Half-flat Structures and Special Holonomy

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Abstract. It was proven by Hitchin that any solution of his evolution equations for a half-flat SU(3)-structure on a compact six-manifold M defines an extension of M to a seven-manifold with holonomy in G_2. We give a new proof, which does not require the compactness of M. More generally, we prove that the evolution of any half-flat G-structure on a six-manifold M defines an extension of M to a Ricci-flat seven-manifold N, for any real form G of $\mathrm{SL}(3,\mathbb{C})$. If G is noncompact, then the holonomy group of N is a subgroup of the noncompact form G_2^* of $\mathrm{G}_2^{\mathbb{C}}$. Similar results are obtained for the extension of nearly half-flat structures by nearly parallel G_2- or G_2^*-structures, as well as for the extension of cocalibrated G_2- and G_2^*-structures by parallel Spin(7)- and $\mathrm{Spin}_0(3,4)$-structures, respectively. As an application, we obtain that any six-dimensional homogeneous manifold with an invariant half-flat structure admits a canonical extension to a seven-manifold with a parallel G_2- or G_2^*-structure. For the group $H_3 \times H_3$, where H_3 is the three-dimensional Heisenberg group, we describe all left-invariant half-flat structures and develop a method to explicitly determine the resulting parallel G_2- or G_2^*-structure without integrating. In particular, we construct three eight-parameter families of metrics with holonomy equal to G_2 and G_2^*. Moreover, we obtain a strong rigidity result for the metrics induced by a half-flat structure (ω,ρ) on $H_3 \times H_3$ satisfying $\omega(\mathfrak{z},\mathfrak{z}) = 0$ where $\mathfrak{z}$ denotes the centre. Finally, we describe the special geometry of the space of stable three-forms satisfying a reality condition. Considering all possible reality conditions, we find four different special Kähler manifolds and one special para-Kähler manifold.

Contents

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Introduction

Following Hitchin [**H1**], a k-form φ on a differentiable manifold M is called *stable* if the orbit of $\varphi(p)$ under $\mathrm{GL}(T_pM)$ is open in $\Lambda^kT_p^*M$ for all $p\in M$. In this paper we are mainly concerned with six-dimensional manifolds M endowed with a stable two-form ω and a stable three-form ρ. A stable three-form defines an endomorphism field J_ρ on M such that $J_\rho^2=\varepsilon\mathrm{id}$, see (1.6). We will assume the following algebraic compatibility equations between ω and ρ:

$$\omega\wedge\rho=0,\quad J_\rho^*\rho\wedge\rho=\frac{2}{3}\omega^3.$$

The pair (ω,ρ) defines an $\mathrm{SU}(p,q)$-structure if $\varepsilon=-1$ and an $\mathrm{SL}(3,\mathbb{R})$-structure if $\varepsilon=+1$. In the former case, the pseudo-Riemannian metric $\omega(J_\rho\cdot,\cdot)$ has signature $(2p,2q)$. In the latter case it has signature $(3,3)$. The structure is called *half-flat* if the pair (ω,ρ) satisfies the following exterior differential system:

$$d\omega^2=0,\quad d\rho=0.$$

In [**H1**], Hitchin introduced the following evolution equations for a time-dependent pair of stable forms $(\omega(t),\rho(t))$ evolving from a half-flat $\mathrm{SU}(3)$-structure $(\omega(0),\rho(0))$:

$$\frac{\partial}{\partial t}\rho=d\omega,\quad \frac{\partial}{\partial t}\hat\omega=d\hat\rho,$$

where $\hat\omega=\frac{\omega^2}{2}$ and $\hat\rho=J_\rho^*\rho$. For compact manifolds M, he showed that these equations are the flow equations of a certain Hamiltonian system and that any solution defined on some interval $0\in I\subset\mathbb{R}$ defines a Riemannian metric on $M\times I$ with holonomy group in G_2. We give a new proof of this theorem, which does not use the Hamiltonian system and does not assume that M is compact. Moreover, our proof yields a similar result for all three types of half-flat G-structures: $G=\mathrm{SU}(3),\mathrm{SU}(1,2)$ and $\mathrm{SL}(3,\mathbb{R})$. For the noncompact groups G we obtain a pseudo-Riemannian metric of signature $(3,4)$ and holonomy group in G_2^* on $M\times I$ (see Theorem 2.3). As an application, we prove that any six-manifold endowed with a real analytic half-flat G-structure can be extended to a Ricci-flat seven-manifold with holonomy group in G_2 or G_2^*, depending on whether G is compact or noncompact, see Corollary 2.6.

More generally, a G-structure (ω,ρ) is called *nearly half-flat* if

$$d\rho=\hat\omega$$

and a G_2- or G_2^*-structure defined by a three-form φ is called *nearly parallel* if

$$d\varphi=*_\varphi\varphi.$$

We prove in Theorem 2.12 that any solution $I\ni t\mapsto\Big(\omega(t)=2\widehat{d\rho}(t),\rho(t)\Big)$ of the evolution equation

$$\dot\rho=d\omega-\varepsilon\hat\rho$$

evolving from a nearly half-flat G-structure $(\omega(0),\rho(0))$ on M defines a nearly parallel G_2- or G_2^*-structure on $M\times I$, depending on whether G is compact or noncompact, see (1.3) for the definition of $\widehat{d\rho}$. For compact manifolds M and $G=\mathrm{SU}(3)$ this theorem was proven by Stock [**St**].

The above constructions are illustrated in Section 3, where we start with a nearly pseudo-Kähler or a nearly para-Kähler six-manifold as initial structure. These structures are both half-flat and nearly half-flat and the resulting parallel or nearly parallel G_2- and G_2^*-structures induce cone or (hyperbolic) sine cone metrics.

In Section 4, we discuss the evolution of invariant half-flat structures on nilmanifolds. Lemma 4.1 shows how to simplify effectively the ansatz for a solution for a number of nilpotent Lie algebras including the direct sum $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$ of two Heisenberg algebras. Focusing on this case, we determine the orbits of the $\mathrm{Aut}(\mathfrak{h}_3 \oplus \mathfrak{h}_3)$-action on non-degenerate two-forms ω on $\mathfrak{h}_3$ which satisfy $d\omega^2 = 0$. Based on this, we describe all left-invariant half-flat structures (ω, ρ) on $H_3 \times H_3$. A surprising phenomenon occurs in indefinite signature. Under the assumption $\omega(\mathfrak{z}, \mathfrak{z}) = 0$, which corresponds to the vanishing of the projection of ω on a one-dimensional space, the geometry of the metric induced by a half-flat structure (ω, ρ) is completely determined (Proposition 4.7) and the evolution turns out to be affine linear (Proposition 4.10). However, this evolution produces only metrics that are decomposable and have one-dimensional holonomy group. On the other hand, we give an explicit formula in Proposition 4.12 for the parallel three-form φ resulting from the evolution for any half-flat structure (ω, ρ) with $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$. In fact, the formula is completely algebraic such that the integration of the differential equation is circumvented. In particular, we give a number of explicit examples of half-flat structures of the second kind on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ which evolve to new metrics with holonomy group equal to G_2 and G_2^*. Moreover, we construct an eight-parameter family of half-flat deformations of the half-flat examples which lift to an eight-parameter family of deformations of the corresponding parallel stable three-forms in dimension seven. Needless to say, those examples of $\mathrm{G}_2^{(*)}$-metrics on $M \times (a, b)$ for which $(a, b) \neq \mathbb{R}$ are geodesically incomplete. However, for M compact with an $\mathrm{SU}(3)$-structure, a conformal transformation produces complete Riemannian metrics on $M \times \mathbb{R}$ that are conformally parallel G_2.

A G_2- or G_2^*-structure defined by a three-form φ is called *cocalibrated* if

$$d *_\varphi \varphi = 0.$$

Hitchin proposed the following equation for the evolution of a cocalibrated G_2-structure $\varphi(0)$:

$$\frac{\partial}{\partial t}(*_\varphi \varphi) = d\varphi.$$

He proved that any solution $I \ni t \mapsto \varphi(t)$ on a compact manifold M defines a Riemannian metric on $M \times I$ with holonomy group in $\mathrm{Spin}(7)$. We generalise also this theorem to noncompact manifolds and show that any solution of the evolution equation starting from a cocalibrated G_2^*-structure defines a pseudo-Riemannian metric of signature $(4, 4)$ and holonomy group in $\mathrm{Spin}_0(3, 4)$, see Theorem 2.13.

Homogeneous projective special pseudo-Kähler manifolds of semisimple groups with compact stabiliser were classified in [**AC1**]. It follows that there is a unique homogeneous projective special pseudo-Kähler manifold with compact stabiliser which admits a transitive action of a real form of $\mathrm{SL}(3, \mathbb{C})$ by automorphisms of the special Kähler structure, namely

$$\frac{\mathrm{SU}(3,3)}{\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))}.$$

Its special Kähler metric is (negative) definite. The above manifold occurred in [**AC1**] as an open orbit of $\mathrm{SU}(3, 3)$ on the projectivised highest weight vector orbit of $\mathrm{SL}(6, \mathbb{C})$ on $\Lambda^3(\mathbb{C}^6)^*$. The space of stable three-forms $\rho \in \Lambda^3(\mathbb{R}^6)^*$, such that $J_\rho^2 = -1$, has also the structure of a special pseudo-Kähler manifold [**H1**]. The underlying projective special pseudo-Kähler manifold is the manifold

$$\frac{\mathrm{SL}(6,\mathbb{R})}{\mathrm{U}(1) \cdot \mathrm{SL}(3,\mathbb{C})}$$

which has noncompact stabiliser and indefinite special Kähler metric. Both manifolds can be obtained from the space of stable three-forms $\rho \in \Lambda^3(\mathbb{C}^6)^*$ by imposing two different reality conditions. In the last section of this paper we determine all homogeneous spaces which can be obtained in this way and describe their special geometric structures. In particular, we calculate the signature of the special Kähler metrics. For the projective special pseudo-Kähler manifold $\mathrm{SL}(6,\mathbb{R})/(\mathrm{U}(1)\cdot\mathrm{SL}(3,\mathbb{C}))$, for instance, we obtain the signature $(6,12)$. Apart from the two above examples, we find two additional special pseudo-Kähler manifolds and also a special para-Kähler manifold. The latter is associated to the space of stable three-forms $\rho \in \Lambda^3(\mathbb{R}^6)^*$, such that $J_\rho^2 = +1$.

1. Algebraic preliminaries

1.1. Stable forms. In this section we will collect some basic facts about stable forms, their orbits and their stabilisers.

PROPOSITION 1.1. *Let V be an n-dimensional real or complex vector space. The general linear group* $\mathrm{GL}(V)$ *has an open orbit in* $\Lambda^k V^*$, $0 \le k \le \left[\frac{n}{2}\right]$, *if and only if* $k \le 2$ *or if* $k = 3$ *and* $n = 6, 7$ *or* 8.

PROOF. The representation of $\mathrm{GL}(V)$ on $\Lambda^k V^*$ is irreducible. In the complex case the result thus follows, for instance, from the classification of irreducible complex prehomogeneous vector spaces, [**KiS**]. The result in the real case follows from the complex case, since the complexification of the $\mathrm{GL}(n,\mathbb{R})$-module $\Lambda^k\mathbb{R}^{n*}$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$-module. □

REMARK 1.2. An open orbit is unique in the complex case, since an orbit which is open in the usual topology is also Zariski-open and Zariski-dense (Prop. 2.2, [**Ki**]). Over the reals, the number of open orbits is finite by a well-known theorem of Whitney.

DEFINITION 1.3. A k-form $\rho \in \Lambda^k V^*$ is called *stable* if its orbit under $\mathrm{GL}(V)$ is open.

PROPOSITION 1.4. *Let* $k \in \{2, n-2\}$ *and n even, or* $k \in \{3, n-3\}$ *and* $n = 6, 7$ *or* 8. *There is a* $\mathrm{GL}(V)$*-equivariant mapping*

$$\phi : \Lambda^k V^* \to \Lambda^n V^*,$$

homogeneous of degree $\frac{n}{k}$, *which assigns a volume form to a stable k-form and which vanishes on non-stable forms. Given a stable k-form ρ, the derivative of ϕ in ρ defines a dual $(n-k)$-form* $\hat\rho \in \Lambda^{n-k}V^*$ *by the property*

$$d\phi_\rho(\alpha) = \hat\rho \wedge \alpha \quad \text{for all } \alpha \in \Lambda^k V^*. \tag{1.1}$$

The dual form $\hat\rho$ is also stable and satisfies

$$(\mathrm{Stab}_{\mathrm{GL}(V)}(\rho))_0 = (\mathrm{Stab}_{\mathrm{GL}(V)}(\hat\rho))_0.$$

A stable form, its volume form and its dual are related by the formula

$$\hat\rho \wedge \rho = \frac{n}{k}\phi(\rho). \tag{1.2}$$

PROOF. We consider the complex case first. As a result of the theory of prehomogeneous vector spaces [**Ki**], the complement of the open orbit is, in our situation, a hypersurface defined by a non-degenerate homogeneous polynomial f which is invariant under $\mathrm{GL}(V)$ up to a non-trivial character. In other words, there is an equivariant mapping from $\Lambda^k V^*$ to $(\Lambda^n V^*)^{\otimes s}$ for some positive integer s. Taking the s-th root, which depends on the choice of an orientation if s is even, we obtain the equivariant map ϕ with the claimed properties. The equivariance under scalar matrices implies that the map ϕ is homogeneous of degree $\frac{n}{k}$.

The derivative

$$\Lambda^k V^* \to (\Lambda^k V^*)^* \otimes \Lambda^n V^* \stackrel{=}{\to} \Lambda^{n-k}V^*\,,\ \rho \mapsto d_\rho\phi \mapsto \hat\rho$$

inherits equivariance from ϕ and is an immersion since f is non-degenerate. Therefore, it maps stable forms to stable forms such that the connected components of the stabilisers are identical. Formula (1.2) is in fact Euler's formula for the homogeneous mapping ϕ.

Since the complexification of the $\mathrm{GL}(n,\mathbb{R})$-module $\Lambda^k\mathbb{R}^{n*}$ is an irreducible $\mathrm{GL}(n,\mathbb{C})$-module, the results in the real case are easily deduced from the complex case. □

In the following, we discuss stable forms, their volume forms and their duals in the cases which are relevant in this article. In each case, V is a real n-dimensional vector space.

$\mathbf{k = 2, n = 2m}$. The orbit of a non-degenerate two-form is open and there is only one open orbit in $\Lambda^2 V^*$. Thus, the stabiliser of a stable two-form ω is isomorphic to $\mathrm{Sp}(2m, \mathbb{R})$. The polynomial invariant is the Pfaffian determinant. We normalise the associated equivariant volume form such that is corresponds to the Liouville volume form

$$\phi(\omega) = \frac{1}{m!}\omega^m.$$

Differentiation of the homogeneous polynomial map $\omega \mapsto \phi(\omega)$ yields

$$\hat{\omega} = \frac{1}{(m-1)!}\omega^{m-1}.$$

$\mathbf{k = (n-2), n = 2m}$. As $\Lambda^{n-2}V^* \cong \Lambda^2 V \otimes \Lambda^n V^*$, there is again only one open orbit. More precisely, an $(n-2)$-form σ is stable if and only if there is a stable two-form ω with $\sigma = \hat{\omega}$ since the mapping $\omega \mapsto \hat{\omega}$ is an equivariant immersion. If m is even, such an ω is unique and we define the volume form $\phi(\sigma) = \phi(\omega)$. If m is odd, we need an orientation on V to uniquely define an associated volume form. We choose the (m-1)-th root ω with positively oriented ω^m and define again $\phi(\sigma) = \phi(\omega)$. In both cases, we find

$$\hat{\sigma} = \frac{1}{m-1}\omega \tag{1.3}$$

with the help of (1.2). The stabiliser of a stable four-form in $\mathrm{GL}^+(V)$ is again the real symplectic group.

$\mathbf{k = 3, n = 6}$. Let V be an oriented six-dimensional vector space and let κ denote the canonical isomorphism

$$\kappa : \Lambda^k V^* \cong \Lambda^{6-k} V \otimes \Lambda^6 V^*.$$

Given any three-form ρ, we define $K : V \to V \otimes \Lambda^6 V^*$ by

$$K_\rho(v) = \kappa((v \lrcorner \rho) \wedge \rho)$$

and the quartic invariant

$$\lambda(\rho) = \frac{1}{6}\mathrm{tr}\,(\mathrm{K}_\rho^2) \ \in (\Lambda^6 \mathrm{V}^*)^{\otimes 2}. \tag{1.4}$$

Recall that, for any one-dimensional vector space L, an element $u \in L^{\otimes 2}$ is defined to be positive, $u > 0$, if $u = s \otimes s$ for some $s \in L$ and negative if $-u > 0$. Therefore, the norm of an element $u \in L^{\otimes 2}$ is well-defined and we set

$$\phi(\rho) = \sqrt{|\lambda(\rho)|} \tag{1.5}$$

for the positively oriented square root. If $\phi(\rho) \neq 0$, we furthermore define

$$J_\rho = \frac{1}{\phi(\rho)} K_\rho. \tag{1.6}$$

Proposition 1.5. *A three-form ρ on an oriented six-dimensional vector space V with volume form ν is stable if and only $\lambda(\rho) \neq 0$. There are two open orbits.*

One orbit consists of all three-forms ρ satisfying one of the following equivalent properties.

(a) $\lambda(\rho) > 0$
(b) There are two uniquely defined real decomposable three-forms α and β such that $\rho = \alpha + \beta$ and $\alpha \wedge \beta > 0$.
(c) The stabiliser of ρ in $\mathrm{GL}^+(V)$ is $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$.
(d) It holds $\lambda(\rho) \neq 0$ and the endomorphism J_ρ is a para-complex structure on V, i.e. $J^2 = \mathrm{id_V}$ and the eigenspaces for the eigenvalues ± 1 are three-dimensional.

(e) There is a basis $\{e_1,...,e_6\}$ *of* V *such that* $\nu = e^{123456} > 0$ *and*

$$\rho = e^{123} + e^{456}$$

where e^{ijk} *is the standard abbreviation for* $e^i \wedge e^j \wedge e^k$. *In this basis, it holds* $\lambda(\rho) = \nu^{\otimes 2}$, $J_\rho e_i = e_i$ *for* $i \in \{1,2,3\}$ *and* $J_\rho e_i = -e_i$ *for* $i \in \{4,5,6\}$.

The other orbit consists of all three-forms ρ *satisfying one of the following equivalent properties.*

(a) $\lambda(\rho) < 0$
(b) There is a unique complex decomposable three-form α *such that* $\rho = \alpha + \bar{\alpha}$ *and* $\mathrm{i}(\bar{\alpha} \wedge \alpha) > 0$.
(c) The stabiliser of ρ *in* $\mathrm{GL}^+(V)$ *is* $\mathrm{SL}(3,\mathbb{C})$.
(d) It holds $\lambda(\rho) \neq 0$ *and the endomorphism* J_ρ *is a complex structure on* V.
(e) There is a basis $\{e_1,...,e_6\}$ *of* V *such that* $\nu = e^{123456} > 0$ *and*

$$\rho = e^{135} - e^{146} - e^{236} - e^{245}.$$

In this basis, it holds $\lambda(\rho) = -4\nu^{\otimes 2}$, $J_\rho e_i = -e_{i+1}$ *and* $J_\rho e_{i+1} = e_i$ *for* $i \in \{1,3,5\}$.

Proof. All properties are proved in section 2 of [**H2**]. The only fact we added is the observation that J_ρ is a para-complex structure if $\lambda(\rho) > 0$ which is obvious in the standard basis. □

It is also possible to introduce a basis describing both orbits simultaneously. Indeed, given a generic stable three-form and an orientation, there is a basis $\{e_1,...,e_6\}$ of V and an $\varepsilon \in \{\pm 1\}$ such that $\nu = e^{123456} > 0$ and

$$\rho_\varepsilon = e^{135} + \varepsilon(e^{146} + e^{236} + e^{245}) \tag{1.7}$$

with $\lambda(\rho) = 4\varepsilon\nu^{\otimes 2}$. Furthermore, it holds $J_\rho e_i = \varepsilon e_{i+1}$, $J_\rho e_{i+1} = e_i$ for $i \in \{1,3,5\}$ and

$$J^*_{\rho_\varepsilon}\rho_\varepsilon = e^{246} + \varepsilon(e^{235} + e^{145} + e^{136}). \tag{1.8}$$

Analogies between complex and para-complex structures are elaborated in a unified language in [**AC2**] and [**SSH**]. In this language, a stable three-form always induces an ε-complex structure J_ρ since $J_\rho^2 = \varepsilon \mathrm{id}$ for the normal form ρ_ε.

Lemma 1.6. *The dual of a stable three-form* $\rho \in \Lambda^3 V^*$ *on an oriented six-dimensional vector space* V *is*

$$\hat{\rho} = J^*_\rho \rho. \tag{1.9}$$

Proof. We already observed that the connected components of the stabilisers of ρ and $\hat{\rho}$ have to be identical. Therefore, since the space of real three-forms invariant under $\mathrm{SL}(3,\mathbb{C})$ respectively $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ is two-dimensional, we can make the ansatz

$$\hat{\rho} = c_1\rho + c_2 J^*_\rho \rho$$

with real constants c_1 and c_2. Computing

$$\frac{6}{3}\phi(\rho) \overset{(1.2)}{=} \hat{\rho} \wedge \rho = c_2\, J^*_\rho\rho \wedge \rho \overset{(1.7,1.8)}{=} 2c_2\,\phi(\rho),$$

we find $c_2 = 1$. By

$$d_\rho\phi(J^*_\rho\rho) = \hat{\rho} \wedge J^*_\rho\rho = c_1\rho \wedge J^*_\rho\rho = -2c_1\phi(\rho),$$

the constant c_1 vanishes if the derivative of λ (recall (1.5)) in ρ in direction of $J^*_\rho\rho$ vanishes. However, using the normal form (1.7) again, we compute $\lambda(\rho + tJ^*_\rho\rho) = 4\varepsilon(-\varepsilon + t^2)^2\,(e^{123456})^{\otimes 2}$ and the assertion follows. □

A convenient way to compute the dual of ρ without determining J_ρ is given by the following corollary. In fact, the corollary explicitly shows the equivalence of the two different definitions of $\rho \mapsto \hat{\rho}$ given in [**H1**] and [**H2**].

COROLLARY 1.7. *If $\lambda(\rho) > 0$ and $\rho = \alpha + \beta$ in terms of decomposables ordered such that $\alpha \wedge \beta > 0$, the dual of ρ satisfies $\hat{\rho} = \alpha - \beta$.*

If $\lambda(\rho) < 0$ and ρ is the real part of a complex decomposable three-form α such that $\mathrm{i}(\bar{\alpha} \wedge \alpha) > 0$, the dual of ρ is the imaginary part of α. In particular, the complex three-form α is a $(3,0)$-form w.r.t. J_ρ.

PROOF. The assertions are easily proved by comparing the claimed formulas for $\hat{\rho}$ with formula (1.9) in the standard bases given in part (e) of Proposition 1.5. $\square$

Finally, we note that for a fixed orientation, it holds

$$\hat{\hat{\rho}} = -\rho \quad \text{and} \quad J_{\hat{\rho}} = -\varepsilon J_\rho. \tag{1.10}$$

$\mathbf{k = 3, n = 7}$. Given any three-form φ, we define a symmetric bilinear form with values in $\Lambda^7 V^*$ by

$$b_\varphi(v, w) = (v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi. \tag{1.11}$$

Since the determinant of a scalar-valued bilinear form is an element of $(\Lambda^7 V^*)^{\otimes 2}$, we have $\det b \in (\Lambda^7 V^*)^{\otimes 9}$. If and only if φ is stable, the seven-form

$$\phi(\varphi) = (\det b_\varphi)^{\frac{1}{9}}$$

defines a volume form, independent of an orientation on V, and the scalar-valued symmetric bilinear form

$$g_\varphi = \frac{1}{\phi(\varphi)} b_\varphi$$

is non-degenerate. Notice that $\phi(\varphi) = \sqrt{\det g_\varphi}$ is the metric volume form.

