right)^2 + (-2p)\left(\frac{p-2}{p}\right)^2 .$$

Proof: The lines L_x,L_y and L_z only intersect themselves and each self-intersection number is $-p$. Now everything follows from the equations (6.3) and (6.4). □

Lemma 7.3. *Let*

$$\mathcal{D}_x = \mathcal{S}_x + \mathcal{G}_x \,, \tag{7.1}$$

where $\mathcal{G}_x = \frac{1}{p}L_x$. Then the divisor $\mathcal{D}_x$ is associated with $\left(\beta^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}[\zeta_p]}}(1)\right)^{\otimes\frac{1}{p^2}}$, or in other words $\mathcal{O}(D_x)^{\otimes p^2} \cong \beta^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}[\zeta_p]}}(1)$; here β is the morphism from Theorem 5.3.

Proof: Let S_x be a cusp and $Q \in \mathbb{P}^1_{\mathbb{Q}(\zeta_p)}$ the corresponding branch point. Since $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{Q}(\zeta_p)}) \cong \mathbb{Z}$ and $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_p)}}(1)$ is a generator of $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{Q}(\zeta_p)})$ any divisor of degree 1 is associated with $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_p)}}(1)$. We choose Q to be this associated divisor. Now

$$\beta^*Q = \sum_{i=1}^{p} pS_i \,,$$

where S_i runs through the cusps lying above Q. It follows from Proposition 4.1 (a) that $\beta^*Q = p^2S_x$ in $\mathrm{Cl}(F_p)_{\mathbb{Q}}$ (remember that S_x is one of the cusps) and so p^2S_x is associated with $\beta^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_p)}}(1)$. Since $\beta^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}[\zeta_p]}}(1)|_{F_p} \cong \beta^*\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}(\zeta_p)}}(1)$ it is clear with Lemma 1.6 that we can choose $\mathcal{D}_x = \mathcal{S}_x + \mathcal{G}_x$ where $\mathcal{G}_x$ is a vertical divisor having support in the special fibre $\mathfrak{F}_p \times_{\mathrm{Spec}\,\mathbb{Z}[\zeta_p]} \mathrm{Spec}\,k(\pi)$. Now let I be the index set from (5.3). Since each component of the special fibre which is different to L is mapped to a single point by β, we have

$$(p^2\mathcal{D}_x) \cdot L_i = 0 \ (\forall i \in I) \tag{7.2}$$

(see [Li2], p. 398: Theorem 2.12 (a)). On the other hand we have

$$p^2 = p^2\mathcal{D}_x \cdot \mathfrak{F}_p \times_{\mathrm{Spec}\,\mathbb{Z}[\zeta_p]} \mathrm{Spec}\,k(\pi) = p^2\mathcal{D}_x \cdot pL \tag{7.3}$$

(see [Li2], p. 388: Remark 1.31.). Solving (7.2) and (7.3) we get $\mathcal{G}_x = \frac{1}{p}L_x$. □

Proposition 7.4. *Let $\mathcal{C}_x = (2g-2)(\mathcal{S}_x + \mathcal{F}_x)$ be a canonical divisors and $\mathcal{D}_x = \mathcal{S}_x + \mathcal{G}_x$ a divisors as in* (7.1)*, where x indicates that this divisor belongs to a cusp S_x. Then*

$$\mathcal{F}_x \cdot \mathcal{F}_x = -\frac{p^3 - 7p^2 + 15p - 8}{p^2(p-3)^2} \,,$$

$$\mathcal{G}_x \cdot \mathcal{G}_x = -(\mathcal{S}_x \cdot \mathcal{G}_x) = -\frac{1}{p} \,.$$

Proof: We have $\mathcal{F}_x^2 = \frac{1}{(2g-2)^2}\left(\mathcal{V}_x^2 + \mathcal{V}_\Sigma^2\right)$. Now Lemma 7.1 and Lemma 7.2 together with $g = \frac{(p-1)(p-2)}{2}$ yield (after simplifying equations) our first claim.
With equation (7.2) we get $\mathcal{S}_x \cdot \mathcal{G}_x = -(\mathcal{G}_x \cdot \mathcal{G}_x)$. Since $\mathcal{G}_x = \frac{1}{p} L_x$ the second claim follows.
□

Now, we successfully prepared all the ingredients to actually calculate some intersection numbers for the Fermat curves.

8. Proof of the main result

Theorem 8.1. *Let $\mathfrak{F}_p$ be the regular model of the fermat curve F_p over* $\operatorname{Spec} \mathbb{Z}[\zeta_p]$ *which was constructed in section 5. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric satisfies*

$$\overline{\omega}^2_{\mathfrak{F}_p, Ar} \leq (2g-2)\left(\log|\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}|^2 + [\mathbb{Q}(\zeta_p):\mathbb{Q}]\,(\kappa_1 \log p + \kappa_2) + \frac{p^2-4p+2}{p(p-3)}\log p\right),$$

where $\kappa_1, \kappa_2 \in \mathbb{R}_+^*$ are positive constants independent of p.

