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Exercise sheet 4

Submit your solutions in the exercise group on 2011-May-09!

Exercise 13: Submanifolds are manifolds. (*10 points*)

Let N be a k-dimensional complex submanifold of a complex manifold M. Prove that the local charts of M which are adapted to N form a holomorphic atlas which turns N (considered as a topological space with respect to the subspace topology of M) into a k-dimensional complex manifold.

Exercise 14: Lemma for the Plücker embedding. (*10 points*)

Let $k \in \{0, \dots, n\}$, let $\Lambda \in \bigwedge^k \mathbb{C}^n$. Prove that $\Lambda \in \bigwedge^k \Sigma_\Lambda$.

(*Hint.* Choose a complementary subspace of Σ_Λ in $\mathbb{C}^n$, then decompose $\bigwedge^k \mathbb{C}^n$.)

Exercise 15: Pullback vector bundles. (*10 points*)

Let M, N be smooth manifolds of dimensions m and n, respectively. Let $f\colon M \to N$ be a smooth map. Let $\pi\colon E \to N$ be a smooth $\mathbb{K}$-vector bundle of rank k. We define

$$f^*E := \{(x, e) \in M \times E \mid f(x) = \pi(e)\} \ ,$$

and we define $\hat{\pi}\colon f^*E \to M$ by $\hat{\pi}(x, e) = x$.

- **a.** Prove that f^*E is a smooth submanifold of $M \times E$. Determine the dimension of f^*E.
- **b.** Prove that $\hat{\pi}$ is smooth and surjective. For every $x \in M$, consider the fiber $(f^*E)_x := \hat{\pi}^{-1}(x)$ and prove that the map $\zeta_x\colon (f^*E)_x \to E_{f(x)}$ given by $(x, e) \mapsto e$ is bijective.
- **c.** On each fiber $(f^*E)_x$, we define a $\mathbb{K}$-vector space structure by declaring ζ_x to be a $\mathbb{K}$-vector space isomorphism. Prove that this turns $\hat{\pi}\colon f^*E \to M$ into a $\mathbb{K}$-vector bundle of rank k over M.

This vector bundle is called the *f-pullback of $\pi\colon E \to N$*. The maps ζ_x are usually regarded as identifications; i.e., one writes $(f^*E)_x = E_{f(x)}$. We will do so in Exercise 16, for example.

Exercise 16: Direct sums of vector bundles. (*10 points*)

Let M be a smooth manifold, let $\pi_E\colon E \to M$ and $\pi_F\colon F \to M$ be smooth $\mathbb{K}$-vector bundles. Prove that there exists a unique smooth $\mathbb{K}$-vector bundle $\pi\colon E \oplus F \to M$ with the following properties:

(1) For all $x \in M$, the $\mathbb{K}$-vector space $(E \oplus F)_x$ is equal to $E_x \oplus F_x$.

(2) When $\phi\colon \pi_E^{-1}(U) \to U \times \mathbb{K}^k$ is a local trivialization of E and $\psi\colon \pi_F^{-1}(U) \to U \times \mathbb{K}^l$ is a local trivialization of F, let $\varphi\colon \pi^{-1}(U) \to U \times (\mathbb{K}^k \oplus \mathbb{K}^l)$ be the map whose restriction to each fiber $(E \oplus F)_x$ with $x \in U$ is given by $E_x \oplus F_x \ni v \oplus w \mapsto (x, \mathrm{pr}(\phi(v)) \oplus \mathrm{pr}(\psi(w)))$; here $\mathrm{pr}\colon U \times \mathbb{K}^m \to \mathbb{K}^m$ denotes projection to the second component. Then φ is a local trivialization of $E \oplus F$.

Determine the rank of $E \oplus F$. Prove that if $f\colon N \to M$ is a smooth map, then $f^*(E \oplus F) = f^*E \oplus f^*F$.