It is known ([**Br**], [**Ha**]) that a stable three-form defines a multiplication "$\cdot$" and a vector cross product "$\times$" on V by the formula

$$\varphi(x, y, z) = g_\varphi(x, y \cdot z) = g_\varphi(x, y \times z), \tag{1.12}$$

such that $(V, \times)$ is isomorphic either to the imaginary octonions $\mathrm{Im}\,\mathbb{O}$ or to the imaginary split-octonions $\mathrm{Im}\,\tilde{\mathbb{O}}$. Thus, there are exactly two open orbits of stable three-forms having isotropy groups

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\varphi) \cong \begin{cases} \mathrm{G}_2 \subset \mathrm{SO}(7), & \text{if } g_\varphi \text{ is positive definite,} \\ \mathrm{G}_2^* \subset \mathrm{SO}(3,4), & \text{if } g_\varphi \text{ is of signature (3,4).} \end{cases} \tag{1.13}$$

There is always a basis $\{e_1, \ldots, e_7\}$ of V such that

$$\varphi = \tau e^{124} + \sum_{i=2}^{7} e^{i\,(i+1)\,(i+3)} \tag{1.14}$$

with $\tau \in \{\pm 1\}$ and indices modulo 7. For $\tau = 1$, the induced metric g_φ is positive definite and the basis is orthonormal such that this basis corresponds to the Cayley basis of $\mathrm{Im}\,\mathbb{O}$. For $\tau = -1$, the metric is of signature (3,4) and the basis is pseudo-orthonormal with e_1, e_2 and e_4 being the three spacelike basis vectors.

The only four-forms having the same stabiliser as φ are the multiples of the Hodge dual $*_{g_\varphi}\varphi$, [**Br**, Propositions 2.1, 2.2]. Since the normal form satisfies $g_\varphi(\varphi, \varphi) = 7$, we have by definition of the Hodge dual $\varphi \wedge *_{g_\varphi}\varphi = 7\,\phi(\varphi)$ and therefore

$$\hat{\varphi} = \frac{1}{3} *_{g_\varphi} \varphi, \tag{1.15}$$

by comparing with (1.2).

LEMMA 1.8. *Let φ be a stable three-form in a seven-dimensional vector space V. Let β be a one-form or a two-form. Then $\beta \wedge \varphi = 0$ if and only if $\beta = 0$.*

PROOF. For the compact case, see also [**Bo**]. If β is a one-form, the proof is very easy. If β is a two-form, we choose a basis such that φ is in the normal form (1.14) and $\beta = \sum_{i<j} b_{i,j}\, e^{ij}$ and compute

$$\begin{aligned}
\beta \wedge \varphi \;=\;& (b_{2,3} - b_{1,6})\, e^{12356} + (b_{2,3} - b_{4,7})\, e^{23457} + (b_{1,6} + b_{4,7})\, e^{14567} \\
+\;& (b_{5,7}\tau + b_{1,2})\, e^{12457} + (b_{3,6} - b_{5,7})\, e^{34567} + (b_{1,2} - b_{3,6}\tau)\, e^{12346} \\
-\;& (b_{3,7}\tau + b_{2,4})\, e^{12347} + (b_{5,6}\tau + b_{2,4})\, e^{12456} + (b_{3,7} + b_{5,6})\, e^{13567} \\
+\;& (b_{2,5} - b_{4,6})\, e^{23456} + (b_{4,6} - b_{1,7})\, e^{13467} - (b_{2,5} + b_{1,7})\, e^{12357} \\
+\;& (b_{4,5} + b_{2,6})\, e^{24567} - (b_{1,3} + b_{2,6})\, e^{12367} + (b_{4,5} + b_{1,3})\, e^{13457} \\
+\;& (b_{3,5} + b_{6,7})\, e^{23567} + (b_{1,4} - b_{3,5}\tau)\, e^{12345} + (b_{6,7}\tau - b_{1,4})\, e^{12467} \\
+\;& (b_{3,4} + b_{1,5})\, e^{13456} + (b_{2,7} - b_{1,5})\, e^{12567} + (b_{3,4} - b_{2,7})\, e^{23467}.
\end{aligned}$$

The five-form is written as a linear combination of linearly independent forms and each line contains exactly three different coefficients of β. Inspecting the coefficient equations line by line, it is easy to see that all coefficients of β vanish if and only if $\beta \wedge \varphi = 0$. □

1.2. Real forms of $\mathrm{SL}(3,\mathbb{C})$**.** By the following proposition, any real form of $\mathrm{SL}(3,\mathbb{C})$ can be written as a simultaneous stabiliser of a stable two-form and a stable three-form.

PROPOSITION 1.9. *Let V be a six-dimensional real vector space. Let $\omega \in \Lambda^2 V^*$ and $\rho \in \Lambda^3 V^*$ be stable forms which are compatible in the sense that*

$$\omega \wedge \rho \;=\; 0. \tag{1.16}$$

Then, we have

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\rho,\omega) \cong \begin{cases} \mathrm{SU}(p,q) \subset \mathrm{SO}(2p,2q)\,, \quad p+q=3\,, & \text{if } \lambda(\rho) < 0, \\ \mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SO}(3,3)\,, & \text{if } \lambda(\rho) > 0, \end{cases}$$

where $\mathrm{SL}(3,\mathbb{R})$ is embedded in $\mathrm{SO}(3,3)$ such that it acts by the standard representation and its dual, respectively, on the maximally isotropic ± 1-eigenspaces of the para-complex structure J_ρ induced by ρ.

PROOF. Let V be oriented by $\phi(\omega) = \frac{1}{6}\omega^3$ and let J_ρ be the unique (para-) complex structure (1.6) associated to the three-form ρ and this orientation. By $\varepsilon \in \{\pm 1\}$, we denote the sign of $\lambda(\rho)$, that is $J_\rho^2 = \varepsilon id_V$. In the basis in which ρ is in the normal form (1.7), it is easy to verify that $\omega \wedge \rho = 0$ is equivalent to the skew-symmetry of J_ρ with respect to ω. Equivalently, the pseudo-Euclidean metric

$$g = g_{(\omega,\rho)} = \varepsilon\, \omega(\cdot, J_\rho \cdot), \tag{1.17}$$

induced by ω and ρ, is compatible with J_ρ in the sense that $g(J_\rho\cdot, J_\rho\cdot) = -\varepsilon g(\cdot,\cdot)$. The stabiliser of the set of tensors $(\omega, J_\rho, g, \rho, J_\rho^*\rho)$ satisfying this compatibility condition is well-known to be $\mathrm{SU}(p,q)$ respectively $\mathrm{SL}(3,\mathbb{R})$. □

We will call a compatible pair of stable forms $(\omega,\rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ normalised if

$$\phi(\rho) = 2\,\phi(\omega) \quad \iff \quad J_\rho^*\rho \wedge \rho = \frac{2}{3}\,\omega^3. \tag{1.18}$$

REMARK 1.10. By our conventions, the metric (1.17) induced by a normalised, compatible pair is of signature either $(6,0)$ or $(2,4)$ or $(3,3)$, where the first number denotes the number of spacelike directions. We emphasise that our conventions are such that

$$\omega = g(\cdot, J_\rho \cdot).$$

This sign choice turned out to be necessary in order to achieve that $\phi(\rho)$ is indeed a positive multiple of $\phi(\omega)$ in the positive definite case.

Sometimes it is convenient to have a unified adapted basis. For a compatible and normalised pair (ω,ρ), there is always a pseudo-orthonormal basis $\{e_1,\dots,e_6\}$ of V with dual basis $\{e^1,\dots,e^6\}$ such that $\rho=\rho_\varepsilon$ is in the normal form (1.7) and

$$\omega \quad = \quad \tau(e^{12}+e^{34})+e^{56} \tag{1.19}$$

for $(\varepsilon,\tau)\in\{(-1,1),(-1,-1),(1,1)\}$. The signature of the induced metric with respect to this basis is

$$(\tau,-\varepsilon\tau,\tau,-\varepsilon\tau,1,-\varepsilon)=\begin{cases}(+,+,+,+,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=1,\\ (-,-,-,-,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=-1,\\ (+,-,+,-,+,-) & \text{for } \varepsilon=1 \text{ and } \tau=1,\end{cases} \tag{1.20}$$

and we have

$$\mathrm{Stab}_{\mathrm{GL}(6,\mathbb{R})}(\omega,\rho)\cong\begin{cases}\mathrm{SU}(3)\subset\mathrm{SO}(6) & \text{for } \varepsilon=-1 \text{ and } \tau=1,\\ \mathrm{SU}(1,2)\subset\mathrm{SO}(2,4) & \text{for } \varepsilon=-1 \text{ and } \tau=-1,\\ \mathrm{SL}(3,\mathbb{R})\subset\mathrm{SO}(3,3) & \text{for } \varepsilon=1.\end{cases}$$

For instance, the following observation is easily verified using the unified basis.

LEMMA 1.11. *Let (ω,ρ) be a compatible and normalised pair of stable forms on a six-dimensional vector space. Then, the volume form $\phi(\omega)$ is in fact a metric volume form w.r.t. to the induced metric $g=g_{(\omega,\rho)}$ and the corresponding Hodge dual of ω and ρ is*

$$*_g\,\omega=-\varepsilon\hat\omega\,,\qquad *_g\rho=-\hat\rho \tag{1.21}$$

1.3. Relation between real forms of $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{G}_2^{\mathbb{C}}$. The relation between stable forms in dimension six and seven corresponding to the embedding $\mathrm{SU}(3)\subset\mathrm{G}_2$ is well-known. We extend this relation by including also the embeddings $\mathrm{SU}(1,2)\subset\mathrm{G}_2^*$ and $\mathrm{SL}(3,\mathbb{R})\subset\mathrm{G}_2^*$ as follows.

PROPOSITION 1.12. *Let $V=W\oplus L$ be a seven-dimensional vector space decomposed as a direct sum of a six-dimensional subspace W and a line L. Let α be a non-trivial one-form in the annihilator W^0 of W and $(\omega,\rho)\in\Lambda^2L^0\times\Lambda^3L^0$ a compatible and normalised pair of stable forms inducing the scalar product $h=h_{(\omega,\rho)}$ given in* (1.17). *Then, the three-form $\varphi\in\Lambda^3V^*$ defined by*

$$\varphi=\omega\wedge\alpha+\rho \tag{1.22}$$

is stable and induces the scalar product

$$g_\varphi=h-\varepsilon\alpha\cdot\alpha \tag{1.23}$$

where ε denotes the sign of $\lambda(\rho)$ such that $J_\rho^2=\varepsilon\mathrm{id}$. The stabiliser of φ in $\mathrm{GL}(V)$ is

$$\mathrm{Stab}_{\mathrm{GL}(V)}(\varphi)\cong\begin{cases}\mathrm{G}_2 & \textit{for } \varepsilon=-1 \textit{ and positive definite } h,\\ \mathrm{G}_2^* & \textit{otherwise.}\end{cases}$$

PROOF. We choose a basis $\{e_1,\dots,e_6\}$ of L^0 such that ω and ρ are in the generic normal forms (1.7) and (1.19). With $e^7=\alpha$, we have

$$\varphi=\tau(e^{127}+e^{347})+e^{567}+e^{135}+\varepsilon(e^{146}+e^{236}+e^{245}). \tag{1.24}$$

The induced bilinear form (1.11) turns out to be

$$b_\varphi(v,w)=(-\varepsilon\tau v^1w^1+\tau v^2w^2-\varepsilon\tau v^3w^3+\tau v^4w^4-\varepsilon v^5w^5+v^6w^6+v^7w^7)e^{1234567}$$

for $v=\sum v^ie_i$ and $w=\sum w^ie_i$. Hence, the three-form φ is stable for all signs of ε and τ and its associated volume form is

$$\phi(\varphi)=(\det b_\varphi)^{\frac19}=-\varepsilon\,e^{1234567}.$$

The formula (1.23) for the metric g_φ induced by φ follows, since the basis $\{e_1,\dots,e_7\}$ of V is pseudo-orthonormal with respect to this metric of signature

$$(\tau,-\varepsilon\tau,\tau,-\varepsilon\tau,1,-\varepsilon,-\varepsilon)=\begin{cases}(+,+,+,+,+,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=1,\\ (-,-,-,-,+,+,+) & \text{for } \varepsilon=-1 \text{ and } \tau=-1,\\ (+,-,+,-,+,-,-) & \text{for } \varepsilon=1 \text{ and } \tau=1.\end{cases} \tag{1.25}$$

The assertion on the stabilisers now follows from (1.13). □

LEMMA 1.13. *Under the assumptions of the previous proposition, the dual four-form of the stable three-form φ is*

$$3\hat{\varphi} = *_\varphi \varphi = -\,\varepsilon\,(\alpha \wedge \hat{\rho} + \hat{\omega}) = \varepsilon\alpha \wedge *_h\rho + *_h\omega, \tag{1.26}$$

where $*_\varphi$ denotes the Hodge dual with respect to the metric g_φ and the orientation induced by $\phi(\varphi)$.

PROOF. In the basis of the previous proof, the Hodge dual of φ is

$$*_\varphi\varphi = -\varepsilon\tau(e^{3456} + e^{1256}) - \varepsilon e^{1234} + \varepsilon\, e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

The second equality follows when comparing this expression with $\varepsilon(e^7 \wedge \hat{\rho} + \frac{1}{2}\omega^2)$ in this basis using (1.8) and (1.19). The first and the third equality are just the formulas (1.15) and (1.21), respectively. □

The inverse process is given by the following construction.

PROPOSITION 1.14. *Let V be a seven-dimensional real vector space and $\varphi \in \Lambda^3 V^*$ a stable three-form which induces the metric g_φ on V. Moreover, let $n \in V$ be a unit vector with $g_\varphi(n,n) = -\varepsilon \in \{\pm 1\}$ and let $W = n^\perp$ denote the orthogonal complement of $\mathbb{R}\cdot n$. Then, the pair $(\omega, \rho) \in \Lambda^2 W^* \times \Lambda^3 W^*$ defined by*

$$\omega = n \lrcorner\, \varphi\,, \qquad \rho = \varphi_{|W}\,, \tag{1.27}$$

is a pair of compatible normalised stable forms. The metric $h = h_{(\omega,\rho)}$ induced by this pair on W satisfies $h = (g_\varphi)_{|W}$ and the stabiliser is

$$\mathrm{Stab}_{\mathrm{GL}(W)}(\omega,\rho) \cong \begin{cases} \mathrm{SU}(3), & \text{if } g_\varphi \text{ is positive definite,} \\ \mathrm{SU}(1,2), & \text{if } g_\varphi \text{ is indefinite and } \varepsilon = -1, \\ \mathrm{SL}(3,\mathbb{R}), & \text{if } \varepsilon = 1. \end{cases}$$

When (V, φ) is identified with the imaginary octonions, respectively, the imaginary split-octonions, by (1.12), the ε-complex structure induced by ρ is given by

$$J_\rho v = -n \cdot v = -n \times v \qquad \text{for } v \in V. \tag{1.28}$$

PROOF. Due to the stability of φ, we can always choose a basis $\{e_1, \ldots, e_7\}$ of V with $n = e_7$ such that φ is given by (1.24) where $\varepsilon = -g_\varphi(n,n)$ and $\tau \in \{\pm 1\}$ depends on the signature of g_φ. As this basis is pseudo-orthonormal with signature given by (1.25), the vector n has indeed the right scalar square and $\{e_1, \ldots, e_6\}$ is a pseudo-orthonormal basis of the complement $W = n^\perp$. Since the pair (ω, ρ) defined by (1.27) is now exactly in the generic normal form given by (1.7) and (1.19), it is stable, compatible and normalised and the induced endomorphism J_ρ is an ε-complex structure. The identity $h = (g_\varphi)_{|W}$ for the induced metric $h_{(\omega,\rho)}$ follows from comparing the signatures (1.25) and (1.20) and the assertion for the stabilisers is an immediate consequence. Finally, the formula for the induced ε-complex structure J_ρ is another consequence of $g = (g_\varphi)_{|W}$ since we have

$$g_\varphi(x, n \times y) \overset{(1.12)}{=} \varphi(x, n, y) = -\omega(x,y) = -h(x, J_\rho y)$$

for all $x, y \in W$. □

Notice that, for a fixed metric h of signature $(2,4)$ or $(3,3)$, the compatible and normalised pairs (ω, ρ) of stable forms inducing this metric are parametrised by the homogeneous spaces $\mathrm{SO}(2,4)/\mathrm{SU}(1,2)$ and $\mathrm{SO}(3,3)/\mathrm{SL}(3,\mathbb{R})$, respectively. Thus, the mapping $(\omega, \rho) \mapsto \varphi$ defined by formula (1.22) yields isomorphisms

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SU}(1,2)} \cong \frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*}\,, \qquad \frac{\mathrm{SO}(3,3)}{\mathrm{SL}(3,\mathbb{R})} \cong \frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*}\,,$$

since the metric h completely determines the metric g_φ by the formula (1.23).

1.4. Relation between real forms of $G_2^{\mathbb{C}}$ and $\mathrm{Spin}(7,\mathbb{C})$. It is possible to extend this construction to dimension eight as follows. Starting with a stable three-form φ on a seven-dimensional space V, we can consider the four-form

$$\Phi = e^8 \wedge \varphi + *_\varphi \varphi. \tag{1.29}$$

on the eight-dimensional space $V \oplus \mathbb{R}e_8$. Although the four-form Φ is not stable, it is shown in [**Br**] that it induces the metric

$$g_\Phi = g_\varphi + (e^8)^2 \tag{1.30}$$

on $V \oplus \mathbb{R}e_8$ and that its stabiliser is

$$\mathrm{Stab}_{\mathrm{GL}(V\oplus\mathbb{R}e_8)}(\Phi) \cong \begin{cases} \mathrm{Spin}(7) \subset \mathrm{SO}(8), & \text{if } g_\varphi \text{ is positive definite,} \\ \mathrm{Spin}_0(3,4) \subset \mathrm{SO}(4,4), & \text{if } g_\varphi \text{ is indefinite.} \end{cases}$$

The index “0” denotes, as usual, the connected component. Starting conversely with a four-form Φ on $V \oplus \mathbb{R}e_8$ such that its stabiliser in $\mathrm{GL}(V \oplus \mathbb{R}e_8)$ is isomorphic to $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$, the process can be reversed by setting $\varphi = e_8 \lrcorner \Phi$. As before, the metric induced by Φ on $V \oplus \mathbb{R}e_8$ is determined by the metric g_φ induced by φ on V. Thus, the indefinite analogue of the well-known isomorphisms

$$\mathbb{RP}^7 \cong \frac{\mathrm{SO}(6)}{\mathrm{SU}(3)} \cong \frac{\mathrm{SO}(7)}{\mathrm{G}_2} \cong \frac{\mathrm{SO}(8)}{\mathrm{Spin}(7)}$$

is given by

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SU}(1,2)} \cong \frac{\mathrm{SO}(3,3)}{\mathrm{SL}(3,\mathbb{R})} \cong \frac{\mathrm{SO}(3,4)}{\mathrm{G}_2^*} \cong \frac{\mathrm{SO}(4,4)}{\mathrm{Spin}_0(3,4)}. \tag{1.31}$$

2. Hitchin’s flow equations

2.1. Half-flat structures and parallel $\mathrm{G}_2^{(*)}$-structures. Now we want to put the algebraic structures considered in the previous section onto smooth manifolds. This is best done in terms of reductions of the bundle of frames of the manifold. This bundle has $\mathrm{GL}(n,\mathbb{R})$ as its structure group if n is the dimension of the manifold. A subbundle whose structure group is a subgroup G of $\mathrm{GL}(n,\mathbb{R})$ is a called a reduction of the frame bundle, or a *G-structure*. For example, if $G \subset \mathrm{O}(p,q)$, for $p+q=n$, the reduction determines a pseudo-Riemannian metric of signature (p,q) and the distinguished frames are orthonormal with respect to this metric. If $G \subset \mathrm{O}(p,q)$, again with $p+q=n$, then a G-structure is called *parallel* if the G-subbundle is invariant under the parallel transport defined by the Levi-Civita connection of the corresponding metric. This is equivalent to the property that the holonomy group of the Levi-Civita is contained in G.

In the following we will consider G-structures that are given by the groups described in the previous sections. According to the notations given there, we denote by $H^{\varepsilon,\tau}$ a real form of $\mathrm{SL}(3,\mathbb{C})$ and $G^{\varepsilon,\tau}$ the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ in which $H^{\varepsilon,\tau}$ is embedded, i.e. $H^{-1,1} = \mathrm{SU}(3) \subset \mathrm{SO}(6)$, $H^{-1,-1} = \mathrm{SU}(1,2) \subset \mathrm{SO}(2,4)$, $H^{1,1} = \mathrm{SL}(3,\mathbb{R}) \subset \mathrm{SO}(3,3)$, $G^{-1,1} = \mathrm{G}_2 \subset \mathrm{SO}(7)$, and $G^{-1,-1} = G^{1,1} = \mathrm{G}_2^* \subset \mathrm{SO}(3,4)$. We will also use the notation $\mathrm{G}_2^{(*)}$ as a shorthand for “G_2 respectively G_2^*”.

An $H^{\varepsilon,\tau}$-structure is equivalent to a pair of everywhere stable forms $\omega \in \Omega^2 M$ and $\rho \in \Omega^3 M$ on M, considered up to rescaling of ρ by a nonzero constant, that satisfy the compatibility condition

$$\rho \wedge \omega = 0 \tag{2.1}$$

corresponding to (1.16) and in addition

$$\phi(\rho) = c\,\phi(\omega)\,, \qquad \text{i.e. } J_\rho^* \rho \wedge \rho = \tfrac{1}{3}\, c\, \omega^3, \tag{2.2}$$

for a positive real constant c. Indeed, if an $H^{\varepsilon,\tau}$-structure is given, these forms are obtained by applying the formulae (1.7) and (1.19) to one of the frames of the $H^{\varepsilon,\tau}$-structure. By construction, the stable forms then satisfy (2.2) with $c = 2$.

On the other hand, if $\omega \in \Omega^2 M$ and $\rho \in \Omega^3 M$ are everywhere stable and satisfy (2.1) and (2.2), we can find a local frame, in which they are in normal form after rescaling ρ by a constant. This frame then determines the $H^{\varepsilon,\tau}$-structure.

Note that stable forms define an $H^{\varepsilon,\tau}$-structure, even if they only satisfy (2.1) but not the second compatibility condition (2.2). In this case ρ can always be rescaled by a smooth function such that (2.2) holds. When we say that the pair of stable forms defines an $H^{\varepsilon,\tau}$-structure, we will always assume that *both* compatibility conditions are satisfied. We will call the $H^{\varepsilon,\tau}$-structure *normalised* if $c = 2$. This seems to be a common normalisation for SU(3)-structures in the literature.

Furthermore, one can show that the $H^{\varepsilon,\tau}$-structure is parallel if and only if ρ, $\hat{\rho}$, and ω are closed. The proof of this fact given in [**H2**, p. 567] generalises to SU(1,2)-structures and also to $\mathrm{SL}(3,\mathbb{R})$-structures, in the latter case using Frobenius' Theorem instead of the Newlander-Nirenberg Theorem. In all cases the parallel $H^{\varepsilon,\tau}$-structure is equivalent to M being a Ricci-flat (para-)Kähler manifold.

Now we consider a weaker condition, that will turn out to be related to parallel $\mathrm{G}_2^{(*)}$-structures.

Definition 2.1. An $H^{\varepsilon,\tau}$-structure (ρ,ω) is called *half-flat* if

$$d\rho = 0 \tag{2.3}$$

$$d\sigma = 0, \tag{2.4}$$

where $2\sigma = \omega^2$.

Similarly, a smooth seven-manifold admits a $\mathrm{G}_2^{(*)}$-structure if and only if there is a stable three-form φ. Again, this structure is parallel if and only if φ is closed and co-closed, i.e. $d\varphi = d{*}\varphi = 0$, where $*$ denotes the Hodge operator with respect to the metric induced by the $\mathrm{G}_2^{(*)}$-structure. For a proof in both cases see [**G1**, Theorem 4.1].