Proof: In Theorem 5.3 we saw that $\beta : F_p \to \mathbb{P}^1$ extends to a morphism as in Assumption 3.7 who fulfills the requirements of Theorem 3.8 (cf. Convention 5.4). Since $\beta^* Q = \sum_{i=1}^p pS_i$ we have $b_j = b_{\max} = p$. The morphism β is of degree p^2. Because $\mathcal{G}_i^2 = \mathcal{G}_j^2$ ($\mathcal{F}_i^2 = \mathcal{F}_j^2$ resp.) for $1 \leq i,j \leq p$ it follows with Proposition 7.4 that in our case the formula (3.6) of Theorem 3.8 becomes

$$\begin{aligned}\sum_{\mathfrak{p} \text{ bad}} a_\mathfrak{p} \log \operatorname{Nm}(\mathfrak{p}) = a_\mathfrak{p} \log \operatorname{Nm}(\mathfrak{p}) &= -2g\mathcal{O}(\mathcal{G}_j)^2 + (2g-2)\mathcal{O}(\mathcal{F}_j)^2 \\ &= -2g{\mathcal{G}_j}^2 \log p + (2g-2){\mathcal{F}_j}^2 \log p \\ &= \frac{2g}{p}\log p - (2g-2)\frac{p^3-7p^2+15p-8}{p^2(p-3)^2}\log p \\ &= \frac{p^2-4p+2}{p(p-3)}\log p.\end{aligned}$$

□

Remark 8.2. In Section 5 we have seen that we get a minimal regular model $\mathfrak{F}_p^{min}$ of F_p if we blow down the component L of the special fibre. Let $\pi : \mathfrak{F}_p \to \mathfrak{F}_p^{min}$ denote this blow-down. Then there exists a vertical divisor $\mathcal{W}$ on $\mathfrak{F}_p$ (with support in the special fibre) such that $\pi^* \omega_{\mathfrak{F}_p^{min}} = \omega_{\mathfrak{F}_p} \otimes \mathcal{O}(\mathcal{W})$. We have

$$\overline{\omega}^2_{\mathfrak{F}_p^{min}, Ar} = \pi^* \overline{\omega}^2_{\mathfrak{F}_p^{min}, Ar} = \overline{\omega}^2_{\mathfrak{F}_p, Ar} + 2\omega_{\mathfrak{F}_p} \cdot \mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W})^2.$$

Lemma 8.3. *With the notation from above we have*

$$2\omega_{\mathfrak{F}_p} \cdot \mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W})^2 = (2p^2 - 10p + 13)\log p.$$

Proof: We start by computing the canonical divisor $\mathcal{K}_x^{min}$ of $\mathfrak{F}_p^{min}$, so the divisor with $\mathcal{O}(\mathcal{K}_x^{min}) \cong \omega_{\mathfrak{F}_p^{min}}$. Let $\tilde{L}_u := \pi L_u$, where $u \in I$ and I is the index set (5.3). In order to compute intersections of the $\tilde{L}_u$ we need to find their pullback and then compute everything on $\mathfrak{F}_p$. We have $\pi^*\tilde{L}_u = L_u$ for $u = \alpha_{i,j}$ and

$$\pi^*\tilde{L}_u = L_u + L$$

for all other u. Indeed, let for instance $u = x$. Then we have $\pi^*\tilde{L}_x = L_x + \mu_x L$, where μ_x is a rational number. It follows that $0 = L \cdot \pi^*\tilde{L}_x = 1 - \mu_x$ (see [Li2], p.398: Theorem 2.12. (a)).
The canonical divisor on $\mathfrak{F}_p^{min}$ is given by

$$\mathcal{K}_x^{min} = (2g-2)(\mathcal{S}_x + \frac{1}{p}\tilde{L}_x)\,.$$

To verify this we just need to proof that $\mathcal{K}_x^{min}$ satisfies the adjunction formula and restricts to the canonical divisor K_x of the generic fibre F_p (see Proposition 2.6). The second property is obviously fulfilled. In order to verify the adjunction formula one has to check that it is valid for each irreducible component of the special fibre. We will illustrate this for the component $\tilde{L}_x$ and leave the rest to the reader since the computations are very similar. We have

$$\begin{aligned}
\mathcal{K}_x^{min} \cdot \tilde{L}_x &= (2g-2)(\mathcal{S}_x \cdot \tilde{L}_x + \frac{1}{p}\tilde{L}_x^2) \\
&= (2g-2)(1 + \frac{1}{p}(L_x + L)^2) \\
&= p(p-3)(1 - \frac{1}{p}(p-1)) = (p-3)
\end{aligned}$$

