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Bachelor of Science Thesis

Quaternionic Analysis, Representation Theory and Physics

(Quaternionische Analysis, Darstellungstheorie und Physik)

by

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Einleitung

Die vorliegende Bachelor-Arbeit basiert auf dem Paper [3] von Igor Frenkel und Matvei Libine desselben Titels. In diesem werden einige Probleme der quaternionischen Analysis unter einem darstellungstheoretischen Zugang behandelt. Ferner werden Anwendungen einiger tieferer Resultate der quaternionischen Analysis auf physikalische Probleme dargestellt.

Im größten Teil dieser Arbeit konzentriere ich mich auf analytische Aspekte von Teilen des Papers, aber im zweiten Kapitel werde ich zumindest teilweise auf darstellungstheoretische und physikalische Aspekte eingehen. In Kapitel 1 werde ich eine Einführung in einige grundlegenden Konzepte der quaternionischen Analysis geben. Es ist in drei Abschnitte eingeteilt.

Im ersten Abschnitt werde ich die Notation einführen, die im Rest der Arbeit verwendet wird, die Quaternionen selbst konstruieren und einige einfache Aussagen über sie beweisen.

Im zweiten Abschnitt werde ich hauptsächlich Änlichkeiten und Unterschiede zwischen klassischer Funktionentheorie und quaternionischer Analysis darstellen. Weiterhin werde ich eine Klasse von Funktionen definieren, die in der quaternionischen Analysis von einem ähnlichen Interesse sind wie die holomorphen Funktionen in der Funktionentheorie. Das Hauptresultat dieses Abschnittes ist eine quaternionische Version der Cauchyschen Integralformel, aus der man Korollare gewinnen kann, die ähnlich sind zu den bekannten Korollaren aus der Cauchyschen Integralformel im komplexen Fall. Diese Darstellung basiert auf den Abschnitten 2.1 und 2.2 des Papers [3].

Im dritten Abschnitt werde ich das Konzept der konformen Transformation auf der Einpunkt-Kompaktifizierung $\hat{\mathbb{H}}$ der Quaternionen einführen, welche eine Untergruppe der Gruppe der Diffeomorphismen von $\hat{\mathbb{H}}$ auf $\hat{\mathbb{H}}$ ist. Die konformen Transformationen stellen eine Verallgemeinerung der aus der Funktionentheorie bekannten Möbius-Transformationen dar und haben ähnliche Eigenschaften. Wir werden sie als relativ natürliche Wirkung der allgemeinen linearen Gruppe $GL(2, \mathbb{H})$ auf quaternionische projektive Räume konstruieren, die wir über einen Diffeomorphismus auf den Raum $\hat{\mathbb{H}}$ übertragen. Das Hauptresultat dieses Abschnittes ist der Satz, dass konforme Transformationen Hyperebenen und Sphären in $\hat{\mathbb{H}}$ erhalten. Dieser Abschnitt ist hauptsächlich eine Erweiterung der Ausführungen im ersten Teil des Abschnittes 2.4 im Paper [3]. Obwohl einige – vor allem rechnerische – Resultate in diesem Abschnitt auf Resultaten aus dem Buch [4] basieren, ist die Behandlung von quaternionischen projektiven Räumen und die Konstruktion der Wirkungen π_ℓ und π_r größtenteils originär.

Nach der Behandlung grundlegender Konzepte der quaternionischen Analysis in Kapitel 1, fahre ich in Kapitel 2 mit Anwendungen der quaternionischen Analysis fort.

Im ersten Abschnitt von Kapitel zwei folge ich dem Abschnitt 2.8 in [3] und zeige eine quaternionische Version der vierdimensionalen Poisson-Formel für den Einheitsball. Dieser Beweis wird Konzepte aus der Darstellungstheorie von Lie-Gruppen benötigen und im Wesentlichen auf eine Anwendung des Schurschen Lemmas heruntergebrochen. Unglücklicherweise habe ich es nicht geschafft, die Einfachheit der dabei betrachteten Darstellung zu zeigen, auf deren Beweis die Autoren von [3] auch nicht näher eingehen. Ferner werde ich nicht die Äquivarianz einer gewissen linearen Abbildung bezüglich dieser Darstellung zeigen. Aus diesen Gründen, ist das in dieser Arbeit dargestellte Argument kein vollständiger Beweis, weshalb ich am Ende des Abschnittes eine Referenz für die Gültigkeit der Poisson-Formel im betrachteten Fall angeben werde, um sie im nächsten Abschnitt sicher nutzen zu können.

