RESEARCH STATEMENT

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1. Description of research area

My research is at the interface of algebraic geometry, representation and invariant theory, and Lie theory. There is also some strong input from homological algebra, and the theory of derived categories/noncommutative algebraic geometry. A unifying theme in my work up to now has been the *explicit* or *classical geometry of varieties*, in particular *rationality questions for algebraic varieties* and *the explicit birational geometry of moduli spaces*, where the term "explicit" means that one wants to go beyond mere existence results and study varieties in terms of defining equations or concrete parametrizations that bring out their geometric or arithmetic features as clearly as possible.

This can be seen in the series of papers, discussed in more detail in Subsection 2.2 below, relating to the so-called *rationality problem in invariant theory* which has been one of my main research interests and to which I have made fundamental contributions; in particular, [B-B10-1] solved a long standing problem in the field, previously worked on by Shepherd-Barron, Katsylo, Dolgachev and others, and [B-B10-2], [BBB10], [BBB12] introduced many new methods and foundational techniques which have been proven fruitful since.

Another very influential recent direction has been the discovery of so-called *phantom* and *quasi-phantom categories* in semiorthogonal decompositions of derived categories, see Subsection 2.3. This concept was pioneered in the articles [BBS12], [BBKS12], and though anticipated by Katzarkov, it came as a surprise to most of the experts (conjectures by Bondal and Kuznetsov said that such structures should not exist) and quickly stimulated a great deal of subsequent research, e.g. [GorOrl], [A-O12], [GS], [Kuz12], [Kuz13]. It is expected that these categories can be used as obstructions to rationality in certain situations, e.g. for conic bundles, and could be regarded as some categorification of Donaldson invariants.

These categories have been discovered in the course of the study of *the irrationality problem for very general cubic fourfolds*. This also lead to some interesting results about Hodge structures on surfaces (Subsection 2.4), disproving a conjecture by V. Kulikov, and served as motivation to take up research directions which propose to study birational automorphism groups through notions of dynamical systems such as entropy and dynamical degrees (Subsection 3.2). It seems to me that such "archetypal" problems serve as beacons to direct one's research toward the discovery of new and significant mathematics; this phenomenon, to give direction to mathematics (which is necessary since there is reason to believe that it is not just a formalism or random game originating from a set of conventional definitions) may be even more important than the actual answer to the initial problems.

Date: May 2014.

2. Summary of results

2.1. **Work on canonical surfaces and flag manifolds.** The article [Boeh05] deals with the question how the ring structure of a Gorenstein A-algebra R is encoded in its minimal free resolution. The result was strengthened in [Boeh08] where it was shown that in the preceding setting one can obtain a resolution of R which is simultaneously Gorenstein symmetric and of Koszul module type. The theory was applied in [Boeh07] to show that the moduli space of canonical surfaces in $\mathbb{P}^4$ with $p_g = 5$, $q = 0$, $K^2 = 11$ satisfying a genericity assumption on the singularities of the canonical image is irreducible, unirational of dimension 38.

The paper [Boeh06] studied derived categories of coherent sheaves on flag manifolds G/P, especially the question of finding small generating sets or if possible complete exceptional sequences in them. In [Boeh06] small generating sets were found for symplectic isotropic Grassmannians, and a structure theorem for the derived categories of quadric bundles was proved. The latter was applied to derived categories of coherent sheaves on orthogonal isotropic Grassmannians. We also outlined a new approach to proving Beilinson-type theorems for flag manifolds G/P, based on a degeneration of the diagonal to a union of products of Schubert varieties and their duals due to M. Brion, and cellular resolutions of monomial ideals. We also confirmed a conjecture of F. Catanese on the structure of $D^b(\mathrm{Coh}(G/P))$ for type A-Grassmannians and quadrics. The conjecture is taken up in recent work of Kaneda and Ye.

2.2. **Work originating from the rationality problem in invariant theory.** The projects pertaining to this problem can be regarded as different facets of a study of birational geometry of group actions. They are at the interface between algebraic geometry, representation theory and group theory. They can be viewed as part of the general question -pursued from the times of Galois to Noether and Grothendieck- inhowfar properties of function fields (two structural operations) can be encoded group-theoretically (one structural operation). This connection has proven to be fruitful in algebraic geometry, arithmetic and topology and K-theory -from the beginnings of the solution theory of algebraic equations up to number theoretic applications in the framework of anabelian geometry.