Note that any orientable hypersurface in a manifold with G_2- or G_2^*-structure admits an $H^{\varepsilon,\tau}$-structure by the algebraic construction described in Proposition 1.14. If the $\mathrm{G}_2^{(*)}$-structure φ is parallel, the induced $H^{\varepsilon,\tau}$-structure is half-flat due to equations (1.22) and (1.26). For the various results on the SU(3)-structures on hypersurfaces in G_2-structures, we refer to [**Cal**], [**Cab**] and references therein.

On the other hand, certain one-parameter families of half-flat structures define parallel $\mathrm{G}_2^{(*)}$-structures.

Proposition 2.2. *Let $H^{\varepsilon,\tau}$ be a real form of $\mathrm{SL}(3,\mathbb{C})$, $G^{\varepsilon,\tau}$ the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ and (ρ,ω) a one-parameter family of $H^{\varepsilon,\tau}$-structures on a six-manifold M with a parameter t from an interval I. Then, the three-form*

$$\varphi = \omega \wedge dt + \rho$$

defines a parallel $G^{\varepsilon,\tau}$-structure on $M \times I$ if and only if the $H^{\varepsilon,\tau}$-structure (ρ,ω) is half-flat for all t and satisfies the following evolution equations

$$\dot{\rho} = d\omega \tag{2.5}$$

$$\dot{\sigma} = d\hat{\rho} \tag{2.6}$$

with $\sigma = \frac{1}{2}\omega^2$.

Proof. Let (ρ,ω) be an $H^{\varepsilon,\tau}$-structure and $\varphi = \omega \wedge dt + \rho$ a stable three-form on $\check{M} := M \times I$. By (1.26), the Hodge-dual of φ is given by

$$*\varphi = \varepsilon\left(\hat{\rho} \wedge dt - \sigma\right).$$

Denoting by $\check{d}$ the differential on $\check{M}$ and by d the differential on M we calculate

$$\check{d}\varphi = d\omega \wedge dt + dt \wedge \dot{\rho} + d\rho = (d\omega - \dot{\rho}) \wedge dt + d\rho \tag{2.7}$$

$$\check{d} * \varphi = \varepsilon\left(d\hat{\rho} \wedge dt - dt \wedge \dot{\sigma} - d\sigma\right) = \varepsilon(d\hat{\rho} - \dot{\sigma}) \wedge dt - \varepsilon d\sigma \tag{2.8}$$

Thus, φ defines a parallel $G^{\varepsilon,\tau}$-structure if and only if the evolution equations (2.5) and (2.6) and the half-flat equations are satisfied. □

The evolution equations (2.5) and (2.6) are the *Hitchin flow equations*, as found in [**H1**] for SU(3)-structures, applied to $H^{\varepsilon,\tau}$-structures. Their solutions (ρ,ω), called *Hitchin flow*, have to satisfy possibly dependent conditions in order to yield a parallel $\mathrm{G}_2^{(*)}$-structure: the evolution equations and the compatibility equations for the family of half-flat structures. The following theorem shows that the evolution equations together with an initial condition already ensure that the family consists of half-flat structures. A special version of this theorem was proved in [**H1**] under the assumption that M is compact and that $H = \mathrm{SU}(3)$.

THEOREM 2.3. *Let (ρ_0, ω_0) be a half-flat $H^{\varepsilon,\tau}$-structure on a six-manifold M. Furthermore, let $(\rho,\omega) \in \Omega^3 M \times \Omega^2 M$ be a one-parameter family of stable forms with parameters from an interval I satisfying the evolution equations (2.5) and (2.6). If $(\rho(t_0), \omega(t_0)) = (\rho_0, \omega_0)$ for a $t_0 \in I$, then (ρ,ω) is a family of half-flat $H^{\varepsilon,\tau}$-structures. In particular, the three-form*

$$\varphi = \omega \wedge dt + \rho \tag{2.9}$$

defines a parallel $G^{\varepsilon,\tau}$-structure on $M \times I$ and the induced metric

$$g_\varphi = g(t) - \varepsilon dt^2, \tag{2.10}$$

has holonomy contained in $G^{\varepsilon,\tau}$, where $g = g(t)$ is the family of metrics on M associated to (ρ,ω).

PROOF. Differentiating the evolution equations (2.5) and (2.6) gives $d\dot{\rho} = d\dot{\sigma} = 0$. The initial condition for t_0 was that (ρ_0, ω_0) is half-flat. This implies

$$\begin{aligned} d\rho &= 0 \\ d\sigma &= 0 \end{aligned}$$

for all $t \in I$. Hence, in order to obtain a family of half-flat structures we have to verify that the compatibility condition (2.1) holds for all $t \in I$.

LEMMA 2.4. *Let M be a six-manifold with $H^{\varepsilon,\tau}$-structure (ρ,ω), $\phi : \Omega^3 M \to \Omega^6 M$ defined pointwise by the map $\phi : \Lambda^3 T_p^* M \to \Lambda^6 T_p^* M$ given in Proposition 1.4 and $\hat{\rho}$ defined by $d\phi_\rho(\xi) = \hat{\rho} \wedge \xi$ for all $\xi \in \Omega^3 M$. If $\mathcal{L}_X$ denotes the Lie derivative, then*

$$\mathcal{L}_X(\phi(\rho)) = \hat{\rho} \wedge \mathcal{L}_X \rho.$$

PROOF. First note that the $\mathrm{GL}(n,\mathbb{R})$-equivariance of the map $\phi : \Lambda^3 T_p^* M \to \Lambda^6 T_p^* M$ implies that the corresponding map $\phi : \Omega^3 M \to \Omega^6 M$ is equivariant under diffeomorphisms. Indeed, if ψ is a (local) diffeomorphism of M we get that

$$\psi^*(\phi(\rho)) = \phi(\psi^* \rho).$$

Let ψ_t be the flow of the vector field X. Then the Lie derivative is given by

$$\mathcal{L}_X(\phi(\rho)) = \frac{d}{dt}\left(\psi_t^* \phi(\rho)\right)|_{t=0} = \frac{d}{dt}\phi(\psi_t^* \rho)|_{t=0} = d\phi_\rho(\mathcal{L}_X \rho),$$

implying the statement. □

LEMMA 2.5. *A stable three-form $\rho \in \Omega^3 M$ on a six-manifold satisfies for any $X \in \mathfrak{X}(M)$*

$$\hat{\rho}_X \wedge \rho = -\hat{\rho} \wedge \rho_X\,, \tag{2.11}$$

$$(d\hat{\rho})_X \wedge \rho = \hat{\rho} \wedge (d\rho)_X\,, \tag{2.12}$$

where ρ_X denotes the interior product of X with the form ρ.

PROOF. In order to verify the first identity, we can assume that $\rho = \rho_p$ is a stable three-form on $V = T_p M$ and $X \in V$ for a $p \in M$. If $\lambda(\rho) < 0$, the stabiliser $\mathrm{SL}(3,\mathbb{C})$ of ρ in $\mathrm{GL}^+(V)$ acts transitively on $V \setminus \{0\}$. If $\lambda(\rho) > 0$, we can decompose V in the ±1-eigenspaces $V^\pm$ of J_ρ. The stabiliser $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$ of ρ in $\mathrm{GL}^+(V)$ acts transitively on the dense open subset $V^+ \setminus \{0\} \times V^- \setminus \{0\} \subset V$ and there is an automorphism exchanging V^+ and V^- which stabilises ρ. Thus, it suffices to verify the first identity for the normal form (1.7), (1.8) and $X = e_1$, which is easy.

For the second identity, using Lemma 2.4 in the second step, we compute

$$\begin{aligned}(d\hat\rho)_X \wedge \rho - \hat\rho \wedge (d\rho)_X &= -d\hat\rho \wedge \rho_X + \hat\rho \wedge d(\rho_X) - \hat\rho \wedge \mathcal{L}_X \rho \\ &= -d(\hat\rho \wedge \rho_X) - \mathcal{L}_X(\phi(\rho)) \\ &= -d(\hat\rho \wedge \rho_X + \phi(\rho)_X) \\ &\overset{(1.2)}{=} -\frac{1}{2} d(\hat\rho \wedge \rho_X + \hat\rho_X \wedge \rho).\end{aligned}$$

Hence, the first identity (2.11) implies (2.12). □

Using this lemma, we calculate the t-derivative of the six-form $\omega_X \wedge \omega \wedge \rho = \sigma_X \wedge \rho$ for any vector field X:

$$\begin{aligned}\frac{\partial}{\partial t}(\sigma_X \wedge \rho) &= \dot\sigma_X \wedge \rho + \sigma_X \wedge \dot\rho \\ &\overset{(2.5),(2.6)}{=} (d\hat\rho)_X \wedge \rho + \sigma_X \wedge d\omega \\ &\overset{(2.12)}{=} \hat\rho \wedge (d\rho)_X + \omega_X \wedge \omega \wedge d\omega \\ &\overset{(2.3),(2.4)}{=} 0.\end{aligned}$$

Together with the initial condition $\omega_0 \wedge \rho_0 = 0$ this implies that $\sigma_X \wedge \rho = 0$ for all $t \in I$ and for all vector fields X. Since ω is non degenerate, the product of any one-form with $\omega \wedge \rho$ vanishes and thus, the compatibility condition $\omega \wedge \rho = 0$ holds for all t.

The preservation of the normalisation (2.2) in time is shown in [**H1**], in the final part of the proof of Theorem 8. The idea is to compute the second derivative of the volume form assigned to a stable three-form. In fact, the proof holds literally for all signatures since all it uses is the first compatibility condition we have just proved. □

Corollary 2.6. *Let M be a real analytic six-manifold with a half-flat $H^{\varepsilon,\tau}$-structure that is given by a pair of analytic stable forms (ω_0, ρ_0).*

- *(i) Then, there exists a unique maximal solution (ω, ρ) of the evolution equations* (2.5), (2.6) *with initial value (ω_0, ρ_0), which is defined on an open neighbourhood $\Omega \subset \mathbb{R} \times M$ of $\{0\} \times M$. In particular, there is a parallel $G^{\varepsilon,\tau}$-structure on Ω.*
- *(ii) Moreover, the evolution is natural in the sense that, given a diffeomorphism f of M, the pullback $(f^*\omega, f^*\rho)$ of the solution with initial value (ω_0, ρ_0) is the solution of the evolution equations for the initial value $(f^*\omega_0, f^*\rho_0)$.*

  In particular, if f is an automorphism of the initial structure (ω_0, ρ_0), then, for all $t \in \mathbb{R}$, f is an automorphism of the solution $(\omega(t), \rho(t))$ defined on the (possibly empty) open set $U_t = \{p \in M \mid (t,p) \in \Omega$ and $(t, f(p)) \in \Omega\}$.
- *(iii) Furthermore, assume that M is compact or a homogeneous space $M = G/K$ such that the $H^{\varepsilon,\tau}$-structure is G-invariant. Then there is a unique maximal interval $I \ni 0$ and a unique solution (ω, ρ) of the evolution equations* (2.5), (2.6) *with initial value (ω_0, ρ_0) on $I \times M$. In particular, there is a parallel $G^{\varepsilon,\tau}$-structure on $I \times M$.*

Proof. If the manifold and the initial structure (ω_0, ρ_0) are analytic, there exists a unique maximal solution of the evolution equations on a neighbourhood Ω of $M \times \{0\}$ in $M \times \mathbb{R}$ by the Cauchy-Kovalevskaya theorem. The naturality of the solution is an immediate consequence of the uniqueness due to the naturality of the exterior derivative. If M is compact, there is a maximal interval I such that the solution is defined on $M \times I$. The same is true for a homogeneous half-flat structure (ω_0, ρ_0) as it is determined by $(\omega_0, \rho_0)_{|p}$ for any $p \in M$. □

We remark that, for a homogeneous half-flat structure (ω_0, ρ_0), the evolution equations reduce to a system of ordinary differential equations due to the naturality assertion of the corollary. This simplification will be used in Section 4.3 to construct metrics with holonomy equal to G_2 and G_2^*.

2.2. Remark on completeness: geodesically complete conformal G_2-metrics. The $\mathrm{G}_2^{(*)}$-metrics arising from the Hitchin flow on a six-manifold N are of the form $(I\times N, dt^2+g_t)$ with an open interval $I=(a,b)$ and a family of Riemannian metrics g_t depending on $t\in I$ (formula (2.10) in Theorem 2.3). As curves of the form $t\mapsto (t,x)$ are geodesics for this metric, they are obviously geodesically incomplete if a or $b\in\mathbb{R}$.

For the *Riemannian* case and *compact* manifolds N, we shall explain how one easily obtains *complete* metrics by a conformal change of the G_2-metric.

LEMMA 2.7. *Let N be a compact manifold with a family g_r of Riemannian metrics. Then the Riemannian metric on $\mathbb{R}\times N$ defined by $h=dr^2+g_r$ is geodesically complete.*

PROOF. Denote by d the distance on $\mathbb{R}\times N$ induced by the Riemannian metric $h=dr^2+g_r$ and by d_r the distance on N induced by g_r. For a curve γ in $M=\mathbb{R}\times N$ we have that the length of $\gamma(t)=(r(t),x(t))$ satisfies

$$\ell(\gamma)=\int_0^1\sqrt{\dot r(t)^2+g_{r(t)}(\dot x(t),\dot x(t))}dt\ \geq\ \int_0^1|\dot r(t)|dt\ \geq\ |r(1)-r(0)|.$$

As the distance of two points $p=(r,x)$ and $q=(s,y)$ is defined as the infimum of the lengths of all curves joining them, this inequality implies that

$$d(p,q)\geq |r-s|. \tag{2.13}$$

Note also that a curve $\gamma(t)=((s-r)t+r,x)$ joining $p=(r,x)$ and $q=(s,x)$ in $\mathbb{R}\times\{x\}$ has length $\ell(\gamma)=|r-s|$ and thus, for such p, q we get that $d(p,q)=|r-s|$. On the other hand, for $p=(r,x)$ and $q=(r,y)$ with the same $\mathbb{R}$-projection r we only get that $d(p,q)\leq d_r(x,y)$.

Since h has Riemannian signature we can use the Hopf-Rinow Theorem and consider a Cauchy sequence $p_n=(r_n,x_n)\in\mathbb{R}\times N$ w.r.t. the distance d. Equation (2.13) then implies that the sequence r_n is a Cauchy sequence in $\mathbb{R}$. Hence, r_n converges to $r\in\mathbb{R}$. Since N is compact, the sequence x_n has a subsequence x_{n_k} converging to $x\in N$. For $p=(r,x)$ and $q_{n_k}:=(r,x_{n_k})$ the triangle inequality implies that

$$d(p,p_{n_k})\ \leq\ d\left(p,q_{n_k}\right)+d\left(q_{n_k},p_{n_k}\right)\ \leq\ d_r(x,x_{n_k})+d\left(q_{n_k},p_{n_k}\right)\ =\ d_r(x,x_{n_k})+|r-r_{n_k}|.$$

Hence, p_{n_k} converges to p. As p_n was a Cauchy sequence, we have found p as a limit for p_n. By the Theorem of Hopf and Rinow, M is geodesically complete. □

The consequence of the lemma is

PROPOSITION 2.8. *Let $(M=I\times N, h=dt^2+g_t)$ be a Riemannian metric on a product of an open interval I and a compact manifold N. Then (M,h) is globally conformally equivalent to a metric on $\mathbb{R}\times N$ that is geodesically complete. The scaling factor depends only on $t\in I$ and is determined by a diffeomorphism $\varphi:\mathbb{R}\to I$.*

PROOF. Let $\varphi:\mathbb{R}\to I$ be a diffeomorphism with inverse $r=\varphi^{-1}$. Changing the coordinate t to r, the metric h on $I\times N$ can be written as

$$h\ =\ \left(\varphi'(r)dr\right)^2+g_{\varphi(r)}\ =\ \varphi'(r)^2\left(dr^2+\frac{1}{\varphi'(r)^2}g_{\varphi(r)}\right).$$

Hence, h is globally conformally equivalent to the metric $dr^2+\frac{1}{\varphi'(r)^2}g_{\varphi(r)}$ on $\mathbb{R}\times N$. By the lemma, this metric is geodesically complete. □

Regarding the solution of the Hitchin flow equations, using Theorem 2.3, Corollary 2.6, and Proposition 2.8 we obtain the following consequence.

COROLLARY 2.9. *Let M be a compact analytic six-manifold with half-flat $\mathrm{SU}(3)$-structure given by analytic stable forms (ρ_0,ω_0). Then there is a complete metric on $\mathbb{R}\times M$ that is globally conformal to the parallel G_2-metric obtained by the Hitchin flow.*

In Example 4.15 of Section 4.3 we will construct explicit examples of this type. Finally, note that due to the Cheeger-Gromoll splitting Theorem, see for example [**Bes**, Theorem 6.79], one cannot expect to obtain by the Hitchin flow irreducible G_2-metrics that are complete without allowing degenerations of g_t.

2.3. Nearly half-flat structures and nearly parallel $\mathrm{G}_2^{(*)}$ -structures. A $\mathrm{G}_2^{(*)}$-structure φ on a seven-manifold N is called *nearly parallel* if

$$d\varphi = \mu *_\varphi \varphi \tag{2.14}$$

for a constant $\mu \in \mathbb{R}^*$. Nearly parallel G_2- and G_2^*-structures are also characterised by the existence of a Killing spinor, refer [**FKMS**] respectively [**Ka1**].

By Proposition 1.14, a $\mathrm{G}_2^{(*)}$-structure on a seven-manifold (N,φ) induces an $H^{\varepsilon,\tau}$-structure (ω,ρ) on an oriented hypersurface in (N,φ). If the $\mathrm{G}_2^{(*)}$-structure is nearly parallel, the $H^{\varepsilon,\tau}$-structure satisfies the equation $d\rho = -\varepsilon\mu\hat{\omega}$ due to the formulas (1.22) and (1.26). This observation motivates the following definition.

Definition 2.10. An $H^{\varepsilon,\tau}$-structure (ω,ρ) on a six-manifold M is called *nearly half-flat* if

$$d\rho = \frac{\lambda}{2}\omega^2 = \lambda\sigma \tag{2.15}$$

for some constant $\lambda \in \mathbb{R}^*$.

The notion of a nearly half-flat SU(3)-structure was introduced in [**FIMU**], where also evolution equations on six-manifolds leading to nearly parallel G_2-structures are considered. For compact manifolds M, it is shown in [**St**] that a solution which is a nearly half-flat SU(3)-structure for a time $t = t_0$ already defines a nearly parallel G_2-structure. In the following, we extend these evolution equations to all possible signatures and give a simplified proof for the properties of the solutions which also holds for non-compact manifolds.

Proposition 2.11. *Let $H^{\varepsilon,\tau}$ be a real form of $\mathrm{SL}(3,\mathbb{C})$, $G^{\varepsilon,\tau}$ the corresponding real form of $\mathrm{G}_2^{\mathbb{C}}$ and (ρ,ω) a one-parameter family of $H^{\varepsilon,\tau}$-structures on a six-manifold M with a parameter t from an interval I. Then, the three-form*

$$\varphi = \omega \wedge dt + \rho$$

defines a nearly parallel $G^{\varepsilon,\tau}$-structure for the constant $\mu \neq 0$ on $M \times I$ if and only if the $H^{\varepsilon,\tau}$-structure (ρ,ω) is nearly half-flat for the constant $-\varepsilon\mu$ for all $t \in I$ and satisfies the evolution equation

$$\dot{\rho} \;=\; d\omega - \varepsilon\mu\hat{\rho}. \tag{2.16}$$

Proof. The assertion follows directly from the following computation, analogously to the proof of Proposition 2.2:

$$\begin{aligned} \check{d}\varphi &= d\omega \wedge dt + dt \wedge \dot{\rho} + d\rho \;=\; (d\omega - \dot{\rho}) \wedge dt + d\rho, \\ \mu * \varphi &= \varepsilon\mu\,(\hat{\rho} \wedge dt - \sigma)\,. \end{aligned}$$

□

The main theorem for the parallel case generalises as follows. Recall (1.3) that for a stable four-form $\sigma = \frac{1}{2}\omega^2 = \hat{\omega}$, the application of the operator $\sigma \mapsto \hat{\sigma}$ yields the stable two-form

$$\hat{\hat{\omega}} = \hat{\sigma} = \frac{1}{2}\omega.$$

Theorem 2.12. *Let (ρ_0,ω_0) be a nearly half-flat $H^{\varepsilon,\tau}$-structure for the constant $\lambda \neq 0$ on a six-manifold M. Let M be oriented such that $\omega_0^3 > 0$. Furthermore, let $\rho \in \Omega^3 M$ be a one-parameter family of stable forms with parameters coming from an interval I such that $\rho(t_0) = \rho_0$ and such that the evolution equation*

$$\dot{\rho} \;=\; \frac{2}{\lambda} d(\widehat{d\rho}) + \lambda\,\hat{\rho} \tag{2.17}$$

is satisfied for all $t \in I$. Then $(\rho, \omega = \frac{2}{\lambda}\widehat{d\rho})$ is a family of nearly half-flat $H^{\varepsilon,\tau}$-structures for the constant λ. In particular, the three-form

$$\varphi = \omega \wedge dt + \rho$$

defines a nearly parallel $G^{\varepsilon,\tau}$-structure for the constant $-\varepsilon\lambda$ on $M \times I$.

Proof. First of all, we observe that $d\rho$ is stable in a neighbourhood of the stable form $d\rho_0 = \lambda\sigma_0$, since stability is an open condition. Furthermore, the operator $d\rho \mapsto \widehat{d\rho}$ is uniquely defined by the orientation induced from ω_0. Therefore, the evolution equation is locally well-defined and we assume that ρ is a solution on an interval I. The only possible candidate for a nearly half-flat structure for the constant λ is $(\rho, \omega = \frac{2}{\lambda}\widehat{d\rho})$ since only this two-form ω satisfies the nearly half-flat equation $\sigma = \hat{\omega} = \frac{1}{\lambda} d\rho$. Obviously, it holds

$$d\sigma = 0 = d\omega \wedge \omega. \tag{2.18}$$

By Proposition 2.11, it only remains to show that this pair of stable forms defines an $H^{\varepsilon,\tau}$-structure, or equivalently, that the compatibility conditions (2.1) and (2.2) are preserved in time. By taking the exterior derivative of the evolution equation, we find

$$\dot{\sigma} = \frac{1}{\lambda} d\dot{\rho} = d\hat{\rho} \tag{2.19}$$

which is in fact the second evolution equation of the parallel case. Completely analogous to the parallel case, the following computation implies the first compatibility condition:

$$\begin{aligned} \frac{\partial}{\partial t}(\sigma_X \wedge \rho) &= \dot{\sigma}_X \wedge \rho + \sigma_X \wedge \dot{\rho} \\ &\overset{(2.17),(2.19)}{=} (d\hat{\rho})_X \wedge \rho + \sigma_X \wedge d\omega + \lambda\, \sigma_X \wedge \hat{\rho} \\ &\overset{(2.12),(2.15)}{=} \hat{\rho} \wedge (d\rho)_X + \omega_X \wedge \omega \wedge d\omega + (d\rho)_X \wedge \hat{\rho} \\ &\overset{(2.18)}{=} 0. \end{aligned}$$

The proof of the second compatibility condition in [**H1**] again holds literally since the term $\hat{\rho} \wedge \dot{\rho} = \hat{\rho} \wedge d\omega$ is the same as in the case of the parallel evolution. $\square$

The system (2.17) of second order in ρ can easily be reformulated into a system of first order in (ω, ρ) to which we can apply the Cauchy-Kovalevskaya theorem. Indeed, a solution (ω, ρ) of the system

$$\dot{\rho} = d\omega + \lambda\hat{\rho}\ , \qquad \dot{\sigma} = d\hat{\rho}, \tag{2.20}$$

with nearly half-flat initial value $(\omega(t_0), \rho(t_0))$ is nearly half-flat for all t and also satisfies the system (2.17). Conversely, (2.17) implies (2.20) with $\sigma = \hat{\omega} = \frac{1}{\lambda} d\rho$.