Im zweiten und letzten Abschnitt von Kapitel 2 gebe ich eine Anwendung der quaternionischen Analysis zur Konstruktion von Eigenfunktionen und somit auch Eigenwerten für den Hamilton-Operator des Wasserstoffatoms. Dieser Abschnitt basiert hauptsächlich auf Abschnitt 2.9 in [3], obwohl es einen kleinen Rechenfehler in diesem Paper gibt, sodass ich die Rechnung in einer korrigierten Version in dieser Arbeit nachvollziehe. Die hierbei dargestellte Konstruktion wird eine Anwendung der quaternionischen Poisson-Formel für den Einheitsball aus dem vorherigen Abschnitt sein. Hierbei werde ich eine bestimmte Klasse von konformen Transformationen, die Cayley-Transformationen, nutzen, um die Poisson-Formel für den Einheitsball in eine Integralformel auf der oberen Halb-Hyperebene in $\mathbb{H}$ umzuwandeln, wobei ich den Rand dieser Halb-Hyperebene mit dem euklidischen Raum $\mathbb{R}^3$ identifiziere werde. Unter Verwendung der Fouriertransformation und der Fourier-Kotransformation kann ich dann diese Integralformel in eine Lösung der Schrödingergleichung für das Coulumb-System umwandeln, was wiederum direkt Eigenwerte des Hamilton-Operators für das Wasserstoffatom liefert.

Introduction

This Bachelor Thesis is based on the paper [3] of the same titel by Igor Frenkel and Matvei Libine, which elaborates on a representation theoretic access to quaternionic analysis. It also shows some applications of deeper results of quaternionic analysis to problems in physics.

The biggest part of this thesis will mainly focus on analytic aspects of parts of the paper, but in the second chapter I will also treat some representation theoretic aspects as well as physical aspects. In chapter 1 of this thesis I will give an introduction to main concepts of quaternionic analysis. It will be devised into three sections.

In the first section I will introduce the notation used throughout this thesis, construct the quaternions itself and show some simple results about them.

The second section will mostly treat the similarities and differences between complex analysis and quaternionic analysis and define the classes of functions that are mainly treated in quaternionic analysis. The main result of this section will be a quaternionic version of the Cauchy integral formula, from which one could derive corollaries similar to the corollaries from the Cauchy integral formula in the complex case. This treatment is based on sections 2.1. and 2.2 of the paper [3].

The third section will introduce the concept of conformal transformations on the one-point compactification $\hat{\mathbb{H}}$ of the quaternions, which form a subgroup of the group of diffeomorphisms from $\hat{\mathbb{H}}$ onto $\hat{\mathbb{H}}$. The conformal transformations are generalizations of the Möbius transformations in the complex case and have similar properties. We will construct them as relatively natural actions of the general linear group $GL(2, \mathbb{H})$ on certain quaternionic projective spaces which we carry over to $\hat{\mathbb{H}}$. The main result of this section will be the proposition that conformal transformations preserve hyper-surfaces and spheres in $\hat{\mathbb{H}}$. This section mainly is an extension of the first part of section 2.4. in the paper [3]. Although some—mainly calculational—results in this chapter are based on results from the book [4], the treatment of quaternionic projective spaces and the construction of the actions π_ℓ and π_r is mostly original work.

After the treatment of basic concepts of quaternionic analysis in chapter 1 I will proceed in chapter 2 with applications of quaternionic analysis.

In the first section of chapter two, I will follow section 2.8 in [3] to show a quaternionic version of the four-dimensional Poisson formula for the unit ball. This proof will rely on concepts from the representation theory of Lie groups and will essentially be broken down to an application of Schur's Lemma. Unfortunately I was not able to show the simplicity of the involved representation, which the authors of [3] just state, and I will not show the equivariance of a certain linear map with respect to this representation. This means that the argument presented in this thesis will not be a full proof. For this reason I will include a reference for the validity of the Poisson formula in order to be safely able to use it in the next section.