Let G be a linear algebraic group, V a generically free G-representation. Then, in general, V/G is not stably rational (i.e. $\mathbb{C}(V)^G$ does not become a purely transcendental extension of $\mathbb{C}$ after adjunction of a certain finite number of indeterminates), if G is not supposed to be connected [Sa], [Bogo87]. However, stable rationality is known for generically free linear quotients of the simply connected groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, G_2 [Bogo86]. On the contrary, the question is very complicated for non-simply connected groups, the prototypical example being $\mathrm{PGL}_n(\mathbb{C})$ [For02]. The picture emerging from this is that the topological resp. connectivity properties of the group G have an impact on the (stable) rationality properties of V/G.
If one asks for the rationality of the quotients V/G, there is first and foremost an exhaustive positive answer in the case of $G = \mathrm{SL}_2(\mathbb{C})$ [Kat83], [Kat84], [Bogo84]. For further history and references to the literature one can consult [Boeh-book].

Let us now summarize our work on the rationality problem: [Boeh09] recasts Katsylo's important work [Kat92/2], [Kat96] in geometric terms, replacing some of his arguments by more geometric ones, e.g. by linking them to classical work of

Clebsch and Salmon.
The article [B-B10-1] proves the rationality of all moduli spaces of plane curves of sufficiently large degree d which presents an important advance over the previous work of Shepherd-Barron [Shep] and Katsylo [Kat89]. The hard part was to check that some data satisfied a genericity requirement, and this was done by showing that the data becomes periodic over a finite field $\mathbb{F}_p$ and using upper-semicontinuity over $\mathrm{Spec}(\mathbb{Z})$.
[B-B10-2] contains a method to calculate matrix representatives for equivariant bilinear maps of $\mathrm{SL}_3(\mathbb{C})$-representations in an algorithmically efficient way and a criterion for the stable rationality of quotients of some Grassmannians by an SL-action. This is subsequently used to prove in particular the rationality of the moduli space of plane curves of degree 34.
[B-B-Kr09] develops algorithmic tools that allow us to prove in this paper the rationality of all moduli spaces of plane curves of degree d, with the possible exception of 15 values of d for which rationality remains unknown.
[Boeh-book] is a detailed account of the topics related to the rationality problem, it also contains some new results on the moduli spaces of plane curves together with a theta-characteristic and a detailed exposition of the relation of the Hesselink stratification of the cone of Hilbert nullforms to the rationality problem. It also contains an account of the proof of the rationality of the moduli space of plane curves of degree 34 in a more conceptual setting. It is planned to include also new results on stable and unramified cohomology of Lie groups.
In [BBB10] we prove the rationality of quotients $V/\mathrm{ASL}_n(\mathbb{C})$, where V is indecomposable and sufficiently large, and in [B-B10-3] the rationality of the moduli space of Lüroth quartics and the related moduli space of Bateman seven-tuples of points in $\mathbb{P}^2$ are proven.
For the classical groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ we obtained in [BBB12] linear bounds in n for the levels of stable rationality of the respective generically free linear quotients, i.e. if G denotes one of these groups, a linear function $d(n)$ such that $V/G \times \mathbb{P}^{d(n)}$is rational for all generically free G-representations V. Up to now one knew bounds that are of the order of magnitude of the dimension of the group (quadratic in n). This improvement of the bounds for levels of stable rationality should in turn make it possible to sharpen the results of [BBB10].
In [BBC11] we obtained the rationality of the tetragonal divisor in $\mathfrak{M}_7$ (whereas the rationality of $\mathfrak{M}_7$ itself is unknown).

2.3. **Work originating from the irrationality problem for very general cubic fourfolds: phantom categories and structural properties of semi-orthogonal decompositions.** The problem to investigate the rationality properties of smooth hypersurfaces $X_d^n \subset \mathbb{P}^{n+1}$ of low degree d is very interesting, hard and "substantial".

Two main results are the Clemens-Griffiths proof of irrationality of cubic threefolds [CG72], using the intermediate Jacobian as an obstruction, and the one by Iskovskikh and Manin [IsMa71] of the irrationality of quartic threefolds, showing that every birational self-map of such a quartic is an isomorphism. For cubic hypersurfaces X_3^n with $n \geq 4$ practically nothing is known except that they are all unirational and for even n there are some rational ones, e.g. those containing two disjoint linear spaces of dimension $n/2$. However, the feeling is that the very general cubic should be irrational.