Therefore, for an initial nearly half-flat structure which satisfies assumptions analogous to those of Corollary 2.6, we obtain existence, uniqueness and naturality of a solution of the system (2.20), or, equivalently, of (2.17).

2.4. Cocalibrated $\mathrm{G}_2^{(*)}$-structures and parallel $\mathrm{Spin}(7)$- and $\mathrm{Spin}_0(3,4)$-structures. In [**H1**], another evolution equation is introduced which relates cocalibrated G_2-structures on compact seven-manifolds M to parallel Spin(7)-structures. As before, we generalise the evolution equation to non-compact manifolds and indefinite metrics.

As we have already seen in Section 1.4, the stabiliser in $\mathrm{GL}(V)$ of a four-form Φ_0 on an eight-dimensional vector space V is $\mathrm{Spin}(7)$ or $\mathrm{Spin}_0(3,4)$ if and only if it can be written as in (1.29) for a stable three-form φ on a seven-dimensional subspace with stabiliser G_2- or G_2^*, respectively. Thus, a $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure on an eight-manifold M is defined by a four-form $\Phi \in \Omega^4 M$ such that $\Phi_p \in \Lambda^4 T_p^* M$ has this property for all p. By formula (1.30) for the metric g_Φ induced by Φ, an oriented hypersurface in (M, Φ) with spacelike unit normal vector field n with respect to g_Φ carries a natural G_2- or G_2^*-structure, respectively, defined by $\varphi = n \lrcorner \Phi$.

A $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure Φ is *parallel* if and only if $d\Phi = 0$. We remark that the proof for the Riemannian case given in [**Sa**, Lemma 12.4] is not hard to transfer to the indefinite case when considering [**Br**, Proposition 2.5] and using the complexification of the two spin groups.

Due to this fact, the induced $\mathrm{G}_2^{(*)}$-structure φ on an oriented hypersurface in an eight-manifold M with parallel $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure Φ is *cocalibrated*, i.e. it satisfies

$$d *_\varphi \varphi = 0. \tag{2.21}$$

Conversely, a cocalibrated $\mathrm{G}_2^{(*)}$-structure can be embedded in an eight-manifold with parallel $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure as follows.

THEOREM 2.13. *Let M be a seven-manifold and $\varphi \in \Omega^3 M$ be a one-parameter family of stable three-forms with a parameter t in an interval I satisfying the evolution equation*

$$\frac{\partial}{\partial t}(*_\varphi \varphi) \quad = \quad d\varphi. \tag{2.22}$$

If φ is cocalibrated at $t = t_0 \in I$, then φ defines a family of cocalibrated G_2- or $\mathrm{G}_2^{(*)}$-structures for all $t \in I$. Moreover, the four-form

$$\Phi = dt \wedge \varphi + *_\varphi \varphi \tag{2.23}$$

defines a parallel $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structure on $M \times I$, respectively, which induces the metric

$$g_\Phi = g_\varphi + dt^2. \tag{2.24}$$

PROOF. Since the time derivative of $d * \varphi$ vanishes when inserting the evolution equation, the family stays cocalibrated if it is cocalibrated at an initial value. As before, we denote by $\check{d}$ the exterior differential on $\check{M} := M \times I$ and differentiate the four-form (2.23):

$$\check{d}\Phi = -dt \wedge d\varphi + d(*\varphi) + dt \wedge \frac{\partial}{\partial t}(*\varphi).$$

Obviously, this four-form is closed if and only the evolution equation is satisfied and the family is cocalibrated. The formula for the induced metric corresponds to formula (1.30). $\square$

As before, the Cauchy-Kovalevskaya theorem guarantees existence and uniqueness of solutions if assumptions analogous to those of Corollary 2.6 are satisfied.

REMARK 2.14. We observe that nearly parallel G_2- and G_2^*-structures are in particular cocalibrated such that analytic nearly half-flat structures in dimension six can be embedded in parallel $\mathrm{Spin}(7)$- or $\mathrm{Spin}_0(3,4)$-structures in dimension eight by evolving them twice with the help of the Theorems 2.12 and 2.13.

3. Evolution of nearly ε-Kähler manifolds

In this section, we consider the evolution of nearly pseudo-Kähler and nearly para-Kähler six-manifolds which can be unified by the notion of a nearly ε-Kähler manifold. The explicit solution of the Hitchin flow yields a simple and unified proof for the correspondence of nearly ε-Kähler manifolds and parallel $\mathrm{G}_2^{(*)}$-structures on cones. We complete the picture by considering similarly the evolution of nearly Kähler structures to nearly parallel $\mathrm{G}_2^{(*)}$-structures on (hyperbolic) sine cones and the evolution of nearly parallel $\mathrm{G}_2^{(*)}$-structures to parallel $\mathrm{Spin}(7)$- and $\mathrm{Spin}_0(3,4)$-structures on cones. Our presentation in terms of differential forms unifies various results in the literature, which were originally obtained using spinorial methods, and applies to all possible real forms of the relevant groups.

3.1. Cones over nearly ε-Kähler manifolds. In the language of [**AC2**] and [**SSH**], an *almost ε-Hermitian manifold* (M^{2m}, g, J) is defined by an almost ε-complex structure J which squares to $\varepsilon\mathrm{id}$ and a pseudo-Riemannian metric g which is ε-Hermitian in the sense that $g(J\cdot, J\cdot) = -\varepsilon g(\cdot,\cdot)$. Consequently, a *nearly ε-Kähler manifold* is defined as an almost ε-Hermitian manifold such that ∇J is skew-symmetric. On a six-manifold M, a nearly ε-Kähler structure (g, J, ω) with $|\nabla J|^2 = 4$ (i.e. of constant type 1 in the terminology of [**G2**]) is equivalent to a normalised $H^{\varepsilon,\tau}$-structure (ω, ρ) which satisfies

$$d\omega \quad = \quad 3\rho\,, \tag{3.1}$$

$$d\hat{\rho} \quad = \quad 4\hat{\omega}. \tag{3.2}$$

This result is well-known for Riemannian signature [**RC**] and is generalised to arbitrary signature in [**SSH**, Theorem 3.14]. In particular, nearly ε-Kähler structures (ω, ρ) in dimension six are half-flat and the structure $(\omega, \hat{\rho})$ is nearly half-flat (for the constant $\lambda = 4$).

PROPOSITION 3.1. *Let (M, h_0) be a pseudo-Riemannian six-manifold of signature $(6,0)$, $(4,2)$ or $(3,3)$ and let $(\bar{M} = M \times \mathbb{R}^+, \bar{g}_\varepsilon = h_0 - \varepsilon dt^2)$ be the timelike cone for $\varepsilon = 1$ and the spacelike cone for $\varepsilon = -1$. There is a one-to-one correspondence between nearly ε-Kähler structures (h_0, J) with $|\nabla J|^2 = 4$ on (M, h_0) and parallel G_2- and G_2^*-structures φ on $\bar{M}$ which induce the cone metric $\bar{g}_\varepsilon$.*

PROOF. This well-known fact is usually proved using Killing spinors, see [**B**], [**Gru**] and [**Ka2**]. We give a proof relying exclusively on the framework of stable forms and the Hitchin flow. For Riemannian signature, this point of view is also adopted in [**ChSa**] and [**Bu**].

The $H^{\varepsilon,\tau}$-structures inducing the given metric h_0 are the reductions of the bundle of orthonormal frames of (M, h_0) to the respective group $H^{\varepsilon,\tau}$. Given any $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0, we consider for $t \in \mathbb{R}^+$ the one-parameter family

$$\omega = t^2 \omega_0\,,\ \rho = t^3 \rho_0, \tag{3.3}$$

which induces the family of metrics $h = t^2 h_0$. By formula (2.10), the metric g_φ on $\bar{M}$ induced by the stable three-form $\varphi = \omega \wedge dt + \rho$ is exactly the cone metric $\bar{g}_\varepsilon$.

It is easily verified that the family (3.3) consists of half-flat structures satisfying the evolution equations if and only if the initial value $(\omega(1), \rho(1)) = (\omega_0, \rho_0)$ satisfies the exterior system (3.1), (3.2). Therefore, the stable three-form φ on the cone $(\bar{M}, \bar{g}_\varepsilon)$ is parallel if and only if the $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0 is a nearly ε-Kähler structure with $|\nabla J|^2 = 4$.

Conversely, let φ be a stable three-form on $\bar{M}$ which induces the cone metric $\bar{g}_\varepsilon$. Since ∂_t is a normal vector field for the hypersurface $M = M \times \{1\}$ satisfying $\bar{g}(\partial_t, \partial_t) = -\varepsilon$, we obtain an $H^{\varepsilon,\tau}$-reduction (ω_0, ρ_0) of h_0 defined by

$$\omega_0 = \partial_t \lrcorner\, \varphi\,, \qquad \rho_0 = \varphi_{|TM} \tag{3.4}$$

with the help of Proposition 1.14. Since the two constructions are inverse to each other, the proposition follows. $\square$

EXAMPLE 3.2. Consider the flat $(\mathbb{R}^{(3,4)} \setminus \{0\}, \langle .,. \rangle)$ which is isometric to the cone $(M^\varepsilon \times \mathbb{R}^+, t^2 h_\varepsilon - \varepsilon dt^2)$ over the pseudo-spheres $M^\varepsilon := \{p \in \mathbb{R}^{(3,4)} \,|\, \langle p, p\rangle = -\varepsilon\}$, $\varepsilon = \pm 1$, with the standard metrics h_ε of constant sectional curvature $-\varepsilon$ and signature $(2,4)$ for $\varepsilon = -1$ and $(3,3)$ for $\varepsilon = 1$. Obviously, a stable three-form φ inducing the flat metric $\langle .,. \rangle$ is parallel if and only if it is constant. Thus, the previous discussion and Proposition 1.14, in particular formula (1.28), yield a bijection

$$\begin{array}{rcl} \mathrm{SO}(3,4)/\,\mathrm{G}_2^* & \to & \{\varepsilon\text{-complex structures } J \text{ on } M^\varepsilon \text{ such that } (h_\varepsilon, J) \text{ is nearly } \varepsilon\text{-Kähler}\} \\ \varphi & \mapsto & J \qquad \text{with} \quad J_p(v) = -p \times v, \qquad \forall\, p \in M^\varepsilon \end{array}$$

where the cross-product $\times$ induced by φ is defined by formula (1.12). In other words, the pseudo-spheres $(M^\varepsilon, h_\varepsilon)$ admit a nearly ε-Kähler structure which is unique up to conjugation by the isometry group $\mathrm{O}(3,4)$ of h_ε. In fact, these ε-complex structures on the pseudo-spheres are already considered in [**Li**] and the nearly para-Kähler property for $\varepsilon = 1$ is for instance shown in [**Be**].

3.2. Sine cones over nearly ε-Kähler manifolds. For Riemannian signature, it has been shown in [**FIMU**] that the evolution of a nearly Kähler SU(3)-structure to a nearly parallel G_2-structure induces the Einstein sine cone metric. This result can be extended as follows. We prefer to consider (hyperbolic) cosine cones since they are defined on all of $\mathbb{R}$ in the hyperbolic case.

PROPOSITION 3.3. *Let (M, h_0) be a pseudo-Riemannian six-manifold.*

(i) If h_0 is Riemannian, or has signature $(2,4)$, respectively, there is a one-to-one correspondence between nearly (pseudo-)Kähler structures (h_0, J) on M with $|\nabla J|^2 = 4$ and nearly parallel G_2-structures, or G_2^*-structures, respectively, for the constant $\mu = -4$ on the spacelike cosine cone

$$(M \times (-\frac{\pi}{2}, \frac{\pi}{2}),\ \cos^2(t) h_0 + dt^2).$$

(ii) If h_0 has signature $(3,3)$, there is a one-to-one correspondence between nearly para-Kähler structures (h_0, J) on M with $|\nabla J|^2 = 4$ and nearly parallel G_2^*-structures for the constant $\mu = 4$ on the timelike hyperbolic cosine cone

$$(M \times \mathbb{R}, -\cosh^2(t)h_0 - dt^2).$$

Proof. (i) Starting with any SU(3)- or SU(1,2)-reduction (ω_0, ρ_0) of h_0, the one-parameter family

$$\omega = \cos^2(t)\omega_0 \, , \; \rho = -\cos^3(t)(\sin(t)\rho_0 + \cos(t)\hat{\rho}_0)$$

with $(\omega(0), \rho(0)) = (\omega_0, -\hat{\rho}_0)$ defines a stable three-form $\varphi = \omega \wedge dt + \rho$ on $M \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $z\Psi_0 = z(\rho_0 + i\hat{\rho}_0)$ is a $(3,0)$-form w.r.t. the induced almost complex structures $J_{\mathrm{Re}\,(z\Psi_0)}$ for all $z \in \mathbb{C}^*$, the structure $J_\rho = J_{\rho_0}$ is constant in t. Thus, the metric g_φ induced by φ is the cosine cone metric. Moreover, it holds $\hat{\rho} = -\cos^3(t)(\sin(t)\hat{\rho}_0 - \cos(t)\rho_0)$ due to Corollary 1.7.

It takes a short calculation to verify that the one-parameter family is nearly half-flat (for the constant $\lambda = -4$) and satisfies the evolution equation (2.16) if and only (ω_0, ρ_0) satisfies the exterior system (3.1), (3.2). Thus, applying Proposition 2.11, the three-form $\varphi = \omega \wedge dt + \rho$ defines a nearly parallel $G^{\varepsilon,\tau}$-structure on $M \times (-\frac{\pi}{2}, \frac{\pi}{2})$ (for the constant $\mu = -4$) if and only if (h_0, J_{ρ_0}) is nearly ε-Kähler with $|\nabla J|^2 = 4$.

The inverse construction is given by (3.4) in analogy to the case of the ordinary cone.

(ii) The proof in the para-complex case is completely analogous if we consider the one-parameter family

$$\omega = \cosh^2(t)\omega_0 \, , \; \rho = -\cosh^3(t)(\sinh(t)\rho_0 + \cosh(t)\hat{\rho}_0)$$

which is defined for all $t \in \mathbb{R}$. We note the following subtleties regarding signs. By Proposition 1.4, we know that the mapping $\rho \mapsto \hat{\rho}$ is homogeneous of degree 1, but not linear. Indeed, by applying Corollary 1.7, we find

$$\sinh(t)\widehat{\rho_0 + \cosh}(t)\hat{\rho}_0 = -\sinh(t)\hat{\rho}_0 - \cosh(t)\rho_0.$$

Using this formula, one can check that $J_\rho = J_{\hat{\rho}_0} = -J_{\rho_0}$ is constant in t such that the metric induced by (ω, ρ) is in fact $h = -\cosh^2(t)h_0$.

□

The fact that the (hyperbolic) cosine cone over a six-manifold carrying a Killing spinor carries again a Killing spinor was proven in [**Ka1**]. By relating spinors to differential forms, these results also imply the existence of a nearly parallel $\mathrm{G}_2^{(*)}$-structures on the (hyperbolic) cosine cone over a nearly ε-Kähler manifold.

Example 3.4. The (hyperbolic) cosine cone of the pseudo-spheres $(M^\varepsilon, h_\varepsilon)$ of Example 3.2 has constant sectional curvature 1, for instance due to [**ACGL**, Corollary 2.3], and is thus (locally) isometric to the pseudo-sphere $S^{3,4} = \{p \in \mathbb{R}^{(4,4)} \mid \langle p, p \rangle = 1\} = \mathrm{Spin}_0(3,4)/\,\mathrm{G}_2^*$.

3.3. Cones over nearly parallel $\mathrm{G}_2^{(*)}$-structures. By Lemma 9 in [**B**], there is a one-to-one correspondence on a Riemannian seven-manifold (M, g_0) between nearly parallel G_2-structures and parallel Spin(7)-structures on the Riemannian cone. In order to illustrate the evolution equations for nearly parallel G_2^*-structures, we extend this result to the indefinite case by applying Theorem 2.13. This is possible since nearly parallel G_2^*-structures are in particular cocalibrated. Again, the fact that the cone over a nearly parallel G_2^*-manifold admits a parallel spinor can be derived from the connection to Killing spinors as observed in [**Ka1**].

Proposition 3.5. *Let (M, g_0) be a pseudo-Riemannian seven-manifold of signature $(3,4)$. There is a one-to-one correspondence between nearly parallel G_2^*-structures for the constant 4 which induce the given metric g_0 and parallel $\mathrm{Spin}_0(3,4)$-structures on $M \times \mathbb{R}^+$ inducing the cone metric $\bar{g} = t^2 g_0 + dt^2$.*

Proof. Let φ_0 be any cocalibrated G_2^*-structure on M inducing the metric g_0. The one-parameter family of three-forms defined by $\varphi = t^3\varphi_0$ for $t \in \mathbb{R}^+$ induces the family of metrics $g = t^2 g_0$ such that the Hodge duals are $*_\varphi\varphi = t^4 *_{\varphi_0} \varphi_0$. By (2.24), the $\mathrm{Spin}_0(3,4)$-structure

$\Psi = dt \wedge \varphi + *_\varphi\varphi$ on $M \times \mathbb{R}^+$ induces the cone metric $\bar{g}$. Conversely, given a $\mathrm{Spin}_0(3,4)$-structure Ψ on the cone $(M \times \mathbb{R}^+, \bar{g})$, we have the cocalibrated G_2^*-structure $\varphi_0 = \partial_t \lrcorner \Psi$ on M, which also induces the given metric g_0. Since the evolution equation (2.22) is satisfied if and only if the initial value φ_0 is nearly parallel for the constant 4 and since the two constructions are inverse to each other, the assertion follows from Theorem 2.13. □

Example 3.6. We consider again the easiest example, i.e. the flat $\mathbb{R}^{(4,4)} \setminus \{0\}$ which is isometric to the cone over the pseudo-sphere $S^{3,4}$. Analogous to Example 3.2, the proposition just proved yields a proof of the fact that the nearly parallel G_2^*-structures for the constant 4 on $S^{3,4}$ are parametrised by $\mathrm{SO}(4,4)/\mathrm{Spin}_0(3,4)$, i.e. by the four homogeneous spaces (1.31). In particular, these structures are conjugated by the isometry group $\mathrm{O}(4,4)$ of $S^{3,4}$.

Summarising the application of the three Propositions 3.1, 3.3 and 3.5 to pseudo-spheres, we find a mutual one-to-one correspondence between

(1) nearly pseudo-Kähler structures with $|\nabla J|^2 \neq 0$ on $(S^{2,4}, g_{can})$,
(2) nearly para-Kähler structures with $|\nabla J|^2 \neq 0$ on $(S^{3,3}, g_{can})$,
(3) parallel G_2^*-structures on $(\mathbb{R}^{(3,4)}, g_{can})$,
(4) nearly parallel G_2^*-structures on the spacelike cosine cone over $(S^{2,4}, g_{can})$,
(5) nearly parallel G_2^*-structures on the timelike hyperbolic cosine cone over $(S^{3,3}, g_{can})$,
(6) nearly parallel G_2^*-structures on $(S^{3,4}, g_{can})$ and
(7) parallel $\mathrm{Spin}_0(3,4)$-structures on $(\mathbb{R}^{(4,4)}, g_{can})$.

This geometric correspondence is reflected in the algebraic fact that the four homogeneous spaces (1.31) are isomorphic.

4. The evolution equations on nilmanifolds $\Gamma \setminus H_3 \times H_3$

Let H_3 be the three-dimensional real Heisenberg group with Lie algebra $\mathfrak{h}_3$. In this section, we will develop a method to explicitly determine the parallel $\mathrm{G}_2^{(*)}$-structure induced by an arbitrary invariant half-flat structure on a nilmanifold $\Gamma \setminus H_3 \times H_3$ without integrating. In particular, this method is applied to construct three explicit large families of metrics with holonomy equal to G_2 or G_2^*, respectively.

4.1. Evolution of invariant half-flat structures on nilmanifolds. Left-invariant half-flat structures (ω_0, ρ_0) on a Lie group G are in one-to-one correspondence with normalised pairs (ω, ρ) of compatible stable forms on the Lie algebra $\mathfrak{g}$ of G which satisfy $d\rho = 0$ and $d\omega^2 = 0$. To shorten the notation, we will speak of a *half-flat structure on a Lie algebra*.

Given as initial value a half-flat structure on a Lie algebra, the evolution equations

$$\dot{\rho} = d\omega \ , \qquad \dot{\sigma} = d\hat{\rho} \ , \tag{4.1}$$

reduce to a system of ordinary differential equations and a unique solution exists on a maximal interval I. Due to the structure of the equation, the solution differs from the initial values by adding exact forms to σ_0 and ρ_0. In other words, an initial value (σ_0, ρ_0) evolves within the product $[\sigma_0] \times [\rho_0]$ of their respective Lie algebra cohomology classes.

Every nilpotent Lie group N with rational structure constants admits a cocompact lattice Γ and the resulting compact quotients $\Gamma \backslash N$ are called nilmanifolds. Recall that a geometric structure on a nilmanifold $\Gamma \backslash N$ is called *invariant* if is induced by a left-invariant geometric structure on N.

Explicit solutions of the Hitchin flow equations on several nilpotent Lie algebras can be found for instance in [**CF**] and [**AS**]. In both cases, a metric with holonomy contained in G_2 has been constructed before by a different method and this information is used to obtain the solution. For a symplectic half-flat initial value, another explicit solution on one of these Lie algebras is given in [**CT**]. In all cases, the solution depends only on one variable.

At least for four nilpotent Lie algebras including $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, a reason for the simple structure of the solutions has been observed in [**AS**]. Indeed, the following lemma shows that the evolution of σ takes place in a one-dimensional space. As usual, we define a nilpotent Lie algebra by giving the image of a basis of one-forms under the exterior derivative, see for instance [**Sa**]. The same reference also contains a list of all six-dimensional nilpotent Lie algebras.

LEMMA 4.1. *Let ρ be a closed stable three-form with dual three-form $\hat\rho$ on a six-dimensional nilpotent Lie algebra $\mathfrak{g}$.*

(i) If $\mathfrak{g}$ is one of the three Lie algebras

$$(0,0,0,0,e^{12},e^{34})\ ,\quad (0,0,0,0,e^{13}+e^{42},e^{14}+e^{23})\ ,\quad (0,0,0,0,e^{12},e^{14}+e^{23}),$$

then $d\hat\rho \in \Lambda^4 U$ for the four-dimensional kernel U of $d : \Lambda^1\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$.

(ii) If $\mathfrak{g}$ is the Lie algebra

$$(0,0,0,0,0,e^{12}+e^{34}),$$

then $d\hat\rho \in \Lambda^4 U$ for the four-dimensional subspace $U = \operatorname{span}\{e^1,e^2,e^3,e^4\}$ of $\ker d$.

REMARK 4.2. The assertion of the lemma is not true for the remaining six-dimensional nilpotent Lie algebras with $b_1 = \dim(\ker d) = 4$ or $b_1 = 5$. In each case, we have constructed a closed stable ρ such that $d\hat\rho$ is not contained in $\Lambda^4(\ker d)$.

In fact, this lemma can also be viewed as a corollary of the following lemma which we will prove first.

LEMMA 4.3. *Let ρ be a closed stable three-form on one of the four Lie algebras of Lemma 4.1 and let U be the four-dimensional subspace of $\ker d$ defined there. In all four cases, the space U is J_ρ-invariant where J_ρ denotes the almost (para-)complex structure induced by ρ.*

PROOF. For $\lambda(\rho) < 0$, the assertion is similar to that of [**AS**, Lemma 2]. However, since the only proof seems to be given for the Iwasawa algebra for integrable J in [**KeS**, Theorem 1.1], we give a complete proof.

Let $\mathfrak{g}$ be one of the three Lie algebras given in part (i) of Lemma 4.1 and $U = \ker d$. Obviously, the two-dimensional image of d lies within $\Lambda^2 U$ in all three cases. By $J = J_\rho$ we denote the almost (para-)complex structure associated to the closed stable three-form ρ. As before, we denote by $\varepsilon \in \{\pm 1\}$ the sign of $\lambda(\rho)$ such that $J_\rho = \varepsilon \mathrm{id}$. Let the symbol i_ε be defined by the property $\mathrm{i}_\varepsilon^2 = \varepsilon$ such that the para-complex numbers and the complex numbers can be unified by $\mathbb{C}_\varepsilon = \mathbb{R}[\mathrm{i}_\varepsilon]$. Thus, a $(1,0)$-form can be defined for both values of ε as an eigenform of J_ρ in $\Lambda^1\mathfrak{g}^* \otimes \mathbb{C}_\varepsilon$ for the eigenvalue i_ε.