The second and last section of chapter two treats an application of the theory developed to this point to construct eigenfunctions and therefore find eigenvalues for the Hamiltonian operator of the Hydrogen atom. This treatment will mainly be based on section 2.9 in [3], although there is a minor miscalculation in this paper, which I will present in a repaired way in this thesis. The construction we will present, will rely on the quaternionic Poisson formula for the unit ball, which was treated in the section before. I will use a certain class of conformal transformations, the Cayley transformations, to translate the Poisson formula on the unit ball into an integral formula on an upper-half-hyperplane in $\mathbb{H}$, where I identify the boundary of this half-hyperplane with the euclidean space $\mathbb{R}^3$. Using Fourier transformations and Fourier cotransformatons I can then translate this integral formula into a solution of the Schrödinger equation for a Coloumb system, from which one directly gets eigenvalues of the Hamiltonian operator of the Hydrogen atom.

Chapter 1

Quaternionic Analysis and Conformal Transformations

1.1 Basic Notations and Results

Before we are able to start the exposition of the subject of this thesis we first have to define basic objects, fix basic notation and develop some very basic results about the space of quaternions.

Definition 1.1.1 (Real and Complex Numbers, Matrix Groups)

1. Let R be a ring with ring-addition $+$ and ring-multiplication $\cdot$ and let 1 denote the neutral element of multiplication. Then we call R a *skew-field*, if and only if
$$\forall\, a \in R: \ \exists\, b \in R: \ a \cdot b = b \cdot a = 1.$$
We call R a *field*, if and only if it is a skew-field and the multiplication is commutative.

2. We will denote the *field of real numbers* by the letter $\mathbb{R}$ and the *field of complex numbers* by the letter $\mathbb{C}$. Furthermore we will denote the imaginary unit of the complex numbers by $\mathfrak{i}$. The *complex conjugate* of an $a \in \mathbb{C}$ will be denoted by $\bar{a}$.

3. Let K be a skew-field and $n \in \mathbb{N}$. Then we denote by $M(n, K)$ the associative K-algebra of $n \times n$-matrices over K and by $GL(n, K)$ the group of invertible $n \times n$-matrices over K.

4. By $SU(n)$ we denote the group of unitary $n \times n$-matrices with determinant 1. In particular we have

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M(2, \mathbb{C}) \mid a, b \in \mathbb{C}, |a| + |b| = 1 \right\}$$

There are different ways to define the skew-field of quaternions. We will follow the way in [3, section 2.3] and define the quaternions as a real subalgebra of $M(2, \mathbb{C})$, while e.g. in [4] the quaternions are defined as a certain Clifford algebra.

Definition 1.1.2 (Quaternions) We define the *ring of quaternions* $\mathbb{H}$ as the following real subalgebra of $M(2, \mathbb{C})$:

$$\begin{aligned} \mathbb{H} := & \left\{ X \in M(2, \mathbb{C}) \mid \exists\, x_0, x_1, x_2, x_3 \in \mathbb{R} : \; X = \begin{pmatrix} x_0 - \mathrm{i}x_3 & -\mathrm{i}x_1 - x_2 \\ -\mathrm{i}x_1 + x_2 & x_0 + \mathrm{i}x_3 \end{pmatrix} \right\} \\ = & \left\{ X \in M(2, \mathbb{C}) \mid \exists\, a, b \in \mathbb{C} : \; M = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \right\} \end{aligned}$$

Remark 1.1.3.
From the definition we see that a basis of $\mathbb{H}$ *as a real vector space is given by*

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 := \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \; e_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; e_3 := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

From now on we will fix the names e_0, e_1, e_2, e_3 *to denote exactly these elements of* $\mathbb{H}$.