For the case of cubic fourfolds most efforts towards establishing this irrationality have been based on an attempt to find a suitable substitute for the intermediate Jacobian $J(Y) = H^{1,2}(Y)/(\mathrm{im}(H^3(Y,\mathbb{Z}))$ of a threefold Y with $h^{1,0}(Y) = h^{3,0}(Y) = 0$, used by Clemens and Griffiths. Their proof is based on two wonderful properties of polarized abelian varieties and, in particular, Jacobians:

(U) (Uniqueness of decomposition) The decomposition of polarized abelian varieties into indecomposable factors is unique.
(I) (Indecomposability in the curve case) Jacobians of curves are indecomposable as polarized abelian varieties (Riemann's theorem).

Namely, Clemens and Griffiths reduce the irrationality of a smooth cubic threefold Y to a statement about curves as follows: if Y were rational, its intermediate Jacobian $J(Y)$ would have to be, using (U), (I) and the behavior of intermediate Jacobians under blow-ups, a product of Jacobians of curves, which one can show it is not (e.g. it is Prym and its polarization divisor is singular in too low codimension).

In [BBS12-2] resp. [ABB13] it is shown that either (U) or (I) fail in the proposals by Kuznetsov [Kuz], [Kuz10] resp. Kulikov [Kul08] to generalize $J(Y)$ to cubic fourfolds. This led to the discovery of interesting new structures in derived categories [BBS12], [BBKS12].

Let X be a (smooth) cubic fourfold. Kuznetsov's proposal suggests that the rôle of $J(Y)$, or more precisely the part of it that remains after throwing away the contribution from Jacobians of curves, is taken by a certain triangulated category $\mathcal{A}_X$ defined by the semi-orthogonal decomposition

$$\mathrm{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\rangle.$$

It is a 2-Calabi-Yau category, and one can show that for certain X, this $\mathcal{A}$ cannot be equivalent to the derived category of a surface or an admissible subcategory in such a category. Moreover, it is indecomposable, so that, using the behavior of semi-orthogonal decompositions under blow-up, one would get irrationality of some cubic fourfolds if one had the analogue of property (U) above: maximal semi-orthogonal decompositions, i.e. those whose constituents cannot be decomposed further, should be unique, up to re-ordering of the pieces and equivalences of categories. It was shown in [BBS12-2] that this is exactly what fails, however: there are maximal exceptional sequences of different lengths in the derived category of the classical Godeaux surface, and this allows one to construct counter-examples to uniqueness of semi-orthogonal decompositions also on some rational fourfolds. Subsequently, Kuznetsov [Kuz13] has found another simple example of this type of failure. Apart from being a corrective to this intensely investigated and widely pursued categorical approach to the irrationality of cubic fourfolds, this raises lots of interesting new questions: can one get a complete overview in certain cases of all possible maximal semi-orthogonal decompositions, up to equivalence? Can one restore uniqueness by endowing the categories with extra structure, such as a stability condition? Although uniqueness fails, does maybe the Calabi-Yau category $\mathcal{A}_X$ have to occur as an indecomposable constituent in every maximal semi-orthogonal decomposition of $\mathrm{D}^b(X)$?

The work on this counterexample and the detailed study of the derived category of the classical Godeaux surface has moreover produced potentially even more interesting objects: (geometric) quasi-phantom and phantom categories. It had been a long-standing conjecture of Bondal that the latter, and a conjecture by Kuznetsov

[Kuz09] that the former should not exist. Here a quasi-phantom category is a non-trivial admissible subcategory $\mathcal{P}$ of some bounded derived category of a smooth variety such that the Hochschild homology vanishes: $\mathrm{HH}_\bullet(\mathcal{P}) = 0$. For a phantom category one requires in addition that also the Grothendieck group $K_0(\mathcal{P}) = 0$. At first sight, this is a pathology because one would like to be able to determine whether a piece in a semi-orthogonal decomposition is zero or not by using invariants like $\mathrm{HH}_\bullet$ or K_0 which are additive on semi-orthogonal decompositions. The article [BBS12] was the first to exhibit a quasi-phantom, again on the classical Godeaux surface. Subsequently, a phantom was found on the Barlow surface [BBKS12] and also on general determinantal Barlow surfaces sufficiently close to the one constructed by Barlow in [Barl1]. This was accompanied by quite a boost of activity in this field: [A-O12] constructed quasi-phantoms on Burniat surfaces and pointed out their moduli or deformation-theoretic relevance, namely that all the moduli of Burniat surfaces are encoded in the variation of the quasi-phantom. This was made precise, among other things, in [Kuz12] who proved that in many cases the second Hochschild cohomology of the surfaces is isomorphic to that of the (quasi-)phantom subcategory. [GS] constructed a quasi-phantom on the Beauville surface and [GorOrl] constructed phantom categories around the same time as [BBKS12] on products of surfaces of general type with $p_g = 0$ and coprime torsion. Recently, Keum and Katzarkov announced the construction of quasi-phantoms on some fake projective planes.

