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Exercise sheet 5

Submit your solutions in the exercise group on 2011-May-16!

The rules for the exercises have changed: From now on, you are allowed to form groups of up to three people. Each group submits a joint solution.

Exercise 17: The Lie bracket. (*10 points*)

Let X, Y be vector fields on a smooth manifold M. Prove that the Lie bracket $[X, Y]$ is a vector field. Prove that with respect to local coordinates $(x_1, \ldots, x_n)$ on M, the Lie bracket of $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$ is

$$[X, Y] = \sum_{i,j=1}^n \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \quad .$$

Exercise 18: The Nijenhuis tensor. (*10 points*)

Let M be a smooth manifold.

a. Let $F \colon \Gamma(TM) \to \Gamma(TM)$ be $\mathbb{R}$-linear. Prove that the following statements are equivalent:

(1) $F(X)_x = F(\tilde{X})_x$ holds for all $x \in M$ and all vector fields $X, \tilde{X}$ on M with $X_x = \tilde{X}_x$.

(2) F is $C^\infty(M, \mathbb{R})$-linear.

b. Let J be an almost complex structure on M. Prove that the map $N_J \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ which defines the Nijenhuis tensor is $C^\infty(M, \mathbb{R})$-bilinear and thus induces indeed a section in $\bigwedge^2 T^*M \otimes TM$.

(to be continued on the next page ⟶)

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Exercise 19: Quaternions and octonions. (*10 points*)

Recall that a (unital) $\mathbb{R}$-*algebra* is an $\mathbb{R}$-vector space A together with a multiplication $\cdot\colon A \times A \to A$ and an element $1 \in A$ such that $(x+y)\cdot z = x\cdot z + y\cdot z$ and $z\cdot(x+y) = z\cdot x + z\cdot y$ and $(rx)\cdot(sy) = (rs)(x\cdot y)$ and $1\cdot x = x = x\cdot 1$ hold for all $x, y, z \in A$ and $r, s \in \mathbb{R}$. An $\mathbb{R}$-algebra is *associative* iff the *associative law* $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ holds for all $x, y, z \in A$. An $\mathbb{R}$-algebra is *alternative* iff the associative law holds whenever two of the three elements $x, y, z \in A$ are equal. For brevity, we omit every "$\cdot$" from now on. For $\mathbb{R}$-algebras A, B, an $\mathbb{R}$-linear map $f\colon A \to B$ is an $\mathbb{R}$-*algebra homomorphism* iff $f(1_A) = 1_B$ and $f(xy) = f(x)f(y)$ hold for all $x, y \in A$.

An *objective algebra* is an $\mathbb{R}$-algebra A together with a sub vector space $\mathrm{Im}(A)$ of A such that $A = \mathbb{R}1 \oplus \mathrm{Im}(A)$. For $x \in A$, the unique $y =: \mathrm{Re}\, x \in \mathbb{R}1$ and $z =: \mathrm{Im}\, x \in \mathrm{Im}(A)$ with $x = y + z$ are called the *real part* and *imaginary part* of x, respectively. The *conjugation* $\overline{}\colon A \to A$ is defined by $x \mapsto \overline{x} := \mathrm{Re}\, x - \mathrm{Im}\, x$. For objective algebras A, B, a map $f\colon A \to B$ is an *objective algebra homomorphism* iff it is an $\mathbb{R}$-algebra homomorphism such that $f(\overline{x}) = \overline{f(x)}$ holds for all $x \in A$; it is an *objective algebra isomorphism* iff it is also bijective.

Let A be an objective algebra. We define an objective algebra $\mathscr{C}(A)$ whose underlying $\mathbb{R}$-vector space is $A \oplus A$ by $(a,b)(c,d) := (ac - \overline{d}b, da + b\overline{c})$ and $1_{\mathscr{C}(A)} := (1_A, 0)$ and $\mathrm{Im}(\mathscr{C}(A)) := \mathrm{Im}(A) \oplus A$.

a. Check that this defines indeed an objective algebra.

b. Check that the map $\iota\colon A \to \mathscr{C}(A)$ given by $a \mapsto (a, 0)$ is an objective algebra homomorphism.

c. Check that $f\colon \mathscr{C}(\mathbb{R}) \to \mathbb{C}$ given by $f(a,b) := a + b\mathrm{i}$ is an objective algebra isomorphism. ($\mathrm{Im}(\mathbb{R})$ is chosen in the only possible way: $\mathrm{Im}(\mathbb{R}) = \{0\}$.)

d. Prove that the $\mathbb{R}$-algebra $\mathbb{H} := \mathscr{C}(\mathbb{C})$ is associative, but (its multiplication is) not commutative. Prove that the elements $\mathrm{i} := (\mathrm{i}, 0)$ and $\mathrm{j} := (0, 1)$ and $\mathrm{k} := (0, \mathrm{i})$ of $\mathrm{Im}(\mathbb{H})$ satisfy $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$. Prove that $\overline{xy} = \overline{y}\,\overline{x}$ and $\mathrm{Re}(xy) = \mathrm{Re}(yx)$ hold for all $x, y \in \mathbb{H}$. (The elements of $\mathbb{H}$ are called *quaternions*. Depending on your linear algebra course, you might already know another definition of $\mathbb{H}$. Whatever definition that is, our equations for $\mathrm{i}, \mathrm{j}, \mathrm{k}$ will yield a short proof that your alternatively defined algebra $\mathbb{H}$ is isomorphic to ours here.)

e. Prove that the $\mathbb{R}$-algebra $\mathbb{O} := \mathscr{C}(\mathbb{H})$ is alternative, but neither associative nor commutative. (The elements of $\mathbb{O}$ are called *octonions*.)

Remark. The "objective" terminology (distinguishing between real and imaginary) is not standard but has been invented here to give short definitions.

Exercise 20: An almost complex structure on S^6. (*10 points*)

In $\mathbb{O} = \mathbb{H}^2 = \mathbb{C}^4 = \mathbb{R}^8$, we identify $\mathrm{Im}(\mathbb{O}) = \{0\} \times \mathbb{R}^7$ with $\mathbb{R}^7$. Note that TS^6 is the sub vector bundle $\{(x,v) \in S^6 \times \mathbb{R}^7 \mid \langle x, v\rangle = 0\}$ of the trivial bundle of rank 7 over S^6; here $\langle .,. \rangle$ denotes the standard scalar product on $\mathbb{R}^7$.

a. We define $J \in \Gamma(\mathrm{End}(TS^6))$ by $J_x(v) := xv$ (multiplication in $\mathbb{O}$) for all $x \in S^6$ and $v \in T_xS^6$. Prove that J is well-defined and an almost complex structure on S^6.

(*Hint.* Check that $\langle x, y\rangle = \mathrm{Re}(\overline{x}y)$ and $\langle x, x\rangle = \overline{x}x$ hold for all $x, y \in \mathbb{H}$ with respect to the standard scalar product on $\mathbb{H} = \mathbb{R}^4$. Then check the same with $\mathbb{O} = \mathbb{R}^8$ instead of $\mathbb{H}$.)

b. Prove that S^6 does not admit a holomorphic atlas whose complex structure is J.

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