Theorem 1.1.4. *(Quaternions as skew-field)*
The ring of Quaternions $\mathbb{H}$ *is a skew-field.*

Proof:
At first we show that $\mathbb{H}$ is an $\mathbb{R}$-subalgebra of $M(2, \mathbb{C})$ and thus especially a subring. It was mentioned in Definition 1.1.2 that $\mathbb{H}$ is a real subvectorspace of $M(2, \mathbb{C})$. Therefore only closedness under multiplication remains to be shown. We can see that by considering $X, Y \in \mathbb{H}$. Then there exist $a_X, b_X, a_Y, b_Y \in \mathbb{H}$ such that

$$X = \begin{pmatrix} a_X & b_X \\ -\overline{b_X} & \overline{a_X} \end{pmatrix}, \; Y = \begin{pmatrix} a_Y & b_Y \\ -\overline{b_Y} & \overline{a_Y} \end{pmatrix}.$$

We see now that

$$
\begin{aligned}
X \cdot Y &= \begin{pmatrix} a_X & b_X \\ -\overline{b_X} & \overline{a_X} \end{pmatrix} \cdot \begin{pmatrix} a_Y & b_Y \\ -\overline{b_Y} & \overline{a_Y} \end{pmatrix} \\
&= \begin{pmatrix} a_X a_Y - b_X \overline{b_Y} & a_X b_Y + b_X \overline{a_Y} \\ -\overline{b_X} a_Y - \overline{a_X}\, \overline{b_Y} & -\overline{b_X} b_Y + \overline{a_X}\, \overline{a_Y} \end{pmatrix} \\
&= \begin{pmatrix} a_X a_Y - b_X \overline{b_Y} & a_X b_Y + b_X \overline{a_Y} \\ -\overline{(a_X b_Y + b_X \overline{a_Y})} & \overline{(a_X a_Y - b_X \overline{b_Y})} \end{pmatrix}
\end{aligned}
$$

Comparing this to the definition of $\mathbb{H}$ we see that $X \cdot Y \in \mathbb{H}$ and thus $\mathbb{H}$ is a subring of $M(2, \mathbb{C})$ and consequently a ring itself.
It remains to be shown that every non-zero element of $\mathbb{H}$ is invertible, i.e. $\mathbb{H} \subseteq GL(2, \mathbb{C})$. To show this we take an arbitrary element of $\mathbb{H}$ and consider its determinant. For $a, b \in \mathbb{C}$ we have:

$$
\det \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} = |a|^2 + |b|^2
$$

Thus we have the following equivalence for all $X \in \mathbb{H}$:

$$
\det X = 0 \;\Leftrightarrow\; X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

Since an element of $M(2, \mathbb{C})$ is invertible if and only if its determinant does not vanish, it is shown that every non-zero element of $\mathbb{H}$ is invertible. Consequently $\mathbb{H}$ is a skew-field. □

Remark 1.1.5. *(Multiplication table)*
In explicit calculations we will use that the multiplication on the quaternions is given as the bilinear extension of the multiplication rules on the distinguished basis elements of $\mathbb{H}$, which are written down in table 1.1.
Thus we get for arbitrary $x_i, y_i \in \mathbb{R}$ with $i \in \{0, 1, 2, 3\}$:

$$
\begin{aligned}
&(x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) \cdot (y_0 e_0 + y_1 e_1 + y_2 e_2 + y_3 e_3) \\
&= (x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3) e_0 + (x_1 y_0 + x_0 y_1 - x_3 y_2 + x_2 y_3) e_1 \\
&\quad + (x_2 y_0 + x_3 y_1 + x_0 y_2 - x_1 y_3) e_2 + (x_3 y_0 - x_2 y_1 + x_1 y_2 + x_0 y_3) e_3
\end{aligned}
$$

Definition 1.1.6 (Operations on quaternions)
Given an element $X \in \mathbb{H}$ we consider the unique $x_0, x_1, x_2, x_3 \in \mathbb{R}$ such that

$$
X = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3.
$$

$\cdot$	e_0	e_1	e_2	e_3
e_0	e_0	e_1	e_2	e_3
e_1	e_1	$-e_0$	e_3	$-e_2$
e_2	e_2	$-e_3$	$-e_0$	e_1
e_3	e_3	e_2	$-e_1$	$-e_0$