We define the J-invariant subspace $W := U \cap J^*U$ of $\mathfrak{g}$ such that $2 \le \dim W \le 4$. In fact, $\dim W = 4$ is equivalent to the assertion. The other two cases are not possible, which can be seen as follows. To begin with, assume that W is two-dimensional. When choosing a complement W' of W in U, we have by definition of W that

$$V = W \oplus W' \oplus J^*W'.$$

We observe that, for $\varepsilon = 1$, the ± 1-eigenspaces of J restricted to $W' \oplus J^*W'$ are both two-dimensional. Therefore, we can choose for both values of ε a basis $\{e^1, e^2, e^3, e^4 = J^*e^1, e^5 = J^*e^2, e^6 = J^*e^3\}$ of V such that e^1, e^2, e^3 and e^4 are closed and $de^5, de^6 \in \Lambda^2 U$. Since $\rho + \mathrm{i}_\varepsilon J_\rho^*\rho$ is a (3,0)-form in both cases, it is possible to change the basis vectors e^1, e^4 within $W \subset \ker d$ such that

$$\rho + \mathrm{i}_\varepsilon J_\rho^*\rho = (e^1 + \mathrm{i}_\varepsilon e^4) \wedge (e^2 + \mathrm{i}_\varepsilon e^5) \wedge (e^3 + \mathrm{i}_\varepsilon e^6)$$

and thus

$$\rho = e^{123} + \varepsilon e^{156} - \varepsilon e^{246} + \varepsilon e^{345}.$$

By construction of the basis, we have that

$$0 = d\rho = -\varepsilon e^1 \wedge de^5 \wedge e^6 + \varepsilon e^1 \wedge e^5 \wedge de^6 + \alpha$$

with $\alpha \in \Lambda^4 U$. As the first two summands are linearly independent and not in $\Lambda^4 U$, we conclude that both $e^1 \wedge de^5$ and $e^1 \wedge de^6$ vanish. Thus, the closed one-form e^1 has the property that the wedge product of e^1 with any exact two-form vanishes. However, an inspection of the standard basis of each of the three Lie algebras in question reveals that such a one-form does not exist on these Lie algebras and we have a contradiction to $\dim W = 2$.

Since a J-invariant space cannot be three-dimensional for $\varepsilon = -1$, the proof is finished for this case. However, if $\varepsilon = 1$, the case $\dim W = 3$ cannot be excluded that easy. Assuming that it is in fact $\dim W = 3$, we choose again a complement W' of W in U and find a decomposition

$$V = W \oplus W' \oplus J^*W' \oplus W''$$

with $J^*W'' = W''$. Without restricting generality, we can assume that J acts trivially on W''. Then, we find a basis for V such that the $+1$-eigenspace of J is spanned by $\{e^1, e^4 + e^5, e^6\}$ and the -1-eigenspace by $\{e^2, e^3, e^4 - e^5\}$, where e^1, e^2, e^3 and e^4 are closed and $e^5 = J^*e^4$. Since the given closed three-form ρ generates this J, it has to be of the form

$$\rho = ae^1 \wedge (e^4 + e^5) \wedge e^6 + be^{23} \wedge (e^4 - e^5)$$

for two real constants a, b. The vanishing exterior derivative

$$d\rho = ae^1 \wedge d(e^{56}) \qquad \mathrm{mod} \quad \Lambda^4 U$$

leads to the same contradiction as in the first case and part (i) is shown.

In fact, the same arguments apply to the Lie algebra of part (ii). The four-dimensional space $U \subset \ker d$ spanned by $\{e^1, ..., e^4\}$ also satisfies $\mathrm{im}\, d \subset \Lambda^2 U$. Going through the above arguments, the only difference is that e^5 or e^6 may be closed. However, at least one of them is not closed and its image under d generates the exact two-forms. Again, there is no one-form $\beta \in U$ such that $\beta \wedge \gamma = 0$ for all exact two-forms γ and the arguments given in part (i) lead to contradictions for both $\dim W = 2$ and $\dim W = 3$. □

Proof of Lemma 4.1. Let ρ be a closed stable three-form on one of the four nilpotent Lie algebras and $U \subset \ker d$ as defined in the lemma. For both values of ε, we can apply Lemma 4.3 and choose two linearly independent closed $(1,0)$-forms E^1 and E^2 within the J_ρ-invariant space $U \otimes \mathbb{C}_\varepsilon$. Considering that $\rho + \mathrm{i}_\varepsilon \hat{\rho}$ is a $(3,0)$-form for both values of ε, there is a third $(1,0)$-form E^3 such that $\rho + \mathrm{i}_\varepsilon \hat{\rho} = \mathrm{E}^{123}$. Since $d\rho = 0$ and $\mathrm{im}\, d \subset \Lambda^2 U$, it follows that the exterior derivative

$$d\hat{\rho} = \varepsilon \mathrm{i}_\varepsilon d(E^{123}) = \varepsilon \mathrm{i}_\varepsilon E^{12} \wedge dE^3$$

is an element of $\Lambda^4 U$. □

4.2. Left-invariant half-flat structures on $H_3 \times H_3$. From now on, we focus on the Lie algebra $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathfrak{h}_3$. Apart from describing all half-flat structures on this Lie algebra, i.e. all initial values for the evolution equations, we give various explicit examples and prove a strong rigidity result concerning the induced metric.

Obviously, pairs of compatible stable forms on a Lie algebra which are isomorphic by a Lie algebra automorphism induce equivalent $H^{\varepsilon,\tau}$-structures on the corresponding simply connected Lie group. Thus, we derive, to begin with, a normal form modulo Lie algebra automorphisms for stable two-forms $\omega \in \Lambda^2\mathfrak{g}^*$ which satisfy $d\omega^2 = 0$.

A basis $\{e_1, e_2, e_3, f_1, f_2, f_3\}$ for $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ such that the only non-vanishing Lie brackets are given by

$$de^3 = e^{12}, \qquad df^3 = f^{12},$$

will be called a *standard basis*. The connected component of the automorphism group of the Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ in the standard basis is

$$\mathrm{Aut}_0(\mathfrak{h}_3 \oplus \mathrm{h}_3) = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ a^t & \det(A) & c^t & 0 \\ 0 & 0 & B & 0 \\ d^t & 0 & b^t & \det(B) \end{pmatrix}, \ \mathrm{A, B} \in \mathrm{GL}(2, \mathbb{R}), \ \mathrm{a, b, c, d} \in \mathbb{R}^2 \right\}. \tag{4.2}$$

We denote by $\mathfrak{g}_i$, $i = 1, 2$, the two summands, by $\mathfrak{z}_i$ their centres and by $\mathfrak{z}$ the centre of $\mathfrak{g}$. The annihilator of the centre is $\mathfrak{z}^0 = \ker \mathrm{d}$ and similarly for the summands by restricting d. We have the decompositions

$$\begin{aligned}
\mathfrak{g}^* &\cong \mathfrak{z}_1^0 \oplus \mathfrak{z}_2^0 \oplus \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0} \oplus \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0},\\
\Lambda^2\mathfrak{g}^* &\cong \Lambda^2(\mathfrak{z}^0) \oplus \underbrace{(\frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0} \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_1} \oplus \underbrace{(\mathfrak{z}_1^0 \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_2} \oplus \underbrace{(\mathfrak{z}_2^0 \wedge \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0})}_{\mathfrak{k}_3} \oplus \underbrace{(\mathfrak{z}_1^0 \wedge \frac{\mathfrak{g}_1^*}{\mathfrak{z}_1^0}) \oplus (\mathfrak{z}_2^0 \wedge \frac{\mathfrak{g}_2^*}{\mathfrak{z}_2^0})}_{\mathfrak{k}_4}.
\end{aligned}$$

By $\omega^{\mathfrak{k}_i}$ we denote the projection of a two-form ω onto one of the spaces $\mathfrak{k}_i$, $i = 1, 2, 3, 4$, defined as indicated in the decomposition. We observe that $\mathfrak{k}_1 = \Lambda^2(\frac{\mathfrak{g}^*}{\mathfrak{z}^0})$ and $\omega^{\mathfrak{k}_1} = 0$ if and only if $\omega(\mathfrak{z}, \mathfrak{z}) = 0$.

Lemma 4.4. *Consider the action of* $\mathrm{Aut}(\mathfrak{h}_3 \oplus \mathfrak{h}_3)$ *on the set of non-degenerate two-forms* ω *on* $\mathfrak{g}$ *with* $d\omega^2 = 0$. *The orbits modulo rescaling are represented in a standard basis by the following two-forms:*

$$\begin{array}{lcl}
\omega_1 = & e^1f^1 + e^2f^2 + e^3f^3, & \textit{if } \omega^{\mathfrak{k}_1} \neq 0,\\
\omega_2 = & e^2f^2 + e^{13} + f^{13}, & \textit{if } d\omega = 0 \iff \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} = 0,\ \omega^{\mathfrak{k}_3} = 0,\\
\omega_3 = & e^1f^3 + e^2f^2 + e^3f^1, & \textit{if } \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} \neq 0,\ \omega^{\mathfrak{k}_3} \neq 0,\ \omega^{\mathfrak{k}_4} = 0,\\
\omega_4 = & e^1f^3 + e^2f^2 + e^3f^1 + e^{13} + \beta f^{13}, & \textit{if } \omega^{\mathfrak{k}_1} = 0,\ \omega^{\mathfrak{k}_2} \neq 0,\ \omega^{\mathfrak{k}_3} \neq 0,\ \omega^{\mathfrak{k}_4} \neq 0,\\
\omega_5 = & e^1f^3 + e^2f^2 + e^{13} + f^{13} & \textit{otherwise,}
\end{array}$$

where $\beta \in \mathbb{R}$ *and* $\beta \neq -1$.

Proof. Let
$$\omega = \sum \alpha_i e^{(i+1)(i+2)} + \sum \beta_i f^{(i+1)(i+2)} + \sum \gamma_{i,j} e^i f^j$$
be an arbitrary non-degenerate two-form expressed in a standard basis. We will give in each case explicitly a change of standard basis by an automorphism of the form (4.2) with the notation

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},\ a^t = (a_5, a_6),\ B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix},\ b^t = (b_5, b_6),\ c^t = (c_1, c_2),\ d^t = (d_1, d_2).$$

First of all, if $\omega^{\mathfrak{k}_1} \neq 0$, the term $\gamma_{3,3}e^3f^3$ is different from zero and we rescale such that $\gamma_{3,3} = 1$. Then, the application of the change of basis

$$\begin{aligned}
a_1 &= 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1,\ a_5 = -\gamma_{1,3},\ a_6 = -\gamma_{2,3},\\
b_1 &= \gamma_{2,2} - \gamma_{2,3}\gamma_{3,2} - \alpha_1\beta_1,\ b_2 = -\gamma_{1,2} - \beta_1\alpha_2 + \gamma_{3,2}\gamma_{1,3},\ b_3 = -\gamma_{2,1} + \gamma_{3,1}\gamma_{2,3} - \beta_2\alpha_1,\\
b_4 &= \gamma_{1,1} - \alpha_2\beta_2 - \gamma_{1,3}\gamma_{3,1},\ b_5 = -\gamma_{3,1}\gamma_{2,2} + \gamma_{3,1}\alpha_1\beta_1 + \gamma_{3,2}\gamma_{2,1} + \gamma_{3,2}\beta_2\alpha_1,\\
b_6 &= \gamma_{3,1}\gamma_{1,2} + \gamma_{3,1}\beta_1\alpha_2 - \gamma_{3,2}\gamma_{1,1} + \gamma_{3,2}\alpha_2\beta_2,\\
c_1 &= \beta_2\gamma_{2,2} - \beta_2\gamma_{2,3}\gamma_{3,2} + \beta_1\gamma_{2,1} - \beta_1\gamma_{3,1}\gamma_{2,3},\ d_1 = -\alpha_2,\\
c_2 &= -\beta_2\gamma_{1,2} + \beta_2\gamma_{3,2}\gamma_{1,3} - \beta_1\gamma_{1,1} + \beta_1\gamma_{1,3}\gamma_{3,1},\ d_2 = \alpha_1,
\end{aligned}$$

transforms ω into $\tilde{\omega} = \tilde{\gamma}_{1,1}(e^1f^1 + e^2f^2 + e^3f^3) + \tilde{\alpha}_3 e^{12} + \tilde{\beta}_3 f^{12}$, $\tilde{\gamma}_{1,1} \neq 0$. This two-form satisfies $d\tilde{\omega}^2 = 0$ if and only if $\tilde{\alpha}_3 = 0, \tilde{\beta}_3 = 0$ and the normal form ω_1 is achieved by rescaling.

Secondly, the vanishing of $d\omega$ corresponds to $\omega^{\mathfrak{k}_1} = 0$, $\omega^{\mathfrak{k}_2} = 0$, $\omega^{\mathfrak{k}_3} = 0$ or $\gamma_{3,3} = \gamma_{1,3} = \gamma_{2,3} = \gamma_{3,1} = \gamma_{3,2} = 0$ in a standard basis. By non-degeneracy, at least one of α_1 and α_2 is not zero and we can always achieve $\alpha_1 = 0, \alpha_2 \neq 0$. Indeed, if $\alpha_1 \neq 0$, we apply the transformation (4.2) with $a_1 = 1$, $a_2 = 1$, $a_4 = \frac{\alpha_2}{\alpha_1}$, $B = \mathbb{1}$ and all remaining entries zero. With an analogous argument, we can assume that $\beta_1 = 0$, $\beta_2 \neq 0$. Since $\gamma_{2,2} \neq 0$ by non-degeneracy, we can rescale ω such that $\gamma_{2,2} = 1$. Now, the transformation of the form (4.2) given by

$$\begin{aligned}
&a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\beta_2,\ b_1 = 1,\ b_2 = 0,\ b_3 = 0,\ b_4 = -\alpha_2,\ a_5 = 0,\\
&a_6 = -\frac{\alpha_3\beta_2}{\alpha_2},\ b_5 = 0,\ b_6 = -\frac{\alpha_2\beta_3}{\beta_2},\ c_1 = \frac{\gamma_{1,1}}{\alpha_2},\ c_2 = -\gamma_{1,2},\ d_1 = 0,\ d_2 = \gamma_{2,1},
\end{aligned}$$

maps ω to a multiple of the normal form ω_2.

Thirdly, we assume that ω is non-degenerate with $\omega^{\mathfrak{k}_1} = 0$, i.e. $\gamma_{3,3} = 0$ and both $\omega^{\mathfrak{k}_2} \neq 0$, i.e. $\gamma_{1,3}$ or $\gamma_{2,3} \neq 0$, and $\omega^{\mathfrak{k}_3} \neq 0$, i.e. $\gamma_{3,1}$ or $\gamma_{3,2} \neq 0$. Similar as before, we can achieve $\gamma_{2,3} = 0$,

$\gamma_{1,3} \neq 0$ by applying, if $\gamma_{2,3} \neq 0$, the transformation (4.2) with $a_1 = 1$, $a_2 = 1$, $a_4 = -\frac{\gamma_{1,3}}{\gamma_{2,3}}$, $B = \mathbb{1}$ and all remaining entries zero. Analogously, we can assume $\gamma_{3,2} = 0$, $\gamma_{3,1} \neq 0$ and rescaling yields $\gamma_{2,2} = 1$, which is non-zero by non-degeneracy. After this simplification, the condition $d\omega^2 = 0$ implies that $\alpha_1 = \beta_1 = 0$ and the transformation

$$\begin{aligned}
a_1 &= 1,\ a_2 = 0,\ a_3 = \frac{\alpha_2\beta_3 - \gamma_{3,1}\gamma_{1,2}}{\gamma_{3,1}},\ a_4 = \gamma_{1,3},\ a_5 = 0,\ a_6 = 0,\ b_1 = 1,\ b_2 = 0,\\
b_3 &= \frac{\beta_2\alpha_3 - \gamma_{1,3}\gamma_{2,1}}{\gamma_{1,3}},\ b_4 = \gamma_{3,1},\ b_5 = \frac{\gamma_{1,2}\gamma_{1,3}\gamma_{2,1}\gamma_{3,1} - \gamma_{1,1}\gamma_{1,3}\gamma_{3,1} - \alpha_2\alpha_3\beta_2\beta_3}{\gamma_{1,3}^2\gamma_{3,1}},\ b_6 = 0,\\
c_1 &= 0,\ c_2 = \beta_3,\ d_1 = 0,\ d_2 = -\alpha_3,
\end{aligned}$$

maps ω to $\tilde{\omega} = e^1 f^3 + e^2 f^2 + e^3 f^1 + \tilde{\alpha}_2 e^{31} + \tilde{\beta}_2 f^{31}$. The condition $\omega^{\mathfrak{k}_4} = 0$ corresponds to $\tilde{\alpha}_2 = 0$, $\tilde{\beta}_2 = 0$, i.e. normal form ω_3. If $\omega^{\mathfrak{k}_4} \neq 0$, we can achieve $\tilde{\alpha}_2 \neq 0$ by possibly changing the summands. Now, the transformation

$$\begin{gathered}
a_1 = 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{1}{\tilde{\alpha}_2},\ a_5 = 0,\ a_6 = 0,\ c_1 = 0,\ c_2 = 0,\\
b_1 = -\frac{1}{\tilde{\alpha}_2},\ b_2 = 0,\ b_3 = 0,\ b_4 = -\frac{1}{\tilde{\alpha}_2},\ b_5 = 0,\ b_6 = 0,\ d_1 = 0,\ d_2 = 0,
\end{gathered}$$

maps $\tilde{\omega}$ to the fourth normal form ω_4.

The cases that remain are $\omega^{\mathfrak{k}_1} = 0$ and either $\omega^{\mathfrak{k}_2} \neq 0, \omega^{\mathfrak{k}_3} = 0$ or $\omega^{\mathfrak{k}_3} = 0, \omega^{\mathfrak{k}_2} \neq 0$. After changing the summands if necessary, we can assume $\omega^{\mathfrak{k}_3} = 0$ and $\omega^{\mathfrak{k}_2} \neq 0$, i.e. $\gamma_{3,1} = \gamma_{3,2} = \gamma_{3,3} = 0$ and at least one of $\gamma_{1,3}$ or $\gamma_{2,3}$ non-zero. As before, we can achieve $\gamma_{2,3} = 0$ by the transformation $a_1 = 1, a_2 = 1, a_4 = -\frac{\gamma_{1,3}}{\gamma_{2,3}}$. Evaluating $d\omega^2 = 0$ yields $\alpha_1 = 0$. Now, non-degeneracy enforces that $\beta_1 \neq 0$ or $\beta_2 \neq 0$, and after another similar transformation $\beta_1 = 0$. Finally, the simplified ω is non-degenerate if and only if $\gamma_{2,2}\alpha_2\beta_2 \neq 0$ and, after rescaling such that $\gamma_{2,2} = 1$, the transformation

$$\begin{aligned}
a_1 &= 1,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{\gamma_{1,3}^2}{\beta_2},\ a_5 = 0,\ a_6 = \frac{\gamma_{1,3}^2(\gamma_{1,3}\gamma_{2,1} - \alpha_3\beta_2)}{\alpha_2\beta_2^2},\\
b_1 &= -\frac{\gamma_{1,3}}{\beta_2},\ b_2 = 0,\ b_3 = 0,\ b_4 = -\alpha_2,\ b_5 = 0,\ b_6 = -\frac{\beta_3\alpha_2}{\beta_2},\\
c_1 &= 0,\ c_2 = -\frac{\gamma_{1,2}\beta_2 + \gamma_{1,3}\beta_3}{\beta_2},\ d_1 = -\frac{\gamma_{1,1}}{\beta_2},\ d_2 = \frac{\gamma_{1,3}^2\gamma_{2,1}}{\beta_2^2},
\end{aligned}$$

maps ω to a multiple of the fifth normal form ω_5. □

Using this lemma, it is possible to describe all half-flat structures (ω, ρ) on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ as follows. In a fixed standard basis such that ω is in one of the normal forms, the equations $d\rho = 0$ and $\omega \wedge \rho = 0$ are linear in the coefficients of an arbitrary three-form ρ. Thus, it is straightforward to write down all compatible closed three-forms for each normal form which depend on nine parameters in each case. The stable forms in this nine-dimensional space are parametrised by the complement of the zero-set of the polynomial $\lambda(\rho)$ of order four. One parameter is eliminated when we require a stable ρ to be normalised in the sense of (1.18). We remark that the computation of the induced tensors J_ρ, $\hat{\rho}$ and $g_{(\omega,\rho)}$ may require computer support, in particular, the signature of the metric is not obvious. However, stability is an open condition: If a single half-flat structure (ω_0, ρ_0) is explicitly given such that ω_0 is one of the normal forms, then the eight-parameter family of normalised compatible closed forms defines a deformation of the given half-flat structure (ω_0, ρ_0) in some neighbourhood of (ω_0, ρ_0).

For instance, the closed three-forms which are compatible with the first normal form

$$\omega = e^1 f^1 + e^2 f^2 + e^3 f^3 \tag{4.3}$$

in a standard basis can be parametrised as follows:

$$\begin{aligned}
\rho = \rho(a_1, \ldots, a_9) &= a_1\, e^{123} + a_2\, f^{123} + a_3\, e^1 f^{23} + a_4\, e^2 f^{13} + a_5\, e^{23} f^1 + a_6\, e^{13} f^2\\
&+ a_7\, (e^2 f^{23} - e^1 f^{13}) + a_8\, (e^{12} f^3 - e^3 f^{12}) + a_9\, (e^{23} f^2 - e^{13} f^1).
\end{aligned} \tag{4.4}$$

The quartic invariant $\lambda(\rho)$ depending on the nine parameters is

$$\begin{aligned}\lambda(\rho) &= (2a_6a_4a_8^2+2a_1a_2a_8^2+2a_8^2a_3a_5-4a_5a_7^2a_6-4a_9^2a_4a_3-4a_9^2a_2a_8+4a_7^2a_8a_1\\ &+ 4a_7a_8^2a_9+a_1^2a_2^2+a_6^2a_4^2+a_3^2a_5^2+a_8^4-2a_6a_4a_3a_5+4a_5a_7a_9a_3+4a_9a_4a_6a_7\\ &- 4a_5a_2a_6a_8+4a_4a_8a_1a_3-4a_9a_2a_1a_7-2a_1a_2a_6a_4-2a_1a_2a_3a_5)\,(\mathrm{e}^{123}f^{123})^{\otimes 2}.\end{aligned}$$

Example 4.5. For each possible signature, we give an explicit normalised half-flat structure with fundamental two-form (4.3). The first and the third example appear in [**SH**]. To begin with, the closed three-form

$$\rho = \frac{1}{\sqrt{2}}(e^{123}-f^{123}-e^1f^{23}+e^{23}f^1-e^2f^{31}+e^{31}f^2-e^3f^{12}+e^{12}f^3) \tag{4.5}$$

induces a half-flat SU(3)-structure (ω,ρ) such that the standard basis is orthonormal. Similarly, the closed three-form

$$\rho = \frac{1}{\sqrt{2}}(e^{123}-f^{123}-e^1f^{23}+e^{23}f^1+e^2f^{31}-e^{31}f^2+e^3f^{12}-e^{12}f^3) \tag{4.6}$$

induces a half-flat $\mathrm{SU}(1,2)$-structure (ω,ρ) such that the standard basis is pseudo-orthonormal with e_1 and e_4 being spacelike. Finally, the closed three-form

$$\rho = \sqrt{2}\,(e^{123}+f^{123}), \tag{4.7}$$

induces a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure (ω,ρ) such that the two $\mathfrak{h}_3$-summands are the eigenspaces of the para-complex structure J_ρ, which is integrable since also $d\hat{\rho}=0$. The induced metric is

$$g = 2\left(e^1\cdot e^4+e^2\cdot e^5+e^3\cdot e^6\right).$$

In fact, half-flat structures with Riemannian metrics are only possible if ω belongs to the orbit of the first normal form.

