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Exercise sheet 2

Submit your solutions in the exercise group on 2011-Apr-18!

Exercise 5: Examples of (pre)sheaves. (*4+4+4+4 points*)

Prove that each of the following data sets $(\mathscr{F}_i, r^i)$ defines a presheaf of $\mathbb{R}$-vector spaces on M. Decide whether it satisfies the separation axiom, whether it satisfies the gluing axiom, and whether it is a sheaf (the answers might depend on M and/or n).

a. Let M be a complex manifold. For each open set $U \subseteq M$, let $\mathscr{F}_0(U)$ be the set of holomorphic functions $U \to \mathbb{C}$. For open sets $U \subseteq V \subseteq M$, let $r^0_{UV} \colon \mathscr{F}_0(V) \to \mathscr{F}_0(U)$ be the usual restriction of functions.

b. Let $n \geq 1$, let $M = \mathbb{R}^n$. For each open set $U \subseteq M$, let $\mathscr{F}_1(U)$ be the set of continuous functions $f \colon U \to \mathbb{R}$ with $\int_U |f(x)| \mathrm{d}x < \infty$. For open sets $U \subseteq V \subseteq M$, let $r^1_{UV} \colon \mathscr{F}_1(V) \to \mathscr{F}_1(U)$ be the usual restriction of functions.

c. Let $x \neq y$, let $M = \{x, y\}$ be equipped with the discrete topology, let $n \geq 2$. Let $\mathscr{F}_2(\varnothing) = \{0\}$ and $\mathscr{F}_2(\{x\}) = \mathscr{F}_2(\{y\}) = \mathbb{R}$ and $\mathscr{F}_2(M) = \mathbb{R}^n$. For $S \subseteq M$, let $r^2_{\varnothing S}$ be the unique map $\mathscr{F}_2(S) \to \{0\}$, and let r^2_{SS} be the identity on $\mathscr{F}_2(S)$. Let $r^2_{\{x\}M} \colon \mathbb{R}^n \to \mathbb{R}$ be the projection to the first coordinate, let $r^2_{\{y\}M} \colon \mathbb{R}^n \to \mathbb{R}$ be the projection to the second coordinate.

d. Let M be a topological space, let $x \in M$, let $n \geq 1$. For each open set $U \subseteq M$, let $\mathscr{F}_3(U)$ be $\mathbb{R}^n$ if $x \in U$, and let it be $\{0\}$ otherwise. For open sets $U \subseteq V \subseteq M$, let $r^3_{UV} \colon \mathscr{F}_3(V) \to \mathscr{F}_3(U)$ be $\mathrm{id}_{\mathbb{R}^n}$ if $x \in U$, and let it be 0 otherwise.

Exercise 6: Kernels, images, cokernels. (*8 points*)

Let $\mathscr{F}, \mathscr{G}$ be presheaves of abelian groups on a topological space M, let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be a presheaf morphism. Prove that the data sets $\ker(\varphi)$ and $\mathrm{im}^{\mathrm{pre}}(\varphi)$ and $\mathrm{coker}^{\mathrm{pre}}(\varphi)$ defined in the lecture are indeed presheaves of abelian groups over M, and that $\ker(\varphi)$ is a sheaf if $\mathscr{F}$ and $\mathscr{G}$ are sheaves.

Exercise 7: Sheafification yields a sheaf. (*8 points*)

Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space M. Prove that the definition of $\mathscr{F}^+$ given in the lecture yields indeed a well-defined sheaf of abelian groups on M. Prove that $\theta \colon \mathscr{F} \to \mathscr{F}^+$ is indeed a well-defined presheaf morphism.

Exercise 8: The sheafification of a sheaf. (*2+6 points*)

Let M be a topological space.

a. Let $\mathscr{F}, \mathscr{G}$ be presheaves of abelian groups on M, let $\varphi \colon \mathscr{F} \to \mathscr{G}$ be a presheaf morphism such that for every open subset U of M, φ_U is bijective. Prove that φ is a presheaf isomorphism.

b. Let $\mathscr{F}$ be a sheaf of abelian groups on M. Prove that the canonical morphism $\theta \colon \mathscr{F} \to \mathscr{F}^+$ is a sheaf isomorphism.

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