Table 1.1: Quaternionic multiplication

1. We define the *quaternionic conjugate* of X as
$$X^+ := x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3.$$
2. We define the *real part* of X as
$$\Re(X) := \frac{X + X^+}{2} = x_0e_0.$$
Analogously we define the *imaginary part* of X as
$$\Im(X) := \frac{X - X^+}{2} = x_1e_2 + x_2e_2 + x_3e_3.$$
3. We define for any two elements $X, Y \in \mathbb{H}$
$$\langle X, Y\rangle := \Re(XY^+) = \Re(X^+Y)$$
and furthermore for $X \in \mathbb{H}$
$$N(X) := \langle X, X\rangle = \Re(XX^+) = XX^+.$$
Thirdly we set for $X \in \mathbb{H}$:
$$|X| := \sqrt{N(X)} = \sqrt{XX^+}$$

Remarks 1.1.7. *Properties of the quaternions*

1. *We use the ring homomorphism*
$$\iota : \mathbb{R} \to \mathbb{H},\ x \mapsto x \cdot e_0$$
to embed the field of real numbers into the skew-field of quaternions. We will usually suppress ι in our notation. With this convention we can consider for any $X \in \mathbb{H}$ the real part $\Re(X)$ as a real number.
2. *We have to keep in mind that unlike in the complex case the imaginary part of a quaternion cannot be treated as a real number.*

3. *Consider the quaternions $X, Y \in \mathbb{H}$. Since $\mathbb{H}$ is a four dimensional $\mathbb{R}$-vector space, we can identify X and Y with elements of $\mathbb{R}^4$ by using the distinguished basis that was fixed in the last part of definition 1.1.2. An evaluation of $\langle X, Y\rangle$ in components shows us that the map*

$$\langle\cdot,\cdot\rangle : \mathbb{H}\times\mathbb{H}\to\mathbb{R}$$

coincides with the usual scalar product on $\mathbb{R}^4$. Thus $\langle\cdot,\cdot\rangle$ endows $\mathbb{H}$ with the structure of a real Hilbert space. Especially this implies that the map

$$|\cdot| : \mathbb{H}\to\mathbb{R}$$

is a norm on $\mathbb{H}$.

4. *To actually calculate the inverse of a quaternion $X \in \mathbb{H}\setminus\{0\}$ we can use the following formula*

$$X^{-1} = \frac{X^+}{N(X)}.$$

To see that this formula is true we just have to mind the uniqueness of the inverse and the fact that

$$\frac{X\cdot X^+}{N(X)} = \frac{N(X)}{N(X)} = 1$$

5. *We have the following relationship between norm and ring multiplication for $X, Y \in \mathbb{H}$:*

$$\begin{aligned} |XY|^2 &= N(X\cdot Y) = (XY)(XY)^+ = (XY)(Y^+X^+) \\ &= X\cdot N(Y)\cdot X^+ = N(X)\cdot N(Y) = |X|^2\,|Y|^2\,, \end{aligned}$$

where we used that quaternions commute with real numbers. Thus we have

$$|XY| = |X|\,|Y|$$

and thus $\mathbb{H}$ especially carries the structure of a Banach algebra. Furthermore we get for $X \in \mathbb{H}\setminus\{0\}$ the relation

$$\left|X^{-1}\right| = |X|^{-1}$$

Proposition 1.1.8. *(Quaternions and $SU(2)$)*
For any $X \in \mathbb{H}$ the following equality holds:

$$N(X) = \det(X).$$

Therefore we have the identity

$$SU(2) = \{X\in\mathbb{H} \mid |X| = 1\},$$

i.e. the group of quaternions with a norm of 1 is exactly the matrix group $SU(2)$.

Proof:
We choose the x_0, x_1, x_2, x_3 such that $X = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$.

$$\begin{aligned} N(X) &= (x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) \cdot (x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3) \\ &= x_0^2 + x_1^2 + x_2^2 + x_3^2. \end{aligned}$$

Here we used the multiplication rules for the basis elements e_0, e_1, e_2, e_3 as given in table 1.1. On the right side we have

$$\begin{aligned} \det(X) &= \det \begin{pmatrix} x_0 - \mathrm{i}x_3 & -\mathrm{i}x_1 - x_2 \\ -\mathrm{i}x_1 + x_2 & x_0 + \mathrm{i}x_3 \end{pmatrix} \\ &= (x_0 - \mathrm{i}x_3)(x_0 + \mathrm{i}x_3) - (-\mathrm{i}x_1 - x_2)(-\mathrm{i}x_1 + x_2) \\ &= x_0^2 + x_3^2 + x_1^2 + x_2^2 \end{aligned}$$