Lemma 4.6. *Let (ω,ρ) be a half-flat* SU(3)*-structure on $\mathfrak{h}_3\oplus\mathfrak{h}_3$. Then it holds $\omega^{\mathfrak{k}_1}\neq 0$. In particular, there is a standard basis such that $\omega=\omega_1=e^1f^1+e^2f^2+e^3f^3$.*

Proof. Suppose that (ω,ρ) is a half-flat SU(3)-structure on $\mathfrak{h}_3\oplus\mathfrak{h}_3$ with $\omega^{\mathfrak{k}_1}=0$. Thus, we can choose a standard basis such that ω is in one of the normal forms $\omega_2,\ldots,\omega_5$ of Lemma 4.3 and ρ belongs to the corresponding nine-parameter family of compatible closed three-forms. We claim that the basis one-form e^1 is isotropic in all four cases which yields a contradiction since the metric of an SU(3)-structure is positive definite. The quickest way to verify the claim is the direct computation of the induced metric, which depends on nine parameters, with the help of a computer. In order to verify the assertion by hand, the following formulas shorten the calculation considerably. For all one-forms α, β and all vectors v, the ε-complex structure J_ρ and the metric g induced by a compatible pair (ω,ρ) of stable forms satisfy

$$\begin{aligned}\alpha\wedge J_\rho^*\beta\wedge\omega^2 &= g(\alpha,\beta)\frac{1}{3}\omega^3,\\ J_\rho^*\alpha(v)\phi(\rho) &= \alpha\wedge\rho\wedge(v\lrcorner\rho),\end{aligned}$$

which is straightforward to verify in the standard basis (1.7), (1.19), cf. also [**SH**, Lemmas 2.1, 2.2]. For instance, for the second normal form ω_2, it holds $e^1\wedge\omega_2^2=-2e^{12}f^{123}$. Thus, by the first formula, it suffices to show that $J_\rho^*e^1(e_3)=e^1(J_\rho e_3)=0$ which is in turn satisfied if $\mathrm{e}^1\wedge\rho\wedge(e_3\lrcorner\rho)=0$ due to the second formula. A similar simplification applies to the other normal forms and we omit the straightforward calculations. □

Moreover, the geometry turns out to be very rigid if $\omega^{\mathfrak{k}_1}=0$. We recall that simply connected para-hyper-Kähler symmetric spaces with abelian holonomy are classified in [**ABCV**], [**C**]. In particular, there exists a unique simply connected four-dimensional para-hyper-Kähler symmetric space with one-dimensional holonomy group, which is defined in [**ABCV**], Section 4. We denote the underlying pseudo-Riemannian manifold as (N^4,g_{PHK}).

PROPOSITION 4.7. *Let (ω, ρ) be a left-invariant half-flat structure with $\omega^{\mathfrak{k}_1} = 0$ on $H_3 \times H_3$ and let g be the pseudo-Riemannian metric induced by (ω, ρ). Then, the pseudo-Riemannian manifold $(H_3 \times H_3, g)$ is either flat or isometric to the product of (N^4, g_{PHK}) and a two-dimensional flat factor. In particular, the metric g is Ricci-flat.*

PROOF. Due to the assumption $\omega^{\mathfrak{k}_1} = 0$, we can choose a standard basis such that ω is in one of the normal forms $\omega_2, \dots, \omega_5$. In each case separately, we do the following. We write down all compatible closed three-forms ρ depending on nine parameters. With computer support, we calculate the induced metric g. For the curvature considerations, it suffices to work up to a constant such that we can ignore the rescaling by $\lambda(\rho)$ which is different from zero by assumption. Now, we transform the left-invariant co-frame $\{e^1, \dots, f^3\}$ to a coordinate co-frame $\{dx_1, \dots, dy_3\}$ by applying the transformation defined by

$$e^1 = dx_1,\ e^2 = dx_2,\ e^3 = dx_3 + x_1 dx_2,\ f^1 = dy_1,\ f^2 = dy_2,\ f^3 = dy_3 + y_1 dy_2, \tag{4.8}$$

such that the metric is accessible for any of the numerous packages computing curvature. The resulting curvature tensor $R \in \Gamma(\mathrm{End}\,\Lambda^2 TM)$, $M = H^3 \times H^3$, has in each case only one non-trivial component

$$R(\partial_{x_1} \wedge \partial_{y_1}) = c\,\partial_{x_3} \wedge \partial_{y_3} \tag{4.9}$$

for a constant $c \in \mathbb{R}$ and R is always parallel. Thus, the metric is flat if $c = 0$ and symmetric with one-dimensional holonomy group if $c \neq 0$, for $H_3 \times H_3$ is simply connected and a naturally reductive homogeneous metric is complete.

Furthermore, it turns out that the metric restricted to $TN := span\{\partial_{x_1}, \partial_{x_3}, \partial_{y_1}, \partial_{y_3}\}$ is non-degenerate and of signature $(2, 2)$ for all parameter values. Thus, the manifold splits in a four-dimensional symmetric factor with neutral metric and curvature tensor (4.9) and the two-dimensional orthogonal complement which is flat. Since a simply connected symmetric space is completely determined by its curvature tensor and the four-dimensional para-hyper-Kähler symmetric space (N^4, g_{PHK}) has the same signature and curvature tensor, the four-dimensional factor is isometric to (N^4, g_{PHK}). Finally, the metric g is Ricci-flat since g_{PHK} is Ricci-flat. □

EXAMPLE 4.8. The following examples define half-flat normalised $\mathrm{SU}(1,2)$-structures with $\omega^{\mathfrak{k}_1} = 0$ in a standard basis. None of the examples is flat. Thus, the four structures are equivalent as $\mathrm{SO}(2,4)$-structures due to Proposition 4.7, but the examples show that the geometry of the reduction to $\mathrm{SU}(1,2)$ is not as rigid.

$$\begin{aligned}
\omega = \omega_2\,, \quad \rho &= e^{12}f^3 + \sqrt{2}e^{13}f^2 + e^1 f^{23} + e^{23}f^1 - e^3 f^{12} + \sqrt{2}f^{123},\\
g &= -(e^2)^2 - (f^2)^2 + 2\,e^1{\cdot}e^3 - 2\sqrt{2}\,e^1{\cdot}f^3 + 2\sqrt{2}\,e^3{\cdot}f^1 - 2\,f^1{\cdot}f^3,\\
&\quad \text{(Ricci-flat pseudo-Kähler since } d\omega = 0,\ d\hat{\rho} = 0);\\
\omega = \omega_3\,, \quad \rho &= e^{123} + e^{12}f^3 + e^{13}f^2 + e^1 f^{12} - 2e^1 f^{23} + e^2 f^{13} - e^3 f^{12},\\
g &= -(e^2)^2 - 2\,(f^2)^2 + 2\,e^1{\cdot}f^1 + 2\,e^1{\cdot}f^3 + 2\,e^2{\cdot}f^2 - 2\,e^3{\cdot}f^1 - 2\,f^1{\cdot}f^3,\\
&\quad (d\omega \neq 0,\ J_\rho \text{ integrable since } d\hat{\rho} = 0);\\
\omega = \omega_4\,, \quad \rho &= \beta\,e^{12}f^3 - \beta\,e^{13}f^2 + \beta\,e^1 f^{23} + \frac{\beta+1}{\beta^3}\,e^{23}f^1 + \frac{\beta^4-\beta-1}{\beta^3}\,e^2 f^{13}\\
&\quad -\beta\,e^3 f^{12} - (\beta^2 + 2\beta)\,f^{123}, \qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
g &= -\frac{1}{\beta^2}\,(e^2)^2 - \beta^2\,(f^2)^2 + 2\beta^2\,e^1{\cdot}f^3 - \frac{2}{\beta^2(\beta+1)}\,e^3{\cdot}f^1 - \frac{2(\beta^4+\beta+1)}{\beta^2}\,f^1{\cdot}f^3;\\
\omega = \omega_5\,, \quad \rho &= e^{12}f^3 + e^{13}f^2 - e^1 f^{23} + e^{23}f^1 - e^3 f^{12} + f^{123}, \qquad (d\omega \neq 0,\ d\hat{\rho} \neq 0),\\
g &= -(e^2)^2 - 2\,(f^2)^2 + 2\,e^1{\cdot}e^3 + 2\,e^2{\cdot}f^2 + 2\,f^1{\cdot}f^3.
\end{aligned}$$

Example 4.9. Moreover, we give examples of half-flat normalised $\mathrm{SL}(3,\mathbb{R})$-structures with $\omega^{\mathfrak{k}_1}=0$. Again, none of the structures is flat.

$$\begin{array}{rcll}
\omega=\omega_2\,, \quad \rho &=& \sqrt{2}\,(\,e^1f^{23}+e^{23}f^1\,), & (d\omega=0,\ d\hat\rho=0),\\
g &=& 2\,e^1{\cdot}e^3-2\,e^2{\cdot}f^2-2\,f^1{\cdot}f^3; &\\
\omega=\omega_3\,, \quad \rho &=& \sqrt{2}\,(\,e^{12}f^3+e^{13}f^2+e^1f^{12}-e^3f^{12}\,), & (d\omega\neq 0,\ d\hat\rho\neq 0),\\
g &=& -2\,(e^1)^2+2\,e^1{\cdot}e^3-2\,e^1{\cdot}f^3+2\,e^2{\cdot}f^2-2\,f^1{\cdot}f^3; &\\
\omega=\omega_4\,, \quad \rho &=& -\sqrt{2\beta+2}\,(\,e^{12}f^3-e^1f^{23}+e^2f^{13}-e^3f^{12}), & (d\omega\neq 0,\ d\hat\rho\neq 0),\\
g &=& -2\,(f^2)^2+2\,e^1{\cdot}e^3+2\,e^1{\cdot}f^3+2\,e^2{\cdot}f^2-2\,e^3{\cdot}f^1-(2\beta+4)\,f^1{\cdot}f^3; &\\
\omega=\omega_5\,, \quad \rho &=& \sqrt{2}\,(\,e^{123}+f^{123}\,), & (d\omega\neq 0,\ d\hat\rho=0),\\
g &=& 2\,e^1{\cdot}f^3+2\,e^2{\cdot}f^2+2\,e^3{\cdot}f^1. &
\end{array}$$

4.3. Solving the evolution equations on $H_3\times H_3$. Due to the preparatory work of the Lemmas 4.1 and 4.3, it turns out to be possible to explicitly evolve every half-flat structure on $\mathfrak{h}_3\oplus\mathfrak{h}_3$ without integrating.

Proposition 4.10. *Let (ω_0,ρ_0) be any half-flat $H^{\varepsilon,\tau}$-structure on $\mathfrak{h}_3\oplus\mathfrak{h}_3$ with $\omega_0^{\mathfrak{k}_1}=0$. Then, the solution of the evolution equations* (4.1) *is affine linear in the sense that*

$$\sigma(t)=\sigma_0+t\,d\hat\rho_0\,,\qquad \rho(t)=\rho_0+t\,d\omega_0 \tag{4.10}$$

and is well-defined for all $t\in\mathbb{R}$.

Proof. Let $\{e_1,\dots,f_3\}$ be a standard basis such that ω_0 is in one of the normal forms $\omega_2,\dots,\omega_5$ of Lemma 4.3 which satisfy $\omega_0^{\mathfrak{k}_1}=0$. By Lemma 4.1 and the second evolution equation, we know that there is a function $y(t)$ with $y(0)=0$ such that

$$\sigma(t)=\sigma_0+y(t)e^{12}f^{12}=\frac{1}{2}\omega_0^2+y(t)e^{12}f^{12}.$$

For each of the four normal forms, the unique two-form $\omega(t)$ with $\frac{1}{2}\omega(t)^2=\sigma(t)$ and $\omega(0)=\omega_0$ is

$$\omega(t)=\omega_0-y(t)e^1f^1.$$

However, the two-form e^1f^1 is closed such that the exterior derivative $d\omega(t)=d\omega_0$ is constant. Therefore, we have $\rho(t)=\rho_0+t\,d\omega_0$ by the first evolution equation. Moreover, the two-form $\omega(t)$ is stable for all $t\in\mathbb{R}$ since it holds $\phi(\omega(t))=\phi(\omega_0)$ for each of the normal forms and for all $t\in\mathbb{R}$. It remains to show that $d\hat\rho(t)$ is constant in all four cases which implies that the function $y(t)$ is linear by the second evolution equation.

As explained in section 4.2, it is easy to write down, for each normal form ω_0 separately, all compatible, closed three-forms ρ_0, which depend on nine parameters. For $\rho(t)=\rho_0+t\,d\omega_0$, we verify with the help of a computer that $\lambda(\rho(t))=\lambda(\rho_0)$ is constant such that $\rho(t)$ is stable for all $t\in\mathbb{R}$ since ρ_0 is stable. When we also calculate $J_{\rho(t)}$ and $\hat\rho(t)=J^*_{\rho(t)}\rho(t)$, it turns out in all four cases that $d\hat\rho(t)$ is constant. This finishes the proof. □

We cannot expect that this affine linear evolution of spaces which have one-dimensional holonomy, due to Proposition 4.7, yields metrics with full holonomy G_2^*. Indeed, due to the following result the geometry does not change significantly compared to the six-manifold.

Corollary 4.11. *Let (ω_0,ρ_0) be a half-flat $H^{\varepsilon,\tau}$-structure on $\mathfrak{h}_3\oplus\mathfrak{h}_3$ with $\omega_0^{\mathfrak{k}_1}=0$ and let g_φ be the Ricci-flat metric induced by the parallel stable three-form φ on $M\times\mathbb{R}$ defined by the solution* (4.10) *of the evolution equations with initial value (ω_0,ρ_0). Then, the pseudo-Riemannian manifold $(M\times\mathbb{R},g_\varphi)$ is either flat or isometric to the product of the four-dimensional para-hyper-Kähler symmetric space (N^4,g_{PHK}) and a three-dimensional flat factor.*

Proof. By formula (1.23), the metric g_φ is determined by the time-dependent metric $g(t)$ induced by $(\omega(t),\rho(t))$. All assertions follow from the analysis of the curvature of g_φ completely analogous to the proof of Proposition 4.7. □

The situation changes completely when we consider the first normal form ω_1 of Lemma 4.3.

PROPOSITION 4.12. *Let (ω_0, ρ_0) be any normalised half-flat $H^{\varepsilon,\tau}$-structure on $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ with $\omega_0^{\mathfrak{k}_1} \neq 0$. There is always a standard basis $\{e_1, \dots, f_3\}$ such that $\omega_0 = e^1 f^1 + e^2 f^2 + e^3 f^3$. In such a basis, we define $(\omega(x), \rho(x))$ by*

$$\begin{aligned} \rho(x) &= \rho_0 + x(e^{12} f^3 - e^3 f^{12}), \\ \omega(x) &= 2\,(\varepsilon\kappa(x))^{-\frac{1}{2}} \left(\frac{1}{4}\varepsilon\kappa(x)\, e^1 f^1 + \frac{1}{4}\varepsilon\kappa(x)\, e^2 f^2 + e^3 f^3 \right), \end{aligned}$$

where $\kappa(x)\,(e^{123} f^{123})^{\otimes 2} = \lambda(\rho(x))$. Furthermore, let I be the maximal interval containing zero such that the polynomial $\kappa(x)$ of order four does not vanish for any $x \in I$. The parallel stable three-form (2.9) on $M \times I$ obtained by evolving (ω_0, ρ_0) along the Hitchin flow (4.1) is

$$\varphi = \frac{1}{2}\sqrt{\varepsilon\kappa(x)}\,\omega(x) \wedge dx + \rho(x).$$

The metric induced by φ, which has holonomy contained in $G^{\varepsilon,\tau}$, is by (2.10) given as

$$g_\varphi = g(x) - \frac{1}{4}\kappa(x) dx^2, \tag{4.11}$$

where $g(x)$ denotes the metric associated to $(\omega(x), \rho(x))$ via (1.6) and (1.17). The variable x is related to the parameter t of the Hitchin flow by the ordinary differential equation (4.14).

PROOF. Since $\omega_0^{\mathfrak{k}_1} \neq 0$, we can always choose a standard basis such that $\omega_0 = e^1 f^1 + e^2 f^2 + e^3 f^3$ is in the first normal form of Lemma 4.3. Then ρ_0 is of the form (4.4).

Moreover, by Lemma 4.1, there is a function $y(t)$ which is defined on an interval containing zero and satisfies $y(0) = 0$ such that the solution of the second evolution equation can be written

$$\sigma(t) = \sigma_0 + y(t) e^{12} f^{12}.$$

The unique $\omega(t)$ that satisfies $\omega(0) = \omega_0$ and $\frac{1}{2}\omega(t)^2 = \sigma(t)$ for all t is

$$\omega(t) = \sqrt{1 - y(t)}\; e^1 f^1 + \sqrt{1 - y(t)}\; e^2 f^2 + \frac{1}{\sqrt{1 - y(t)}}\; e^3 f^3.$$

Since

$$d\omega(t) = \frac{1}{\sqrt{1 - y(t)}} (e^{12} f^3 - e^3 f^{12}), \tag{4.12}$$

there is another function $x(t)$ with $x(0) = 0$ such that the solution of the first evolution equation can be written

$$\rho(t) = \rho_0 + x(t)(e^{12} f^3 - e^3 f^{12}). \tag{4.13}$$

This three-form is compatible with $\omega(t)$ for all t, as one can easily see from (4.4). Furthermore, the solution is normalised by Theorem 2.3, which implies

$$\sqrt{\varepsilon\lambda(\rho(t))} = \phi(\rho(t)) = 2\phi(\omega(t)) = -2\sqrt{1 - y(t)}\, e^{123} f^{123}.$$

Hence, we can eliminate $y(t)$ by

$$y(t) = 1 - \frac{1}{4}\varepsilon\kappa(x(t)).$$

We remark that the normalisation of $\rho_0 = \rho(0)$ corresponds to $\kappa(0) = 4\varepsilon$. Comparing (4.12) and (4.13), the evolution equations are equivalent to the single ordinary differential equation

$$\dot{x} = \frac{2}{\sqrt{\varepsilon\kappa(x(t))}} \tag{4.14}$$

for the only remaining parameter $x(t)$. In fact, we do not need to solve this equation in order to compute the parallel $\mathrm{G}_2^{(*)}$-form when we substitute the coordinate t by x via the local diffeomorphism $x(t)$ satisfying $dt = \frac{1}{2}\sqrt{\varepsilon\kappa(x(t))}\; dx$. Inserting all substitutions into the formulas (2.9) and (2.10) for the stable three-form φ on $M \times I$ and the induced metric g_φ, all assertions of the proposition follow immediately from Theorem 2.3. □

EXAMPLE 4.13. The invariant $\kappa(x)$ and the induced metric $g(x)$ for the three explicit half-flat structures of Example 4.5 are the following.
If (ω_0, ρ_0) is the SU(3)-structure (4.5), it holds

$$\begin{aligned}
\kappa(x) &= (x-\sqrt{2})^3(x+\sqrt{2}), \qquad I=(-\sqrt{2},\sqrt{2}),\\
g(x) &= (1-\tfrac{1}{2}\sqrt{2}\,x)\left((e^1)^2+(e^2)^2-4\kappa(x)^{-1}(e^3)^2+(e^4)^2+(e^5)^2-4\kappa(x)^{-1}(e^6)^2\right)\\
&+ \sqrt{2}\,x(1-\tfrac{1}{2}\sqrt{2}\,x)\left(e^1\cdot e^4+e^2\cdot e^5+4\kappa(x)^{-1}e^3\cdot e^6\right).
\end{aligned}$$

If (ω_0, ρ_0) is the SU(1, 2)-structure (4.6), we have

$$\begin{aligned}
\kappa(x) &= (x-\sqrt{2})(x+\sqrt{2})^3, \qquad I=(-\sqrt{2},\sqrt{2}),\\
g(x) &= (1+\tfrac{1}{2}\sqrt{2}\,x)\left((e^1)^2-(e^2)^2+4\kappa(x)^{-1}(e^3)^2+(e^4)^2-(e^5)^2+4\kappa(x)^{-1}(e^6)^2\right)\\
&- \sqrt{2}\,x(1+\tfrac{1}{2}\sqrt{2}\,x)\left(e^1\cdot e^4+e^2\cdot e^5+4\kappa(x)^{-1}e^3\cdot e^6\right).
\end{aligned}$$

And for the $SL(3,\mathbb{R})$-structure (4.7), it holds

$$\begin{aligned}
\kappa(x) &= (2+x^2)^2, \qquad I=\mathbb{R},\\
g(x) &= (2+x^2)\left(e^1\cdot e^4+e^2\cdot e^5\right)+4(2-x^2)\kappa(x)^{-1}e^3\cdot e^6+4\sqrt{2}\,x\kappa(x)^{-1}\left((e^3)^2-(e^6)^2\right).
\end{aligned}$$

THEOREM 4.14. *Let $(\omega(x), \rho(x))$ be the solution of the Hitchin flow with one of the three half-flat structures (ω_0, ρ_0) of Example 4.5 as initial value (see Proposition 4.12 for the explicit solution and Example 4.13 for the corresponding metric $g(x)$, defined for $x \in I$).*

Then, the holonomy of the metric g_φ on $M \times I$ defined by formula (4.11) *equals* G_2 *for the* SU(3)*-structure* (ω_0, ρ_0) *and* G_2^* *for the other two structures.*

Moreover, restricting the eight-parameter family of half-flat structures given by (4.4) *to a small neighbourhood of the initial value (ρ_0, ω_0) yields in each case an eight-parameter family of metrics of holonomy equal to* G_2 *or* G_2^*.

PROOF. For all three cases, we can apply the transformation (4.8) and calculate the curvature R of the metric g_φ defined by (4.11). Carrying this out with the package "tensor" contained in Maple 10, we obtained that the rank of the curvature viewed as endomorphism on two-vectors is 14. This implies that the holonomy of g_φ in fact equals G_2 or G_2^*.

The assertion for the eight-parameter family is an immediate consequence. Indeed, by construction, the rank of the curvature endomorphism is bounded from above by 14 and being of maximal rank is an open condition. □

To conclude this section we address the issue of completeness and use the Riemannian family in Example 4.13 and Corollary 2.9 to construct a complete conformally parallel G_2-metric on $\mathbb{R} \times (\Gamma\backslash H_3 \times H_3)$.

EXAMPLE 4.15. Let H_3 be the Heisenberg group and $N = \Gamma\backslash H_3 \times H_3$ be a compact nilmanifold given by a lattice Γ. Let us denote by $x : I \to (-\sqrt{2}, \sqrt{2})$ the maximal solution to the equation

$$\dot{x}(t) = \frac{2}{\sqrt{(\sqrt{2}-x(t))^3(x(t)+\sqrt{2})}},$$

with initial condition $x(0) = 0$, defining the t-dependent family of Riemannian metrics

$$\begin{aligned}
g_t &= \frac{\sqrt{2}-x(t)}{\sqrt{2}}\left((e^1)^2+(e^2)^2+(e^4)^2+(e^5)^2\right)+x(t)\left(\sqrt{2}-x(t)\right)\left(e^1\cdot e^4+e^2\cdot e^5\right)\\
&+\frac{2\sqrt{2}}{(\sqrt{2}-x(t))^2(x(t)+\sqrt{2})}\left((e^3)^2+(e^6)^2\right)-\frac{4x(t)}{(\sqrt{2}-x(t))^2(x(t)+\sqrt{2})}e^3\cdot e^6.
\end{aligned}$$

If $\varphi : \mathbb{R} \to I$ is a diffeomorphism, then the metric

$$dr^2 + \frac{1}{\varphi'(r)^2}g_{\varphi(r)}$$

is globally conformally parallel G_2 and geodesically complete.