Naturally, the study of (quasi-)phantom categories is still in its infancy, and interesting open questions abound: what are some characterizing structural properties of phantoms? Can they be used to better understand (the moduli spaces of) surfaces of general type with $p_g = q = 0$? This is conceivable, and every such surface could conjecturally have a (quasi-)phantom. How can phantoms be interpreted from the point of view of homological mirror symmetry? Is it possible to rule out in certain simple cases, such as that of $\mathbb{P}^n$, the existence of a semi-orthogonal decomposition containing a phantom as one of its constituents?

2.4. **Work originating from the irrationality problem for very general cubic fourfolds: Hodge theory of surfaces.** Kulikov in [Kul08] suggested another candidate for a substitute of the intermediate Jacobian $J(Y)$ of a threefold Y. Since polarized abelian varieties are the same thing as integral polarized weight 1 Hodge structures, it is natural to look at integral polarized weight 2 Hodge structures here, in particular, the one on the primitive middle cohomology of a very general cubic fourfold X. Kulikov proves that, if X were rational, then there would be a surface S such that the integral polarized Hodge structure (IPHS) $-\mathcal{T}_S(-1)$, where $\mathcal{T}_S$ is the transcendental lattice of S, splits off the IPHS on the primitive cohomology of the cubic as a proper summand. He then conjectures that the IPHS $\mathcal{T}_S$ of any surface should be indecomposable; this is the natural analogue of property (I) of subsection 2.1, the indecomposability of the Jacobian of a curve as polarized abelian variety. However, in the absence of a geometric object such as the theta divisor in the weight 2 case, there is some potential for Kulikov's conjecture to fail, and in [ABB13] it is shown that it actually does fail (on the sextic Fermat surface).

Kulikov subsequently suggested another conjecture to the effect that such decomposability should not occur in families. This is very interesting and still open, but note that Izadi's work [Izadi99] shows that the Hodge structures coming from the cubics are Prym and do occur as summands in a variation of Hodge structure coming from a family of surfaces if one is allowed to work over $\mathbb{Z}[1/\sqrt{2}]$.

2.5. **Cremona groups and general structure theory of rational varieties.** If X is a rational variety, then, by definition, a general point $x \in X$ has a Zariski open neighborhood $U \ni x$ which is isomorphic to some open subset of $\mathbb{A}^n$. In [Gro89], Gromov asked whether this is true for *every* point in X if one makes the (obviously necessary) assumption that X be smooth. Call a smooth rational variety *uniformly rational* if Gromov's question has a positive answer for it. Clearly, this is so for rational varieties X which are *quasi-homogeneous*, i.e. such that for every nonempty Zariski open U and every point $x \in X$ there exists a birational map $X \dashrightarrow X$ defined at x and mapping a neighborhood of x isomorphically into U. For example, cubic hypersurfaces are quasi-homogeneous (the class of reflections in points $p \in X$ is sufficient to check the quasi-homogeneity). Thus one sees that Gromov's question is closely related to a question about the existence of sufficiently many Cremona transformations in $\mathrm{Bir}(\mathbb{P}^n)$ of a prescribed type.

In [BB13] it is shown that small and big resolutions of nodal cubic threefolds are uniformly rational if they are algebraic. On the contrary, it is also shown there that Gromov's question has a negative answer if one asks it in the more general setting of Moishezon manifolds: there are examples of non-algebraic Moishezon manifolds which are bimeromorphic to $\mathbb{P}^n$, but where for some points x there is no bimeromorphic map $X \dashrightarrow \mathbb{P}^n$ which is defined at and an isomorphism around x. Moreover, [BB13] also derives some necessary conditions for Gromov's question to always have a positive answer for algebraic varieties; they boil down to asking whether $\mathrm{Bir}(\mathbb{P}^n)$ is sufficiently large and are closely related to the question which subvarieties of $\mathbb{P}^N$ are Cremona equivalent to each other.