Thus we see that both sides are equal as proposed.
To see the second part of the proposition we first note that by definition of N and $|\cdot|$ we have for all $X \in \mathbb{H}$:

$$N(X) = 1 \Leftrightarrow |X| = 1.$$

Therefore we have by the definition of $\mathbb{H}$

$$\{X \in \mathbb{H} \mid |X| = 1\} = \left\{ X \in M(2, \mathbb{C}) \mid \det X = 1 \wedge \exists\, a, b \in \mathbb{C} : X = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \right\}$$

and this is exactly the definition of $SU(2) \subseteq M(2, \mathbb{C})$. □

After we have considered the basic properties we will now introduce some notation from topology in order to be able to determine the continuity of basic quaternionic functions.

Definition 1.1.9 (Interior, topological closure and boundary)
Let X be a topological space and $S \subseteq X$ an arbitrary subset. Then we will denote the *interior* of S by $\mathring{S}$ and the *topological closure* of S by $\overline{S}$. Furthermore we will denote the *boundary* of S by $\partial S := \overline{S} \setminus \mathring{S}$.

Definition 1.1.10 (Spheres and Balls)

1. For closed balls in $\mathbb{R}^n$ we write

   $$B_R^n(x) := \{y \in \mathbb{R}^n \mid \langle x - y, x - y \rangle \leq R^2\}$$

   with $n \in \mathbb{N}$, $R \in \mathbb{R}_{>0}$, $x \in \mathbb{R}^n$ and the usual scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Furthermore we set

   $$S_R^n(x) := \partial B_R^{n+1}(x).$$

2. For the sake of simplifying our notation we set $B^n_R := B^n_R(0)$ and $S^n_R := S^n_R(0)$.

In the following we will prove the continuity of basic quaternionic functions that have been defined in this section.

Proposition 1.1.11. *(Continuity of basic operations) The following functions are continuous:*

1.

$$\begin{aligned} \cdot^+ :\ \mathbb{H} &\longrightarrow \mathbb{H} \\ X &\longmapsto X^+ \end{aligned}$$

2.

$$\begin{aligned} N :\ \mathbb{H} &\longrightarrow \mathbb{R}_{\geq 0} \\ X &\longmapsto N(X) \end{aligned}$$

3.

$$\begin{aligned} \cdot^{-1} :\ \mathbb{H} \setminus \{0\} &\longrightarrow \mathbb{H} \setminus \{0\} \\ X &\longmapsto X^{-1} \end{aligned}$$

4.

$$\begin{aligned} \cdot :\ \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (X, Y) &\longmapsto X \cdot Y \end{aligned}$$

Proof:

1. At first we see the following identity for $X, Y \in \mathbb{H}$:

$$N(X - Y) = (X - Y)(X - Y)^+ = (X - Y)(X^+ - Y^+) = N(X^+ - Y^+).$$

With this we get for arbitrary $\varepsilon \in \mathbb{R}_{>0}$ and $X \in \mathbb{H}$:

$$\begin{aligned} \left(\cdot^+\right)^{-1} (\mathring{B}^4_\varepsilon(X)) &= \{Y^+ \in \mathbb{H} \mid N(Y - X) < \varepsilon^2\} \\ &= \{Y^+ \in \mathbb{H} \mid N(Y^+ - X^+) < \varepsilon^2\} \\ &= \mathring{B}^4_\varepsilon(X^+). \end{aligned}$$

This proves the continuity of $\cdot^+$.

2. As mentioned in the definition of N we have for any $X \in \mathbb{H}$ the identity $N(X) = \langle X, X\rangle$. And since $\mathbb{H}$ endowed with the scalar product $\langle \cdot, \cdot \rangle$ is a Hilbert space, the map

$$N: \mathbb{H} \to \mathbb{R}_{\geq 0},\, X \mapsto \langle X, X\rangle$$

is continuous.

3. For any $X \in \mathbb{H} \setminus \{0\}$ we have the identity

$$X^{-1} = \frac{X^+}{N(X)}.$$

Since $\mathbb{H}$ is a topological real vector space, the multiplication with real scalars is continuous. By this and parts 1 and 2 of this proof we see that the map $\cdot^+$ is a composition of continuous functions and thus continuous itself.