5. Special geometry of real forms of the symplectic $\mathrm{SL}(6,\mathbb{C})$-module $\wedge^3\mathbb{C}^6$

Homogeneous projective special Kähler manifolds of semisimple groups with possibly indefinite metric and compact stabiliser were classified in [**AC1**]. This includes the case of manifolds with (positive or negative) definite metrics, for which the stabiliser is automatically compact. Projective special Kähler manifolds with negative definite metric play an important role in supergravity and string theory. The space of local deformations of the complex structure of a Calabi Yau three-fold, for instance, is an example of a projective special Kähler manifold with negative definite metric. As a particular result of the classification [**AC1**], there is an interesting one-to-one correspondence between complex simple Lie algebras $\mathfrak{l}$ of type A, B, D, E, F and G and homogeneous projective special Kähler manifolds of semisimple groups with negative definite metric. The resulting spaces are certain Hermitian symmetric spaces of non-compact type. The homogeneous projective special Kähler manifold associated to the complex simple Lie algebra of type E_6, for instance, is precisely the Hermitian symmetric space $\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3))$.

Under the above assumptions, the homogeneous projective special Kähler manifold G/K is realised as an open orbit of a real semisimple group G acting on a smooth projective algebraic variety $X\subset P(V)$, where V is the complexification of a real symplectic module V_0 of G and the cone $C(X):=\{v\in V\,|\,\pi(v)\in X\}\subset V$ over X is Lagrangian. Here $\pi: V\setminus\{0\}\to P(V)$ denotes the canonical projection. In fact, $C(X)$ is the orbit of the highest weight vector of the $G^{\mathbb{C}}$-module V under the complexified group $G^{\mathbb{C}}$. In the case $G/K=\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3))$ the symplectic module is given by $V=\wedge^3\mathbb{C}^6$. It was shown in [**BC**] that the real symplectic G-module V_0 always admits a homogeneous quartic invariant λ, which is related to the hyper-Kähler part of the curvature tensor of a symmetric quaternionic Kähler manifold associated to the given complex simple Lie algebra $\mathfrak{l}$. Moreover, the level sets $\{\lambda=c\}$ are proper affine hyperspheres for $c\neq 0$ and the affine special Kähler manifold M underlying the homogeneous projective special Kähler manifold $\bar{M}=G/K$ can be realised as one of the open orbits of $\mathbb{R}^*\cdot G$ on V_0 [**BC**].

In the following we shall describe all real forms (G,V_0) of the $\mathrm{SL}(6,\mathbb{C})$-module $V=\wedge^3\mathbb{C}^6$ and study the affine special geometry of the corresponding open orbits of $\mathbb{R}^*\cdot G$. As a consequence, we obtain a list of projective special Kähler manifolds, which admit a transitive action of a real form of $\mathrm{SL}(6,\mathbb{C})$ by automorphisms of the special Kähler structure. Besides the unique stationary compact example

$$\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3)),$$

we obtain the homogeneous projective special Kähler manifolds

$$\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(2,1)\times\mathrm{U}(1,2)),\quad \mathrm{SU}(5,1)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2,1))\quad\text{and}\quad \mathrm{SL}(6,\mathbb{R})/\,(\mathrm{U}(1)\cdot\mathrm{SL}(3,\mathbb{C})),$$

which are symmetric spaces with indefinite metrics and non-compact stabiliser. The Hermitian signature of the metric is $(4,5)$, $(6,3)$ and $(3,6)$, respectively. The latter result $(3,6)$ corrects Proposition 7 in [**H1**], according to which the Hermitian signature of the underlying affine special Kähler manifold $\mathrm{GL}^+(6,\mathbb{R})/\,\mathrm{SL}(3,\mathbb{C})$ is $(1,9)$. The correct Hermitian signature of the affine special Kähler manifold is $(4,6)$.

Finally, we find that one of the two open orbits of $\mathrm{SL}(6,\mathbb{R})$ on $\wedge^3\mathbb{R}^6$ carries affine special para-Kähler geometry, the geometry of $N=2$ vector multiplets on Euclidian rather than Minkowskian space-time [**CMMS**]. The corresponding homogeneous projective special para-Kähler manifold is the symmetric space

$$\mathrm{SL}(6,\mathbb{R})/\,\mathrm{S}(\mathrm{GL}(3,\mathbb{R})\times\mathrm{GL}(3,\mathbb{R})).$$

5.1. The symplectic $\mathrm{SL}(6,\mathbb{C})$-module $V=\wedge^3\mathbb{C}^6$ and its Lagrangian cone $C(X)$ of highest weight vectors. We consider the 20-dimensional irreducible $\mathrm{SL}(6,\mathbb{C})$-module $V=\wedge^3\mathbb{C}^6$ equipped with a generator ν of the line $\wedge^6\mathbb{C}^6$. The choice of ν determines an $\mathrm{SL}(6,\mathbb{C})$-invariant symplectic form Ω, which given by

$$\Omega(v,w)\nu=v\wedge w,\quad v,w\in V. \tag{5.1}$$

The highest weight vectors in V are precisely the non-zero decomposable three-vectors. They form a cone $C(X) \subset V$ over a smooth projective variety $X \subset P(V)$, namely the Grassmannian $Gr_3(\mathbb{C}^6)$ of complex three-planes in $\mathbb{C}^6$. The group $\mathrm{SL}(6,\mathbb{C})$ acts transitively on the cone $C(X)$ and, hence, on the compact variety

$$X \cong \mathrm{SL}(6,\mathbb{C})/P \cong \mathrm{SU}(6)/\,\mathrm{S}(\mathrm{U}(3)\times \mathrm{U}(3)),$$

where $P = \mathrm{SL}(6,\mathbb{C})_x \subset \mathrm{SL}(6,\mathbb{C})$ is the stabiliser of a point $x \in X$ (a parabolic subgroup).

Proposition 5.1. *The cone $C(X) = \{v \in \wedge^3\mathbb{C}^6\backslash\{0\} \,|\, v$ is decomposable$\} \subset V$ is Lagrangian.*

Proof. Let $(e_1,\dots,e_6)$ be a basis of $\mathbb{C}^6$ and put $p = e_{123}$. Then

$$T_pC(X) = \mathrm{span}\{e_{ijk} \,|\, \#\{i,j,k\}\cap\{1,2,3\} \geq 2\}$$

is ten-dimensional and is clearly totally isotropic with respect to Ω. □

5.2. Real forms (G, V_0) of the complex module $(\mathrm{SL}(6,\mathbb{C}), V)$. Let G be a real form of the complex Lie group $\mathrm{SL}(6,\mathbb{C})$. There exists a G-invariant real structure τ on $V = \wedge^3\mathbb{C}^6$ if and only if $G = \mathrm{SL}(6,\mathbb{R})$, $\mathrm{SU}(3,3)$, $\mathrm{SU}(5,1)$. In the first case τ is simply complex conjugation with respect to $V_0 = \wedge^3\mathbb{R}^6$. In order to describe the real structure in the other two cases, we first endow $\mathbb{C}^6$ with the standard $\mathrm{SU}(p,q)$-invariant pseudo-Hermitian form $\langle\cdot,\cdot\rangle$. The pseudo-Hermitian form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^6$ induces an $\mathrm{SU}(p,q)$-invariant pseudo-Hermitian form γ on V such that

$$\gamma(v_1\wedge v_2\wedge v_3, w_1\wedge w_2\wedge w_3) = \det(\langle v_i, w_j\rangle), \tag{5.2}$$

for all $v_1,\dots,w_3 \in \mathbb{C}^6$. Then we define an $\mathrm{SU}(p,q)$-invariant anti-linear map $\tau : V \to V$ by the equation

$$\tau := \sqrt{-1}\gamma^{-1}\circ\Omega.$$

Notice that $\Omega : V \to V^*, v \mapsto \Omega(\cdot, v)$ is linear, whereas $\gamma : V \to V^*, v \mapsto \gamma(\cdot, v)$ and $\gamma^{-1} : V^* \to V$ are anti-linear.

Proposition 5.2. *The anti-linear map τ is an $\mathrm{SU}(p,q)$-invariant real structure on $V = \wedge^3\mathbb{C}^6$ if and only if $p - q \equiv 0 \pmod 4$. In that case, the $\mathrm{SU}(p,q)$-invariant pseudo-Hermitian form $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$ on V has signature $(10,10)$. Otherwise, τ is an $\mathrm{SU}(p,q)$-invariant quaternionic structure on V.*

Proof. We present the calculations in the relevant cases $(p,q) = (3,3)$ and $(p,q) = (5,1)$. The calculations in the other cases are similar.

Case $(p,q) = (3,3)$. Let $(e_1,e_2,e_3,f_1,f_2,f_3)$ be a unitary basis of $(\mathbb{C}^6,\langle\cdot,\cdot\rangle) = \mathbb{C}^{3,3}$, such that $\langle e_i,e_i\rangle = -\langle f_i,f_i\rangle = 1$. We consider the following basis of V:

$$\begin{aligned}&(e_{123}, e_1\wedge f_{12}, e_1\wedge f_{13}, e_1\wedge f_{23}, e_2\wedge f_{12}, e_2\wedge f_{13}, e_2\wedge f_{23}, e_3\wedge f_{12}, e_3\wedge f_{13}, e_3\wedge f_{23},\\ &f_{123}, e_{23}\wedge f_3, -e_{23}\wedge f_2, e_{23}\wedge f_1, -e_{13}\wedge f_3, e_{13}\wedge f_2, -e_{13}\wedge f_1, e_{12}\wedge f_3, -e_{12}\wedge f_2, e_{12}\wedge f_1).\end{aligned}$$

With respect to that basis and $\nu = e_{123}\wedge f_{123}$ we have

$$\gamma = \begin{pmatrix} \mathbb{1}_{10} & 0 \\ 0 & -\mathbb{1}_{10}\end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & \mathbb{1}_{10} \\ -\mathbb{1}_{10} & 0\end{pmatrix}, \quad \tau = \sqrt{-1}\gamma^{-1}\circ\Omega = \sqrt{-1}\begin{pmatrix} 0 & \mathbb{1}_{10} \\ \mathbb{1}_{10} & 0\end{pmatrix}. \tag{5.3}$$

This implies $\tau^2 = \mathrm{Id}$, since τ is anti-linear.

Case $(p,q) = (5,1)$. Let $(e_1,\dots,e_5,f)$ be a unitary basis of $(\mathbb{C}^6,\langle\cdot,\cdot\rangle) = \mathbb{C}^{5,1}$, such that $\langle e_i,e_i\rangle = -\langle f,f\rangle = 1$. With respect to the basis

$$\begin{aligned}&(e_{123}, e_{124}, e_{125}, e_{134}, e_{135}, e_{145}, e_{234}, e_{235}, e_{245}, e_{345},\\ &e_{45}\wedge f, -e_{35}\wedge f, e_{34}\wedge f, e_{25}\wedge f, -e_{24}\wedge f, e_{23}\wedge f, -e_{15}\wedge f, e_{14}\wedge f, -e_{13}\wedge f, e_{12}\wedge f)\end{aligned} \tag{5.4}$$

of V and $\nu = e_{12345}\wedge f$ we have again the formulas (5.3) and $\tau^2 = \mathrm{Id}$. □

G/H	Hermitian signature
$\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(3))$	(0,9)
$\mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(2,1)\times\mathrm{U}(1,2))$	(4,5)
$\mathrm{SU}(5,1)/\,\mathrm{S}(\mathrm{U}(3)\times\mathrm{U}(2,1))$	(6,3)
$\mathrm{SL}(6,\mathbb{R})/\,(\mathrm{U}(1)\cdot\mathrm{SL}(3,\mathbb{C}))$	(3,6)

TABLE 1. *Homogeneous projective special Kähler manifolds* $\bar{M} = G/H$ *of real simple groups* G *of type* A_5. *Notice that* $\dim_{\mathbb{C}} \bar{M} = 9$.

5.3. Classification of open G-orbits on the Grassmannian X and corresponding special Kähler manifolds. For each of the real forms (G, V_0) obtained in the previous section, we will now describe all open orbits of the real simple Lie group G on the Grassmannian $X = Gr_3(\mathbb{C}^6) = \{E \subset \mathbb{C}^6 \text{ a three-dimensional subspace}\} \hookrightarrow P(V)$, $E \mapsto \wedge^3 E$. We will also describe the projective special Kähler structure of these orbits $\bar{M} \subset P(V)$ and the (affine) special Kähler structure of the corresponding cones $M = C(\bar{M}) \subset V$. The resulting homogeneous projective special Kähler manifolds are listed in Table 1. Let us first recall some definitions and constructions from special Kähler geometry.

5.3.1. *Basic facts about special Kähler manifolds.*

DEFINITION 5.3. A *(pseudo-)Kähler manifold* (M, J, g) is a pseudo-Riemannian manifold (M, g) endowed with a parallel skew-symmetric complex structure $J \in \Gamma(\mathrm{End}\, TM)$. The symplectic form $\omega = g(\cdot, J\cdot)$ is called *Kähler form.* A *special Kähler manifold* (M, J, g, ∇) is a (pseudo-)Kähler manifold (M, J, g) endowed with a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and $d^\nabla J = 0$, where $d^\nabla J$ is the exterior covariant derivative of the vector valued one-form J.

A *conical special Kähler manifold* (M, J, g, ∇, ξ) is a special Kähler manifold (M, J, g, ∇) endowed with a timelike or a spacelike vector field ξ such that $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection. The vector field ξ is called *Euler vector field.*

The vector fields ξ and $J\xi$ generate a free holomorphic action of a two-dimensional Abelian Lie algebra. If the action can be integrated to a free holomorphic $\mathbb{C}^*$-action such that the quotient map $M \to \bar{M} := M/\mathbb{C}^*$ is a holomorphic submersion, then $\bar{M}$ is called a *projective special Kähler manifold*. We will see now that $\bar{M}$ carries a canonical (pseudo-)Kähler metric $\bar{g}$ compatible with the induced complex structure J on $\bar{M} = M/\mathbb{C}^*$. Multiplying the metric g with -1 if necessary, we can assume that $g(\xi, \xi) > 0$. Then $S = \{p \in M \mid g(\xi(p), \xi(p)) = 1\}$ is a smooth hypersurface invariant under the isometric S^1-action generated by the Killing vector field $J\xi$ and we can recover $\bar{M}$ as the base of the circle bundle $S \to \bar{M} = S/S^1$. Then $\bar{M}$ carries a unique pseudo-Riemannian metric $\bar{g}$ such that $S \to \bar{M}$ is a Riemannian submersion. $(\bar{M}, J, \bar{g})$ is in fact the Kähler quotient of (M, J, g) by the S^1-action generated by the Hamiltonian Killing vector field $J\xi$.

Next we explain the extrinsic construction of special Kähler manifolds from [**ACD**]. Let (V, Ω) be a complex symplectic vector space of dimension $2n$ endowed with a real structure τ such that

$$\overline{\Omega(v, w)} = \Omega(\tau v, \tau w), \quad \text{for all} \quad v, w \in V. \tag{5.5}$$

Then the pseudo-Hermitian form

$$\gamma := \sqrt{-1}\Omega(\cdot, \tau\cdot) \tag{5.6}$$

has signature (n, n).

DEFINITION 5.4. A holomorphic immersion $\phi : M \to V$ from an n-dimensional complex manifold (M, J) into V is called

(i) *nondegenerate* if $\phi^*\gamma$ is nondegenerate,
(ii) *Lagrangian* if $\phi^*\Omega = 0$ and
(iii) *conical* if $\phi(p) \in d\phi(T_pM)$ and $\gamma(\phi(p), \phi(p)) \neq 0$ for all $p \in M$.

THEOREM 5.5. [**ACD**]

(i) *Any nondegenerate Lagrangian immersion $\phi : M \to V$ induces on the complex manifold (M, J) the structure of a special Kähler manifold (M, J, g, ∇), where $g = \mathrm{Re}\,\phi^*\gamma$ and ∇ is determined by the condition $\nabla\phi^*\alpha = 0$ for all $\alpha \in V^*$ which are real valued on V^τ.*

(ii) *Any conical nondegenerate Lagrangian immersion $\phi : M \to V$ induces on (M, J) the structure of a conical special Kähler manifold (M, J, g, ∇, ξ). The vector field ξ is determined by the condition $d\phi\,\xi(p) = \phi(p)$.*

5.3.2. *The case $G = \mathrm{SL}(6, \mathbb{R})$.* Using the complex conjugation $\tau : v \mapsto \bar{v}$ on $\mathbb{C}^6$ we can decompose X into G-invariant real algebraic subvarieties $X_{(k)} = X_{(k)}(\tau) := \{E \in X \mid \dim(E \cap \bar{E}) = k\} \subset X$, where $E \subset \mathbb{C}^6$ runs through all three-dimensional subspaces and $k \in \{0, 1, 2, 3\}$. Notice that only $X_{(0)} \subset X$ is open.

PROPOSITION 5.6. *The group $\mathrm{SL}(6, \mathbb{R})$ acts transitively on the open real subvariety $X_{(0)} = \{E \in X \mid E \cap \bar{E} = 0\} \subset X$.*

PROOF. Given bases (e_1, e_2, e_3), (e_1', e_2', e_3') of $E, E' \in X_{(0)}$, respectively, let φ be the linear transformation, which maps the basis $(e_1, e_2, e_3, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ of $\mathbb{C}^6$ to the basis $(e_1', e_2', e_3', \bar{e}_1', \bar{e}_2', \bar{e}_3')$ of $\mathbb{C}^6$. Then $\varphi \in \mathrm{SL}(6, \mathbb{R})$ and $\varphi E = E'$. □

THEOREM 5.7. *The group $\mathrm{SL}(6, \mathbb{R})$ has a unique open orbit $X_{(0)} \cong \mathrm{SL}(6, \mathbb{R})/\left(\mathrm{U}(1)\cdot\mathrm{SL}(3, \mathbb{C})\right)$ on the highest weight variety $P(V) \supset X \cong Gr_3(\mathbb{C}^6)$ of $V = \wedge^3\mathbb{C}^6$. The cone $V \supset M = C(X_{(0)}) \cong \mathrm{GL}^+(6, \mathbb{R})/\mathrm{SL}(3, \mathbb{C})$ carries an $\mathrm{SL}(6, \mathbb{R})$-invariant special Kähler structure of Hermitian signature $(4, 6)$, which induces on $\bar{M} = X_{(0)}$ the structure of a homogeneous projective special Kähler manifold $\bar{M}$ of Hermitian signature $(3, 6)$.*

PROOF. Let $e_1, e_2, e_3 \in \mathbb{C}^6$ be three vectors which span a three-dimensional subspace $E \subset \mathbb{C}^6$ such that $E \in X_{(0)}$. Then $\mathbb{C}^6 = E \oplus \bar{E}$ and the tangent space of $M = C(X_{(0)})$ at $p = e_{123}$ is given by $T_pM = \wedge^3 E \oplus \wedge^2 E \wedge \bar{E}$. We choose the real generator $\nu = \sqrt{-1}e_{123} \wedge \bar{e}_{123} \in \wedge^3\mathbb{R}^6$ and compute $\gamma = \sqrt{-1}\Omega(\cdot, \tau\cdot)$ on T_pM using the formula (5.1). The matrix of $\gamma|_{T_pM}$ with respect to the basis $(e_{123}, e_{12} \wedge \bar{e}_3, e_{13} \wedge \bar{e}_2, e_{23} \wedge \bar{e}_1, e_{12} \wedge \bar{e}_1, e_{13} \wedge \bar{e}_1, e_{12} \wedge \bar{e}_2, -e_{23} \wedge \bar{e}_3, e_{23} \wedge \bar{e}_2, e_{13} \wedge \bar{e}_3)$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_3 \\ 0 & 0 & \mathbb{1}_3 & 0 \end{pmatrix}. \tag{5.7}$$

This shows that γ has signature $(4, 6)$ on T_pM. Since $\mathrm{GL}^+(6, \mathbb{R})$ acts transitively on M and preserves the pseudo-Hermitian form γ up to a positive factor ($\mathrm{SL}(6, \mathbb{R})$ acts isometrically), the signature of the restriction of γ to M does not depend on the base point. This shows that the inclusion $M \subset V$ is a holomorphic conical *nondegenerate* Lagrangian immersion. By Theorem 5.5, it induces a conical special (pseudo-)Kähler structure (J, g, ∇, ξ) on M. It follows that the image $\bar{M} = \pi(M) = X_{(0)} \cong \mathrm{SL}(6, \mathbb{R})/\left(\mathrm{U}(1) \cdot \mathrm{SL}(3, \mathbb{C})\right)$ of M under the projection $\pi : V \setminus \{0\} \to P(V)$ is a homogeneous projective special Kähler manifold. The induced pseudo-Hermitian form $\bar{\gamma}$ on $T_{\pi(p)}\bar{M}$ has signature $(3, 6)$. The latter statement follows from formula $\bar{\gamma}(d\pi_p X, d\pi_p Y) = \frac{\gamma(X,Y)}{\gamma(p,p)}$ for $X, Y \in T_pM \cap p^\perp \subset V$ (see [**AC1**]), since $\gamma(p, p) = 1$ for $p = e_{123}$. □

5.3.3. *The case $G = \mathrm{SU}(3, 3)$.* Using the pseudo-Hermitian form $h = \langle\cdot, \cdot\rangle$ on $\mathbb{C}^6$ invariant under $G = \mathrm{SU}(3, 3)$ we can decompose the Grassmannian $X = Gr_3(\mathbb{C}^6)$ into the G-invariant real algebraic subvarieties $X_{(k)} = X_{(k)}(h) := \{E \in X \mid \mathrm{rk}(h|_E) = k\}$, $k \in \{0, 1, 2, 3\}$. Notice that only $X_{(3)} \subset X$ is open and that it can be decomposed further according to the possible signatures of $h|_E$:

$$X_{(s,t)} := \{E \in X \mid E \text{ has signature } (s, t)\},$$

where $(s, t) \in \{(3, 0), (2, 1), (1, 2), (0, 3)\}$.

THEOREM 5.8. *The group* $\mathrm{SU}(3,3)$ *has precisely four open orbits on the highest weight variety* $P(V) \supset X \cong Gr_3(\mathbb{C}^6)$ *of* $V = \wedge^3\mathbb{C}^6$*, namely* $X_{(3,0)}$, $X_{(2,1)}$, $X_{(1,2)}$ *and* $X_{(0,3)}$. *In all four cases the cone* $M = C(X_{(s,t)}) \subset V$ *carries an* $\mathrm{SU}(3,3)$*-invariant special Kähler structure.*

$$C(X_{(3,0)}) \cong \mathbb{R}^* \cdot \mathrm{SU}(3,3)/\,\mathrm{SU}(3) \times \mathrm{SU}(3)$$

has Hermitian signature $(1,9)$. $C(X_{(0,3)}) \cong \mathbb{R}^* \cdot \mathrm{SU}(3,3)/\,\mathrm{SU}(3) \times \mathrm{SU}(3)$ *has Hermitian signature* $(9,1)$. *For* $\{s,t\} = \{2,1\}$,

$$C(X_{(s,t)}) \cong \mathbb{R}^* \cdot \mathrm{SU}(3,3)/\,\mathrm{SU}(2,1) \times \mathrm{SU}(1,2)$$

has Hermitian signature $(5,5)$. *In all cases, the conical special Kähler manifold* $M = C(X_{(s,t)})$ *induces on* $\bar{M} = X_{(s,t)}$ *the structure of a homogeneous projective special Kähler manifold* $\bar{M}$. *For* $\{s,t\} = \{3,0\}$,

$$\bar{M} = X_{(s,t)} \cong X_{(3,0)} \cong \mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(3))$$

has Hermitian signature $(0,9)$, *for* $\{s,t\} = \{2,1\}$,

$$\bar{M} = X_{(s,t)} \cong X_{(2,1)} \cong \mathrm{SU}(3,3)/\,\mathrm{S}(\mathrm{U}(2,1) \times \mathrm{U}(1,2))$$

has Hermitian signature $(4,5)$. *The special Kähler manifolds* $C(X_{(s,t)})$ *and* $C(X_{(t,s)})$ *are equivalent. In fact, they are related by a holomorphic* ∇*-affine anti-isometry, which induces a holomorphic isometry between the corresponding projective special Kähler manifolds.*

PROOF. $X_{(3)} \subset X$ is Zariski open and is decomposed into the four open (in the standard topology) orbits $X_{(s,t)}$ of $G = \mathrm{SU}(3,3)$. Let $(e_1, e_2, e_3, f_1, f_2, f_3)$ be a unitary basis of $(\mathbb{C}^6, h)$, such that $\langle e_i, e_i\rangle = -\langle f_i, f_i\rangle = 1$. Then $X_{(3,0)}$, $X_{(2,1)}$, $X_{(1,2)}$, $X_{(0,3)}$ are the G-orbits of the lines generated by the elements $e_{123}, e_1 \wedge f_{12}, e_{12} \wedge f_1$, $f_{123} \in V$, respectively.