2.6. **Stable and unramified group cohomology.**

The stable cohomology $H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ of a finite group G can be defined as the image of the group cohomology $H^*(G, \mathbb{Z}/p)$ in the Galois cohomology of the function field $k(V/G)$ for some generically free linear G-representation V (k being the ground field). See e.g. [Bogo93], [Bogo95] for background, as well as [GMS], which revolves around the closely related notions of cohomological invariants and negligible classes of Serre. Let us just mention that there is an important subring $H^*_{\mathrm{nr}}(G, \mathbb{Z}/p) \subset H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ of unramified elements which constitute natural obstructions to the stable rationality of V/G (and have been used as such in [Sa], [Bogo87], [CTO], [CT], [GMS] and elsewhere), and that the quotients V/G play a rôle for stable cohomology similar to Eilenberg-MacLane spaces in topology.

In [BB11] the stable cohomology $H^*_{\mathrm{s}}(\mathfrak{A}_n, \mathbb{Z}/p)$ for the alternating groups $\mathfrak{A}_n$ and odd primes p is completely determined via a new inductive residue-theoretic method which allows one to show that the stable cohomology of each group in a class of groups $\mathcal{C}$ is detected by abelian subgroups; essentially, $\mathcal{C}$ should be closed under taking centralizers of elements and all groups in $\mathcal{C}$ should have stably rational quotients (or trivial unramified cohomology). In [BB12] the stable cohomology of iterated wreath products of groups $\mathbb{Z}/p$ is computed, showing that if $\mathcal{C}$ is a class of groups as before, then closing $\mathcal{C}$ under wreath products with groups $\mathbb{Z}/p$, isoclinism, and taking finite direct products, gives a class of the same type. Moreover, it is shown there that isoclinic groups have isomorphic unramified cohomology groups.

These methods presumably allow one to compute the stable cohomology of much wider classes of groups, e.g. many finite groups of Lie type.

3. Research projects and future directions

They are divided into several blocks.

3.1. **Projects related to Hodge-theoretic and categorical ramifications of the irrationality problem for cubic fourfolds.** As already pointed out in Subsection 2.3, the deeper understanding of structural features of phantom or quasi-phantom categories is very desirable; a basic question is to find a "nice" strong generator in them. One way to get such is to project a strong generator on the ambient X into the (quasi-)phantom $\mathcal{P}$. By [Orl09], on a surface, one can take $\mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2}$ for a very ample line bundle $\mathcal{L}$ on X, as the strong generator in the ambient derived category. It is intended to carry out this procedure on the classical Godeaux surface, Barlow surface or some Burniat surfaces. This is likely to lead to quite unappealing and complicated complexes in $\mathcal{P}$, but possibly after subsequent modification, they become sufficiently nice to allow some structural conclusions about $\mathcal{P}$.

Another project is to produce (quasi-)phantoms on varieties which are not directly derived from surfaces of general type with $p_g = 0$ (of course, one can always take products of these, or blow up a variety Y in such a surface). A candidate would be the unirational, but not stably rational Artin-Mumford examples [A-M79] which are some conic bundles over rational surfaces.

A more far-reaching question is to elicit the reason for the appearance of phantoms: a starting point here could be to more conceptually connect the derived categories of the surfaces of general type with $p_g = 0$ which have so far been shown to contain phantoms, to the derived categories of their Del Pezzo partners: those surfaces of general type are fake (homology) del Pezzos. It is conceivable that the phantoms are a symptom of the exotism of the complex structure on the same underlying topological manifold. A different approach would be to relate the existence of (quasi-)phantoms to the deformation-theoretic behavior of the varieties under consideration since this is also used in the proof of their existence [BBKS12].

A different set of problems is connected to the failure of the Jordan-Hölder property for semi-orthogonal decompositions discussed in Subsection 2.3. It is intended to investigate if the Jordan-Hölder property holds on del Pezzo surfaces or if there are possibly maximal decompositions on del Pezzo surfaces some of whose pieces are not generated by an exceptional object. Moreover, one should try to see if at least some features are always *common* to all maximal semi-orthogonal decompositions on a given variety: for example, if one of them contains a Calabi-Yau category, i.e. one where the Serre functor is just a shift, is this maybe true for all others as well?

Let us pass to questions and projects related to Kulikov's Hodge-theoretic approach to irrationality of very general cubic fourfolds as mentioned in Subsection 2.4. An important point here is to given an answer to the following: if $\mathcal{S} \to B$ is a projective family of surfaces over some analytic base space B such that for a very general fiber $\mathcal{S}_b$ the IPHS on the transcendental lattice of $\mathcal{S}_b$ is decomposable, is it true that the image of the period map of this family is always a point? This statement would imply irrationality of a very general cubic fourfold [ABB13]. Potential counterexamples could be produced by deforming the Fermat sextic surface suitably, or smoothing some normal-crossing surfaces with decomposable mixed Hodge structure.

