Prof. Dr. Vicente Cortés
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Exercise sheet 6

Submit your solutions in the exercise group on 2011-May-23!

Exercise 21: The Dolbeault operator. (*10 points*)

Let M be a complex manifold, let $k \in \mathbb{N}_0$. Prove that the definitions of the operators

$$\overline{\partial}\colon \Gamma(\textstyle\bigwedge^k T^*M^{\mathbb{C}}) \to \Gamma(\textstyle\bigwedge^{k+1} T^*M^{\mathbb{C}}) \quad \text{and}$$
$$\partial\colon \Gamma(\textstyle\bigwedge^k T^*M^{\mathbb{C}}) \to \Gamma(\textstyle\bigwedge^{k+1} T^*M^{\mathbb{C}})$$

given in the lecture do not depend on the choice of local coordinates. Prove that the principal symbols $\sigma_1(\overline{\partial})$ and $\sigma_1(\partial)$ are given by $\sigma_1(\overline{\partial})_\xi(\eta) = \xi^{0,1} \wedge \eta$ and $\sigma_1(\partial)_\xi(\eta) = \xi^{1,0} \wedge \eta$.

Exercise 22: Differential operators and the principal symbol. (*10 points*)

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let E, F be $\mathbb{K}$-vector bundles over a manifold M, let $m \in \mathbb{N}_0$, let $L\colon \Gamma(E) \to \Gamma(F)$ be a ($\mathbb{K}$-linear) differential operator of order m.

a. The definition of the notion *differential operator* says that every $x \in M$ has a neighborhood U on which there exist local coordinates χ of M and vector bundle trivializations ϕ of $E|_U$ and ψ of $F|_U$ such that with respect to $\chi, \phi, \psi,\ L$ has the form described in the lecture:

$$L(s)_i = \sum_{j=1}^{n} \sum_{|\alpha| \le m} a^j_{\alpha i} \partial^\alpha s_j \quad . \tag{1}$$

Prove that this implies that for every $x \in M$ and for *every* triple (χ, ϕ, ψ) of local coordinates and local vector bundle trivializations on an open subset U of M, there exists a neighborhood $\tilde{U} \subseteq U$ of x such that with respect to $\chi, \phi, \psi,\ L$ has the form (1) on $\tilde{U}$.

b. Prove that the principal symbol of L does not depend on the choice of local coordinates and local trivializations used in its definition.

(to be continued on the next page ⟶)

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Exercise 23: Covariant derivatives and the Levi-Civita connection. (*10 points*)

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let E be a $\mathbb{K}$-vector bundle over a manifold M. A *covariant derivative* (synonymously: *linear connection*) on E is a $\mathbb{K}$-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that $\nabla(fs) = f\nabla s + \mathrm{d}f \otimes s$ holds for all $s \in \Gamma(E)$ and $f \in C^\infty(M, \mathbb{K})$.

a. Let ∇ be a covariant derivative on E. Prove that there exists a unique $R^\nabla \in \Gamma(\bigwedge^2 T^*M \otimes \mathrm{End}(E))$ such that $R^\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$ holds for all $X, Y \in \Gamma(TM)$ and $s \in \Gamma(E)$. (R^∇ is called the *curvature (tensor) of* ∇.)

b. Let g be a Riemannian metric on M. Prove that there exists a unique covariant derivative ∇ on TM such that the *Koszul formula*

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X)$$

holds for all $X, Y, Z \in \Gamma(TM)$. This ∇ is called the *Levi-Civita connection* of g.

c. Prove that the Levi-Civita connection satisfies

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$
$$[X,Y] = \nabla_X Y - \nabla_Y X$$

for all $X, Y, Z \in \Gamma(TM)$, and that it is the only linear connection on TM with this property.

Exercise 24: Divergence and Laplacian of a Riemannian metric. (*10 points*)

Let g be a Riemannian metric on a manifold M, let $f \in C^\infty(M, \mathbb{R})$, let $X \in \Gamma(TM)$. The *gradient* $\mathrm{grad} f \in \Gamma(TM)$ of f (with respect to g) is the vector field on M such that $g(\mathrm{grad} f, V) = \mathrm{d}f(V)$ holds for all $V \in TM$. The *divergence* $\mathrm{div} X \in C^\infty(M, \mathbb{R})$ of X (with respect to g) is the function on M such that for every $x \in M$, $\mathrm{div} X\,(x)$ is the trace of the endomorphism of T_xM given by $v \mapsto \nabla_v X$, where ∇ is the Levi-Civita connection of g. The *Laplacian* of f (with respect to g) is the function $\Delta f := \mathrm{div}\,\mathrm{grad} f \in C^\infty(M, \mathbb{R})$.

Compute the principal symbols of the differential operators $C^\infty(M, \mathbb{R}) \ni f \mapsto \mathrm{grad} f \in \Gamma(TM)$ and $\Gamma(TM) \ni X \mapsto \mathrm{div} X \to C^\infty(M, \mathbb{R})$ and $C^\infty(M, \mathbb{R}) \ni f \mapsto \Delta f \in C^\infty(M, \mathbb{R})$.

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