For $p = e_{123}$ and $M = C(X_{(3,0)})$, we calculate

$$T_pM = \mathrm{span}\{e_{123}, f_i \wedge e_{jk}\}.$$

From (5.2) we see that the matrix of $\gamma|_{T_pM}$ with respect to the basis $(e_{123}, f_i \wedge e_{jk})$ is $\mathrm{diag}(1, -\mathbb{1}_9)$. Therefore γ has signature $(1,9)$ on T_pM. Since $\gamma(e_{123}, e_{123}) = 1$, the signature of the induced pseudo-Hermitian form $\bar{\gamma}$ on $T_p\bar{M}$ is $(0,9)$.

For $p = e_{12} \wedge f_1$ and $M = C(X_{(2,1)})$, we obtain

$$T_pM = \mathrm{span}\{e_1 \wedge f_{12}, e_2 \wedge f_{12}, e_2 \wedge f_{13}, e_1 \wedge f_{13}, e_{123}, e_{ij} \wedge f_1, e_{12} \wedge f_i\}.$$

The matrix of $\gamma|_{T_pM}$ with respect to the above basis is $\mathrm{diag}(\mathbb{1}_5, -\mathbb{1}_5)$. Therefore γ has signature $(5,5)$ on T_pM. We have $\gamma(e_{12} \wedge f_1, e_{12} \wedge f_1) = -1$ and the signature of the induced pseudo-Hermitian form $\bar{\gamma}$ on $T_p\bar{M}$ is $(4,5)$.

The linear transformation which sends the vectors e_i to f_i and f_i to e_i induces a linear map $\varphi : V \to V$, which interchanges the cone $C(X_{(s,t)})$ with $C(X_{(t,s)})$ and maps γ to $-\gamma$. This shows that γ has signature $(9,1)$ and $(5,5)$ on $C(X_{(0,3)})$ and $C(X_{(1,2)})$, respectively. As a consequence, the induced pseudo-Hermitian form $\bar{\gamma}$ on $\bar{M} = X_{(0,3)}$, $X_{(1,2)}$, has still signature $(0,9)$ and $(4,5)$, respectively.

It follows from these calculations that the inclusion $M = C(X_{(s,t)}) \subset V$ is a holomorphic conical nondegenerate Lagrangian immersion. By Theorem 5.5, it induces a conical special (pseudo-)Kähler structure (J, g, ∇, ξ) on M and $\bar{M} = X_{(s,t)} \subset P(V)$ is a projective special Kähler manifold.

The above linear anti-isometry $\varphi : (V, \gamma) \to (V, \gamma)$ maps Ω to $-\Omega$, and, hence, preserves the real structure $\tau = \sqrt{-1}\gamma^{-1} \circ \Omega$. As a result, it maps the special Kähler structure (J, g, ∇) of $C(X_{(s,t)})$ to $(J', -g', \nabla')$, where (J', g', ∇') is the special Kähler structure of $C(X_{(t,s)})$. In particular, it induces a holomorphic isometry $X_{(s,t)} \cong X_{(t,s)}$. □

5.3.4. *The case* $G = \mathrm{SU}(5,1)$. Let $h = \langle\cdot,\cdot\rangle$ be the standard pseudo-Hermitian form of signature $(5,1)$ on $\mathbb{C}^6$, which is invariant under $G = \mathrm{SU}(5,1)$. Let us fix a unitary basis $(e_1, \ldots, e_5, f)$ of $(\mathbb{C}^6, h)$, such that $\langle e_i, e_i\rangle = -\langle f, f\rangle = 1$. As in the previous subsection, $X = Gr_3(\mathbb{C}^6)$ is decomposed into the G-invariant real algebraic subvarieties $X_{(k)} = X_{(k)}(h)$, of which $X_{(3)} \subset X$ is Zariski open. $X_{(3)}$ is now the union of the two open G-orbits $X_{(3,0)}$ and $X_{(2,1)}$. $X_{(3,0)}$ is the orbit of the line $\mathbb{C}e_{123} \in P(V)$ and $X_{(2,1)}$ is the orbit of $\mathbb{C}e_{45} \wedge f \in P(V)$.

Theorem 5.9. *The group* $\mathrm{SU}(5,1)$ *has precisely two open orbits on the highest weight variety* $P(V) \supset X \cong Gr_3(\mathbb{C}^6)$ *of* $V = \wedge^3\mathbb{C}^6$, *namely* $X_{(3,0)}$ *and* $X_{(2,1)}$. *In both cases the cone* $M = C(X_{(s,t)}) \subset V$ *carries an* $\mathrm{SU}(5,1)$*-invariant special Kähler structure.*

$$C(X_{(3,0)}) \cong \mathbb{R}^* \cdot \mathrm{SU}(5,1)/\,\mathrm{SU}(3) \times \mathrm{SU}(2,1)$$

has Hermitian signature $(7,3)$.

$$C(X_{(2,1)}) \cong \mathbb{R}^* \cdot \mathrm{SU}(5,1)/\,\mathrm{SU}(3) \times \mathrm{SU}(2,1)$$

has Hermitian signature $(3,7)$. *In both cases, the conical special Kähler manifold* $M = C(X_{(s,t)})$ *induces on* $\bar{M} = X_{(s,t)}$ *the structure of a homogeneous projective special Kähler manifold* $\bar{M}$.

$$\bar{M} = X_{(3,0)} \cong \mathrm{SU}(5,1)/\,\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2,1))$$

and

$$\bar{M} = X_{(2,1)} \cong \mathrm{SU}(5,1)/\,\mathrm{S}(\mathrm{U}(3) \times \mathrm{U}(2,1))$$

have both Hermitian signature $(6,3)$. *The special Kähler manifolds* $C(X_{(3,0)})$ *and* $C(X_{(2,1)})$ *are equivalent. In fact, they are related by a holomorphic* ∇*-affine anti-isometry, which induces a holomorphic isometry between the corresponding projective special Kähler manifolds.*

Proof. For $p = e_{123}$ and $M = C(X_{(3,0)})$, we have

$$T_pM = \mathrm{span}\{e_{123}, e_{234}, e_{235}, e_{134}, e_{135}, e_{124}, e_{125}, e_{12} \wedge f, e_{13} \wedge f, e_{23} \wedge f\}$$

and the restriction of γ to T_pM is represented by the matrix $\mathrm{diag}(\mathbb{1}_7, -\mathbb{1}_3)$ with respect to the above basis. This shows that the inclusion $M = C(X_{(3,0)}) \subset V$ is a holomorphic conical nondegenerate Lagrangian immersion. By Theorem 5.5, it induces a conical special (pseudo-)Kähler structure (J, g, ∇, ξ) of Hermitian signature $(7,3)$ on M and $\bar{M} = X_{(3,0)} \subset P(V)$ is a projective special Kähler manifold of Hermitian signature $(6,3)$. The anti-isometry relating $C(X_{(3,0)})$ and $C(X_{(2,1)})$ is induced by the linear map $\varphi : V \to V$ which has the matrix

$$\begin{pmatrix} 0 & \mathbb{1}_{10} \\ \mathbb{1}_{10} & 0 \end{pmatrix},$$

with respect to the basis (5.4). □

5.4. The homogeneous projective special para-Kähler manifold $\mathrm{SL}(6,\mathbb{R})/\,\mathrm{S}(\mathrm{GL}(3,\mathbb{R}) \times \mathrm{GL}(3,\mathbb{R}))$**.** Let us first briefly recall the necessary definitions and constructions from special para-Kähler geometry, see [**CMMS**] for more details.

5.4.1. *Basic facts about special para-Kähler manifolds.*

Definition 5.10. A *para-Kähler manifold* (M, J, g) is a pseudo-Riemannian manifold (M, g) endowed with a parallel skew-symmetric endomorphism field $J \in \Gamma(\mathrm{End}\,TM)$ such that $J^2 = \mathrm{Id}$. A *special para-Kähler manifold* (M, J, g, ∇) is a para-Kähler manifold (M, J, g) endowed with a flat torsion-free connection ∇ such that $\nabla\omega = 0$ and $d^\nabla J = 0$, where $\omega = g(\cdot, J\cdot)$.

A *conical special para-Kähler manifold* (M, J, g, ∇, ξ) is a special para-Kähler manifold (M, J, g, ∇) endowed with a timelike or a spacelike vector field ξ such that $\nabla\xi = D\xi = \mathrm{Id}$, where D is the Levi-Civita connection.

It follows from the definition of a para-Kähler manifold that the eigenspaces of J are of the same dimension and involutive. An endomorphism field $J \in \Gamma(\mathrm{End}\,TM)$ with these properties is called a *para-complex structure* on M. The pair (M, J) is then called a *para-complex manifold*. A smooth map $f : (M, J_M) \to (N, J_N)$ between para-complex manifolds is called *para-holomorphic* if $df \circ J_M = J_N \circ df$. The skew-symmetry of J in the definition of a para-Kähler manifold implies that the eigenspaces of J are totally isotropic of dimension $n = \frac{1}{2}\dim M$. In particular, M is of even dimension $2n$ and g is of signature (n, n).

On any conical special para-Kähler manifold, the vector fields ξ and $J\xi$ generate a free para-holomorphic action of a two-dimensional Abelian Lie algebra. If the action can be integrated to a free para-holomorphic action of a Lie group A such that the quotient map $M \to \bar{M} := M/A$ is a para-holomorphic submersion, then $\bar{M}$ is called a *projective special para-Kähler manifold*. The quotient $\bar{M}$ carries a canonical para-Kähler metric $\bar{g}$ compatible with the induced para-complex structure J on $\bar{M} = M/A$.

Next we explain the extrinsic construction of special para-Kähler manifolds. Recall that a *para-complex vector space* V of dimension n is simply a free module $V \cong C^n$ over the ring $C = \mathbb{R}[e]$, $e^2 = 1$, of *para-complex numbers*. Notice that C^n is a para-complex manifold with the para-complex structure $v \mapsto ev$ and any para-complex manifold of real dimension $2n$ is locally isomorphic to C^n. An $\mathbb{R}$-linear map $\tau : V \to V$ on a para-complex vector space is called *anti-linear* if $\tau(ev) = -e\tau(v)$ for all $v \in V$. An example is the para-complex conjugation $C^n \to C^n$, $z = x + ey \mapsto \bar{z} := x - ey$. Let (V, Ω) be a para-complex symplectic vector space of dimension $2n$ endowed with a real structure (i.e. an anti-linear involution) τ such that (5.5) holds true. Then

$$\gamma := e\Omega(\cdot, \tau\cdot) \tag{5.8}$$

is a para-Hermitian form and $g_V := \operatorname{Re}\gamma$ is a flat para-Kähler metric on V.

Definition 5.11. A para-holomorphic immersion $\phi : M \to V$ from para-complex manifold (M, J) of real dimension 2n into V is called

(i) *nondegenerate* if $\phi^*\gamma$ is nondegenerate,
(ii) *Lagrangian* if $\phi^*\Omega = 0$ and
(iii) *conical* if $\phi(p) \in d\phi(T_pM)$ and $\gamma(\phi(p), \phi(p)) \neq 0$ for all $p \in M$.

Theorem 5.12. [**CMMS**]

(i) *Any nondegenerate para-holomorphic Lagrangian immersion $\phi : M \to V$ induces on the para-complex manifold (M, J) the structure of a special para-Kähler manifold (M, J, g, ∇), where $g = \operatorname{Re}\phi^*\gamma$ and ∇ is determined by the condition $\nabla\phi^*\alpha = 0$ for all $\alpha \in V^*$ which are real valued on V^τ.*
(ii) *Any conical nondegenerate para-holomorphic Lagrangian immersion $\phi : M \to V$ induces on (M, J) the structure of a conical special para-Kähler manifold (M, J, g, ∇, ξ). The vector field ξ is determined by the condition $d\phi\,\xi(p) = \phi(p)$.*

5.4.2. *The (affine) special para-Kähler manifold as a para-complex Lagrangian cone.* Now we consider the real symplectic module $V_0 = \wedge^3\mathbb{R}^6$ of $G = \mathrm{SL}(6, \mathbb{R})$. For convenience, the standard basis of $\mathbb{R}^6$ is denoted by $(e_1, e_2, e_3, f_1, f_2, f_3)$. The para-complexification $V := V_0 \otimes C = \wedge^3\mathbb{C}^6 \cong C^{20}$ of V_0 is a para-complex symplectic vector space endowed with a real structure τ such that $V^\tau = V_0$ and (5.5). We put $u_i := e_i + ef_i$ and consider the orbit

$$V \subset M = \mathrm{GL}^+(6, \mathbb{R})p \cong \mathrm{GL}^+(6, \mathbb{R})/\,\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$$

of the element $p = u_1 \wedge u_2 \wedge u_3$.

Theorem 5.13. *$M = \mathrm{GL}^+(6, \mathbb{R})p \subset V$ is a nondegenerate para-complex Lagrangian cone. The inclusion $M \subset V$ induces on M an $\mathrm{SL}(6, \mathbb{R})$-invariant special para-Kähler structure. The image $\bar{M} = \pi(M) \cong \mathrm{SL}(6, \mathbb{R})/\,\mathrm{S}(\mathrm{GL}(3, \mathbb{R}) \times \mathrm{GL}(3, \mathbb{R}))$ under the projection $\pi : V' \to P(V')$ is a homogeneous projective special para-Kähler manifold of real dimension 18. Here $V' \subset V$ stands for the subset of nonisotropic vectors.*

Proof. Using the formulas (5.1) and (5.8) with $\nu = eu_{123} \wedge \bar{u}_{123} = -8e_{123} \wedge f_{123}$ we compute: $\gamma(p, p) = 1$. This shows that $M = \mathrm{GL}^+(6, \mathbb{R})p \subset V'$ consists of spacelike vectors. The tangent $T_pM \subset V$ has the following basis:

$$(u_{123}, \bar{u}_1 \wedge u_{23}, \bar{u}_2 \wedge u_{31}, \bar{u}_3 \wedge u_{12}, \bar{u}_2 \wedge u_{23}, \bar{u}_2 \wedge u_{12}, \bar{u}_3 \wedge u_{23}, \bar{u}_1 \wedge u_{13}, \bar{u}_3 \wedge u_{13}, -\bar{u}_1 \wedge u_{12}).$$

The restriction of Ω to T_pM is zero in view of (5.1). The para-Hermitian form $\gamma|_{T_pM}$ is represented by the matrix (5.7). This shows that $\gamma|_{T_pM}$ is nondegenerate. Hence, the inclusion $M \subset V$ is a conical para-holomorphic nondegenerate Lagrangian immersion. In virtue of Theorem 5.12 it induces an $\mathrm{SL}(6, \mathbb{R})$-invariant conical special para-Kähler structure (J, g, ∇, ξ) on M, which in turn induces a homogeneous projective special para-Kähler structure on $\bar{M} = \pi(M) \subset P(V')$. $\square$

5.4.3. *The special para-Kähler manifold as an open orbit of* $\mathrm{GL}^+(6,\mathbb{R})$ *on* $\wedge^3\mathbb{R}^6$. The conical special Kähler manifold $M = \mathrm{GL}^+(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C})$ described in Theorem 5.7 as a complex Lagrangian cone $M \subset V_0 \otimes \mathbb{C}$ can be identified with the open $\mathrm{GL}^+(6,\mathbb{R})$-orbit $\{\lambda < 0\} \subset V_0 = \wedge^3\mathbb{R}^6$, where λ stands for the quartic $\mathrm{SL}(6,\mathbb{R})$-invariant (1.4):

Proposition 5.14. *The projection* $\rho : V_0 \otimes \mathbb{C} \to V_0$, $v \mapsto \mathrm{Re}\, v$, *induces a* $\mathrm{GL}^+(6,\mathbb{R})$-*equivariant diffeomorphism from the Lagrangian cone* $C(X_{(0)}) \subset V_0 \otimes \mathbb{C}$ *described in Theorem 5.7 onto* $\{\lambda < 0\} \subset V_0$:

$$C(X_{(0)}) \cong \{\lambda < 0\} \cong \mathrm{GL}^+(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C}).$$

Proof. This follows from the fact that λ is negative on the real part of a non-zero decomposable $(3,0)$-vector, since $\{\lambda < 0\} \cong \mathrm{GL}^+(6,\mathbb{R})/\mathrm{SL}(3,\mathbb{C})$ is connected, see Proposition 1.5. □

In that picture the complex structure is less obvious than in the complex Lagrangian picture but the flat connection and symplectic (Kähler) form are simply the given structures of the symplectic vector space V_0. The complex structure is then obtained from the metric, which is the Hessian of the function $f = \sqrt{|\lambda|}$. (We consider λ as a scalar invariant by choosing a generator of $\wedge^6\mathbb{R}^6$.) This route was followed by Hitchin in [**H1**].

The other open $\mathrm{GL}^+(6,\mathbb{R})$-orbit $\{\lambda > 0\} \subset V_0$ cannot be obtained as the real image of a $\mathrm{GL}^+(6,\mathbb{R})$-orbit on the complex Lagrangian cone $C(X) \subset V_0 \otimes \mathbb{C}$ over the highest weight variety $X \subset P(V_0 \otimes \mathbb{C})$. In fact, $\mathrm{GL}^+(6,\mathbb{R})$ has only one open orbit on X, see Theorem 5.7, and that orbit maps to $\{\lambda < 0\} \subset V_0$ under the projection $V_0 \otimes \mathbb{C} \to V_0$. Instead we have:

Proposition 5.15. *The projection* $\rho : V_0 \otimes C \to V_0$, $v \mapsto \mathrm{Re}\, v = \frac{v+\bar{v}}{2}$, *induces a* $\mathrm{GL}^+(6,\mathbb{R})$-*equivariant diffeomorphism from the para-complex Lagrangian cone* $M = \mathrm{GL}^+(6,\mathbb{R})p \subset V_0 \otimes C$, $p = u_{123}$, *described in Theorem 5.13 onto the open* $\mathrm{GL}^+(6,\mathbb{R})$-*orbit* $\{\lambda > 0\} \subset V_0$:

$$M \cong \{\lambda > 0\} \cong \mathrm{GL}^+(6,\mathbb{R})/(\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})).$$

Proof. It suffices to check that $\mathrm{Re}\, u_{123} \in \{\lambda > 0\}$. This follows from the expression

$$\begin{aligned} 2\mathrm{Re}\, u_{123} &= ((e_1 + ef_1) \wedge (e_2 + ef_2) \wedge (e_3 + ef_3) + (e_1 - ef_1) \wedge (e_2 - ef_2) \wedge (e_3 - ef_3)) \\ &= ((e_1 + f_1) \wedge (e_2 + f_2) \wedge (e_3 + f_3) + (e_1 - f_1) \wedge (e_2 - f_2) \wedge (e_3 - f_3)), \end{aligned}$$

since a three-vector belongs to $\{\lambda > 0\}$ if and only if it can be written as the sum of two decomposable three-vectors which have a non-trivial wedge product, see Proposition 1.5. □

Let us denote by ∇ the standard flat connection of the vector space V_0, by ξ the position vector field, by ω its $\mathrm{SL}(6,\mathbb{R})$-invariant symplectic form and by X_f the Hamiltonian vector field associated to the function $f = \sqrt{\lambda}$. Then we have:

Theorem 5.16. *The data* ($J = \nabla X_f$, $g = \omega \circ J$, ∇, ξ) *define on* $U = \{\lambda > 0\} \subset V_0$ *an* $\mathrm{SL}(6,\mathbb{R})$-*invariant conical special para-Kähler structure.*

Proof. Any three-vector $\psi \in U$ can be written uniquely as $\psi^+ + \psi^-$ with decomposable three-vectors $\psi^\pm$ such that $\psi^+ \wedge \psi^- = f(\psi)\nu$, cf. (1.2) and Corollary 1.7. Differentiation at ψ in direction of a vector $\xi \in V_0$ yields

$$(df_\psi \xi)\nu = (\psi^+ - \psi^-) \wedge \xi = \omega(\psi^+ - \psi^-, \xi)\nu,$$

that is

$$X_f(\psi) = \psi^+ - \psi^-. \tag{5.9}$$

Using this equation, we can calculate $J = \nabla X_f$ by ordinary differentiation in the vector space V_0. The result is that J acts as identity on the subspace $\wedge^3 E_+ \oplus \wedge^2 E_+ \wedge E_- \subset V_0 = \wedge^3\mathbb{R}^6$ and as minus identity on the subspace $\wedge^3 E_- \oplus \wedge^2 E_- \wedge E_+$ where $E_\pm = \mathrm{span}\{\alpha \lrcorner\, \psi^\pm \mid \alpha \in \wedge^2(\mathbb{R}^6)^*\}$ denotes the support of the three-vectors ψ^+ and ψ^-. This shows that $J^2 = \mathrm{Id}$ and that J is skew-symmetric with respect to ω. To prove that the data (J, $g = \omega \circ J$, ∇, ξ) define on $U = \{\lambda > 0\} \subset V_0$ an $\mathrm{SL}(6,\mathbb{R})$-invariant conical special para-Kähler structure, it suffices to show that under the map $\rho : V_0 \otimes C \to V_0$ these data correspond to the conical special para-Kähler structure on $M = \mathrm{GL}^+(6,\mathbb{R})p \subset V_0 \otimes C$, $p = u_{123}$, described in Theorem 5.13. It follows from Proposition 5.15

and the definition of the structures on M that the data (ω, ∇, ξ) on U correspond to the symplectic structure, flat connection and Euler vector field of the conical special para-Kähler manifold M. One can check by a simple direct calculation that the endomorphism J on $T_{\rho(p)}U$ corresponds to multiplication by $e \in C$ on $T_pM \subset V_0 \otimes C$. This proves the theorem.

Alternatively, we give now a direct argument which avoids the use of Theorem 5.13. The structure J on U satisfies

$$d^\nabla J = d^\nabla \nabla X_f = (d^\nabla)^2 X_f = 0,$$

since ∇ is flat. This easily implies the integrability of J by expanding the brackets in the Nijenhuis tensor using that ∇ has zero torsion. In view of the fact that J is skew-symmetric for ω, we conclude that $(U, J, g = \omega \circ J)$ is para-Kähler. Finally, the flat torsion-free connection ∇ satisfies not only $d^\nabla J = 0$ but also $\nabla\omega = 0$, since the two-form ω on V_0 is constant. This proves that (U, J, g, ∇) is special para-Kähler. Now we check that (U, J, g, ∇, ξ) is a conical special para-Kähler manifold, that is $\nabla\xi = D\xi = \mathrm{Id}$. It is clear that $\nabla\xi = \mathrm{Id}$, since ξ is the position vector field in V_0. To prove the second equation, we first remark that the Levi-Civita connection is given by

$$D = \nabla + \frac{1}{2} J\nabla J.$$

(It suffices to check that D is metric and torsion-free.) Therefore, the equation $D\xi = \mathrm{Id}$ is reduced to $\nabla_\xi J = 0$. Let us first prove that ξ is para-holomorphic, that is $L_\xi J = 0$. By homogeneity of f and ω, we have the Lie derivatives

$$L_\xi f = 2f, \quad L_\xi df = 2df, \quad L_\xi \omega = 2\omega, \quad L_\xi \omega^{-1} = -2\omega^{-1}$$

and, hence,

$$L_\xi X_f = 0.$$

The latter equation implies

$$L_\xi J = L_\xi(\nabla X_f) = 0,$$

since ξ is an affine (and even linear) vector field. Using $\nabla_\xi - L_\xi = \nabla\xi = \mathrm{Id}$ we get that

$$\nabla_\xi J = L_\xi J + [\mathrm{Id}, J] = 0.$$

□

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