There is a beautiful byroad issuing from this Hodge-theoretic set-up: this is to see inhowfar the Tate-resolution methods in [EFS03] give a quick and possibly more powerful approach to proving infinitesimal Torelli theorems, and if one can use this method and combine it with results from [BC04] to produce examples of surfaces S, possibly ones isogeneous to a product of curves, where K_S is very ample, $q = 0$, but infinitesimal Torelli fails. In a similar spirit, it is intended to investigate if known inequalities in surface theory can be sharpened or complemented in a geometrically relevant way by using Boij-Söderberg theory [ES08].

3.2. **Projects related to Cremona groups and general structure theory of rational varieties.** It is intended to further pursue Gromov's question described in Subsection 2.5, verifying it for wider classes of smooth threefolds; finding potential examples of smooth rational varieties which are not uniformly rational is closely related to problems which are probably amenable to some of the techniques developed in [MelPol09], [MelPol12] and [Mel12]: for example, which obstructions, if any, are there for a rational surface S in $\mathbb{P}^3$ to be Cremona equivalent to a scroll? Notice also that Gromov's question seems to be related to the *contraction problem*: given a smooth divisor $D \subset X$ in a smooth algebraic variety X such that $f : D \to Z$ is a $\mathbb{P}^r$-bundle over a smooth variety Z and the conormal bundle of D in X restricts to $\mathcal{O}_{\mathbb{P}^r}(1)$ on every fiber of f, does there exist a modification $X \to Y$ blowing down D to $Z \subset Y$ with Y smooth and *algebraic*. By [Moish67] one knows that this is always possible if one allows Y to be a Moishezon manifold, or equivalently by [Art70], a regular algebraic space. However, Gromov's question does not have a positive answer in this more general context as mentioned in 1.3. Moreover, one knows that if X is projective, $D = \mathbb{P}^{\dim X - 1}$, Z a point, then Y exists as a projective variety (by Castelnuovo's argument). So this is the easiest case to start with. One also knows by [Ishii77], that in the projective set-up the missing condition for contractibility of D to a projective Y is the existence of a globally generated line bundle $\mathcal{L}$ on X whose restriction to D is the pull-back under f of an ample line bundle on Z. The relation to Gromov's question is that the hard part in proving that it always has a positive answer is to show (with the preceding notation): X uniformly rational $\implies$ Y uniformly rational. On the other hand, it is known and not too hard to see that a blow-up of a uniformly rational variety in a smooth center is again uniformly rational [BB13]. Thus a basic difficulty may be to characterize D as above (i.e., contractible to something smooth algebraic) birationally, i.e. if $\mathbb{P}^n \dashrightarrow X$ is a birational map, what can one say about the strict transform $\overline{D}$ of D in $\mathbb{P}^n$? What are its singularities, or the singularities of the pair $(\mathbb{P}^n, \overline{D})$? In the case where Z is a point, does there exist a Cremona transformation $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ mapping $\overline{D}$ to a point?

A different approach to Gromov's question could be the following: for Y smooth and rational, $y \in Y$ a point, find a family of rational curves $\{C_t\}$ through y such that each C_t is smooth at y and every tangent direction in T_yY is realized by exactly one curve in the family. This means one wants to find what would be the lines in $\mathbb{A}^n$ through y. This is reminiscent of the approach in [Mori79].

In the context of the irrationality problem for cubic fourfolds briefly exposed in 2.3, it is natural to consider the family of all smooth cubic fourfolds over its quasi-projective parameter space. This leads to the following question which should be settled: given a projective family $\mathcal{X} \to B$ such that every (geometric) fiber $\mathcal{X}_b$, $b \in B$ a (closed) point, is rational, is it true that $\mathcal{X}$ is birationally isotrivial? That is, is there an étale cover B' of B such that $\mathcal{X} \times_B B'$ is rational over B'? In other words, is the generic fiber $\mathcal{X}_{k(B)}$ geometrically rational (rational over $\overline{k(B)}$) in this case? This can be seen as a birational version of the triviality theorem of Grauert-Fischer. Some techniques from [BF13] may be helpful in proving such a statement.