(see [Li2], p.398: Theorem 2.12. (c) for the second equality). On the other hand is

$$2p_a(\tilde{L}_x) - 2 - \tilde{L}_x^2 = -2 - (L_x + L)^2 = (p-3)$$

and so the formula is valid for $\tilde{L}_x$.
The pullback of the canonical divisor is now

$$\pi^*\mathcal{K}_x^{min} = (2g-2)(\mathcal{S}_x + \frac{1}{p}L_x + \frac{1}{p}L)$$

and an easy computation shows that

$$\mathcal{W} = -\lambda_y L_y - \lambda_z L_z - \frac{(2-p)}{p}L_x - \mathcal{V}_\Sigma + \frac{2g-2}{p}L$$

fulfills $\pi^*\mathcal{K}_x^{min} = \mathcal{K}_x + \mathcal{W}$. It follows that we have to compute $(2\mathcal{K}_x \cdot \mathcal{W} + \mathcal{W}^2)\log p$ in order to get $2\omega_{\mathfrak{F}_p} \cdot \mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W})^2$. Since we have $\mathcal{W} \cdot (2\mathcal{K}_x + \mathcal{W}) = \mathcal{W} \cdot (\mathcal{K}_x + \pi^*\mathcal{K}_x^{min})$ we may compute $\mathcal{W}\cdot\mathcal{K}_x$ and $\mathcal{W}\cdot\pi^*\mathcal{K}_x^{min}$. Using the adjunction formula and linearity we get

$$\begin{aligned}\mathcal{W}\cdot\mathcal{K}_x &= (p-2)\left(-\lambda_y-\lambda_z-\left(\frac{2-p}{p}\right)\right)-\mathcal{V}_\Sigma\cdot\mathcal{K}_x-\left(\frac{2g-2}{p}\right)\\ &= 3\left(\frac{(p-2)^2}{p}\right)-\mathcal{V}_\Sigma^2-\left(\frac{p(p-3)}{p}\right)\\ &= (p-2)^2-(p-3)\,.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\mathcal{W}\cdot\pi^*\mathcal{K}_x^{min} &= \mathcal{W}\cdot(p(p-3)\mathcal{S}_x+(p-3)L_x+(p-3)L)\\ &= (p-2)(p-3)-(p-2)(p-3)+(p-3)^2+(p-3)\mathcal{W}\cdot L\\ &= (p-3)^2+(p-3)\left(-\lambda_y-\lambda_z-\frac{2-p}{p}+\frac{p-2}{p}(p-3)-(p-3)\right)\\ &= (p-3)^2+(p-3)(p-2)-(p-3)^2=(p-2)(p-3)\end{aligned}$$

and so $2\omega_{\mathfrak{F}_p}\cdot\mathcal{O}(\mathcal{W})+\mathcal{O}(\mathcal{W})^2=(2p^2-10p+13)\log p$. □

Theorem 8.4. *Let $\mathfrak{F}_p^{min}$ be the minimal regular model of the fermat curve F_p over $\operatorname{Spec}\mathbb{Z}[\zeta_p]$ from section 5. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric satisfies*

$$\overline{\omega}^2_{\mathfrak{F}_p^{min},Ar}\leq(2g-2)\left(\log|\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}|^2+[\mathbb{Q}(\zeta_p):\mathbb{Q}]\left(\kappa_1\log p+\kappa_2\right)+\frac{3p^2-14p+15}{p(p-3)}\log p\right),$$

where $\kappa_1,\kappa_2\in\mathbb{R}_+^*$ are positive constants independent of p.

Proof: Follows directly from Theorem 8.1 and Lemma 8.3. □

Corollary 8.5. *With the notation from the previous theorem we have:*

$$\overline{\omega}^2_{\mathfrak{F}_p^{min},Ar}\leq(2g-2)\left((p-1)\left(\kappa_1\log p+\kappa_2\right)+\frac{2p^2-p-5}{p}\log p\right)$$

Proof: It is a well known fact that $\Delta_{\mathbb{Q}(\zeta_p)|\mathbb{Q}}=(-1)^{\frac{p-1}{2}}p^{p-2}$ and $[\mathbb{Q}(\zeta_p):\mathbb{Q}]=p-1$ and so Theorem 8.4 yields

$$\begin{aligned}\overline{\omega}^2_{\mathfrak{F}_p,\mathrm{Ar}} &\leq (2g-2)\left(\log p^{2p-4}+(p-1)\left(\kappa_1\log p+\kappa_2\right)+\frac{3p^2-14p+15}{p(p-3)}\log p\right)\\ &= (2g-2)\left((p-1)\left(\kappa_1\log p+\kappa_2\right)+\frac{2p^2-p-5}{p}\log p\right)\end{aligned}$$

□

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