In connection with the irrationality problem for very general cubic fourfolds, there is another project which is potentially harder than the preceding two: is there a way to show that the birational automorphism group of a very general cubic fourfold is, for a suitable notion of size, "smaller" than the Cremona group of $\mathbb{P}^4$? The recent applications of the theory of complex dynamical systems to the structure of Cremona groups as in [CL13] and [BC13], as well as input from the Sarkisov program may give clues as to how to approach this. Here are some more details on what is envisaged in this respect: to a d-dimensional projective algebraic variety X and a birational map $f : X \dashrightarrow X$ one can associate a d-tuple of real numbers $\lambda_i(f) \geq 1$, $i = 1, \ldots, d$, called the dynamical degrees of f; there are several different ways of defining these. For example, one can look at the induced linear action $f^* : H^{i,i}(X, \mathbb{R}) \to H^{i,i}(X, \mathbb{R})$ on the Hodge pieces of the cohomology, and let $\varrho_i(f)$ be the spectral radius of this map. Then the λ_i describe the growth rate of the spectral radii of the iterates of f:

$$\lambda_i(f) := \liminf_{n\to\infty} \left(\varrho_i(f^n)\right)^{\frac{1}{n}} .$$

From a more algebraic point of view, one has with H a hyperplane section of X

$$\lambda_i(f) = \liminf_{n\to\infty} \left((f^n)^*(H^i).H^{d-i}\right)^{\frac{1}{n}} ,$$

which explains the name dynamical degrees. For more of their foundational properties, see e.g. [Guedj10]. These quantities are a measure of the dynamical complexity of f and invariants under birational conjugacy; in particular, the maximum of the logarithms of the $\lambda_i(f)$ is an upper bound for the topological entropy $h(f)$ (which is not invariant under birational conjugacy), and conjecturally equal to it on some birational model of X. For birational f one always has $\lambda_d(f) = 1$ (one can define dynamical degrees more generally for a dominant rational map, λ_d always measures the topological degree), so

$$\Lambda(X) := \{(\lambda_1(f), \ldots, \lambda_{d-1}(f)) \mid f \in \mathrm{Bir}(X)\} \subset \mathbb{R}^{d-1}$$

measures the birational complexity of X from a dynamical point of view. It is called the dynamical spectrum. It is reasonable to expect that the topological or metric properties of $\Lambda(X)$ may help to distinguish irrational and rational varieties: already in the case of Riemann surfaces (complex dimension 1), holomorphic self-maps of hyperbolic or Euclidean surfaces display a rather tame behavior, and the intricacy of Julia sets and related notions only shows up on the Riemann sphere. For algebraic surfaces, results in [BC13] show that $\Lambda(X)$ is discrete for irrational surfaces whereas for $\mathbb{P}^2$ it is comparatively complicated: it is closed, has accumulation points, in fact infinite Cantor-Bendixson rank, and to the right of every dynamical degree

$\lambda_1(f) \in \Lambda(\mathbb{P}^2)$ there is a small interval free of further dynamical degrees, a so-called "gap". Moreover, $\lambda_i(f)$ is always a particular type of algebraic integer in the surface case, a Salem or Pisot number. It follows from results of [Mull02], [Mull07] and [BC13] that the smallest gap is $]1, \lambda_L[$, where λ_L is the Lehmer number (approximately 1.17628081).

One should try to determine features of the spectrum of $\mathbb{P}^3$ and $\mathbb{P}^4$ and compare them to spectra of the general cubic hypersurfaces of the same dimensions. This may lead to a proof of the irrationality of a very general cubic fourfold. More precisely, it is conceivable that there are gaps in $\Lambda(X)$ also in the higher-dimensional case, and that the smallest gap after 1 has a special significance. If so, it is planned to determine or bound its size in terms of geometric properties of X. The proofs in [Mull02], [Mull07] use the action of the birational automorphism group of $\mathbb{P}^2$ on the Picard-Manin space of $\mathbb{P}^2$ via an infinite Weyl group, and the combinatorial theory of Coxeter groups. It is possible that the first part, essentially an incarnation of Noether's factorization theorem, can be replaced by factorization into Sarkisov links in the higher-dimensional case, and that, using a suitable analogue of the Picard-Manin space, one may use techniques from combinatorial and/or geometric group theory to extract the necessary information on the gap sizes from the otherwise very "infinite" (e.g. not countably generated in degrees ≥ 1) birational automorphism groups.

Lastly, there is a related question from [BKK13] which will be further investigated: a potential characterization of unirational varieties within the class of rationally connected varieties; namely, X being unirational is conjecturally equivalent to X being *stably birationally infinitely transitive*. The latter means that after adjoining finitely many indeterminates, $k(X)$ has a model Y on which the part $\mathrm{SAut}(Y)$ of the automorphism group generated by subgroups isomorphic to $\mathbb{G}_a$ acts infinitely transitively, in the sense that any tuple of points on Y can be carried into every other such tuple of the same cardinality by such an automorphism.

3.3. **Projects related to stable and unramified group cohomology.** As already mentioned in Subsection 2.6, the results of [BB12] should be applied to show that the stable cohomology of finite groups of Lie type is detected by abelian subgroups, and to possibly compute it completely afterwards. This requires an understanding of the Sylow subgroup structure as well as the conjugacy classes of abelian subgroups in these groups.

Another project consists in a detailed study of the notions of essential dimension and stable cohomological dimension of an algebraic group G: here the first is defined as the smallest possible value of $\dim(Y/G)$ where Y is a G-compression of a generically free linear G-representation; here a G-compression of a generically free G-variety X is a dominant G-equivariant rational map $X \dashrightarrow Y$ where Y is another generically free G-variety. On the other hand, the stable cohomological dimension is the maximum integer n such that there exists a finite G-module M with $H^n_{\mathrm{s}}(G, M) \neq 0$. One goal is to show, for example, that the stable cohomological dimension of a finite p-group is bounded by the minimal length of a normal series for G with cyclic quotients and to give examples where strict inequality holds. Moreover, the essential dimension always bounds the stable cohomological dimension, and it will be interesting to study when and why one can have strict inequality.

A question of great interest here is to find, for a given finite group G a discrete group Γ with a surjection $\Gamma \to G$ which yields stabilization of the cohomology of G, i.e. such that the kernel of $H^*(G, \mathbb{Z}/p) \to H^*(\Gamma, \mathbb{Z}/p)$ consists precisely of the unstable classes. This will be further studied with special regard to the question how far fundamental groups of the open parts of toric representations are from accomplishing this goal.

Lastly it is also proposed to complete the determination of the stable cohomology of simple algebraic groups.

3.4. **Projects related to rationality properties of generically free linear group quotients.** In a broad sense, one wants to find further evidence for the empirical rule that the topological (connectivity) properties of an algebraic group G govern the (stable) rationality of quotients V/G. A truly conceptual approach to this question will most likely require the development of fundamentally new methods, which maybe take their point of departure from theories which connect topological and holomorphic/algebraic properties, like Gromovs h-principle or the more recent work of Voevodsky-Rost on the Bloch-Kato conjecture.

It is planned to generalize the results of the article [B-B10-1] as far as possible to the case of hypersurfaces of sufficiently large degree d in $\mathbb{P}^n$, i.e. to investigate the rationality properties of the moduli space $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee)/\mathrm{SL}_{n+1}(\mathbb{C})$ for large d (as long as stable rationality of generically free $\mathrm{PGL}_{n+1}(\mathbb{C})$-quotients is unknown, it seems reasonable to restrict to the case where d and $n+1$ are relatively prime). The methods of [B-B10-1] (the construction of suitable covariants etc.) should carry over to the more general case to obtain rationality for all sufficiently large d for fixed n. That which presumably requires new ideas is a proof that for all n there is a $d(n)$ such that $\mathbb{P}(\mathrm{Sym}^d(\mathbb{C}^{n+1})^\vee)/\mathrm{SL}_{n+1}(\mathbb{C})$ is rational for $d \geq d(n)$.

In [BBB12] bounds for the levels of stable rationality for generically free linear quotients V/G, G a classical group, were obtained which grow linearly with the rank of the group G; this should be applied to prove a more precise result about affine groups $\mathrm{ASL}_n(\mathbb{C})$: in this case, [BBB10] shows that generically free linear quotients $V/\mathrm{ASL}_n(\mathbb{C})$ are always rational if only $\dim V$ is sufficiently large and V is indecomposable. But the afore-mentioned result should enable one to classify the exceptions explicitly and to substantially sharpen the result. One should try to obtain statements of a similar type for larger classes of nilpotent extensions of semi-simple algebraic groups, e.g. jet-groups.

One of the most important open questions here is obviously the one if there exist nonrational quotients V/G for a connected G. It is not at all clear at this point how this could be approached, but, to return to the previously used image, it also serves as a beacon. A closely related problem is the stable rationality of the groups of the Spin-series as well as the exceptional groups (other than G_2). This may be more amenable to begin with.

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