4. As mentioned before, $\mathbb{H}$ carries the structure of a Banach algebra and thus its algebra multiplication is continuous.

□

Now we will introduce some analytic notions we will need to develop concepts of quaternionic analysis. Here we mainly follow the notation of [1].

Definition 1.1.12 (Standard analytic notions)

1. Let S and S' be two arbitrary sets. Then we denote by $\mathrm{Map}(S, S')$ the set of all functions from S to S'.

2. Let U be an open subset of a Banach space and V a Banach space, then we denote by $\mathcal{C}^n(U, V)$ with $n \in \mathbb{N} \cup \{\infty\}$ the *space of n-times continuously differentiable functions* from U to V.

3. Analogously for two manifolds M, N of class $\mathcal{C}^n$ for $n \in \mathbb{N} \cup \{\infty\}$ we denote with $\mathcal{C}^n(M, N)$ the *space of n-times continuously differentiable functions* from M to N.
In the case that M and N are diffeomorphic manifolds of class $\mathcal{C}^\infty$ we denote with $\mathrm{Diff}(M, N)$ the group of diffeomorphisms from M onto N of class $\mathcal{C}^\infty$.

4. Let $m \in \mathbb{N}, n \in \mathbb{N} \cup \{\infty\}$, $U \subseteq \mathbb{R}^m$ an open subset and V a finite-dimensional Banach space. For $f \in \mathcal{C}^n(U, V)$ we denote *the ℓ-th partial derivative of f* for $\ell \in \{1, \ldots, m\}$ by $\partial_\ell f$.

5. Let V be a vector space. Then we denote by $\Lambda^\bullet(V) := \bigoplus_{n\in\mathbb{N}} \Lambda^n V$ the *graded exterior algebra of V*.

6. For M a manifold of class $\mathcal{C}^n$ with $n \in \mathbb{N}^* \cup \{\infty\}$ we have the bundles

$$TM := \bigcup_{x \in M} T_x M$$
$$T^*M := \bigcup_{x \in M} T_x^* M$$
$$\Lambda^p(T^*M) := \bigcup_{x \in M} \Lambda^p(T_x^* M)$$

of class $(n-1)$ for $p \in \mathbb{N}$.

7. Let M be a manifold of class $\mathcal{C}^n$ with $n \in \mathbb{N} \cup \{\infty\}$ and (E, π) a bundle of class $\mathcal{C}^n$ over M. Then we denote by $\Gamma(E)$ the *set of sections of class* $\mathcal{C}^n$ *of* (E, π).

8. Let M be a manifold of class $\mathcal{C}^1$ then we denote by $\mathcal{L}^n(M, \mathbb{C})$ the space of functions $f : M \to \mathbb{C}$ such that the surface integral

$$\int_M |f|^n \, \mathrm{d}S$$

is finite.

In the next lemma we will state the smoothness of basic quaternionic functions.

Lemma 1.1.13. *(Smoothness of basic operations) The following functions are of class* $\mathcal{C}^\infty$*:*

1.

$$\begin{aligned} \cdot^+ : \ \mathbb{H} &\longrightarrow \mathbb{H} \\ X &\longmapsto X^+ \end{aligned}$$

2.

$$\begin{aligned} N : \ \mathbb{H} &\longrightarrow \mathbb{R}_{\geq 0} \\ X &\longmapsto N(X) \end{aligned}$$

3.

$$\begin{aligned} \cdot^{-1} : \ \mathbb{H} \setminus \{0\} &\longrightarrow \mathbb{H} \setminus \{0\} \\ X &\longmapsto X^{-1} \end{aligned}$$

4.

$$\begin{aligned} \cdot : \ \mathbb{H} \times \mathbb{H} &\longrightarrow \mathbb{H} \\ (X, Y) &\longmapsto X \cdot Y \end{aligned}$$

Proof:

1. Considering this map in components we get:

$$\begin{aligned} \cdot^+ :\ & \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \\ & \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_0 \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \end{aligned}$$

   This map is in every component just a monomial and therefore of class $\mathcal{C}^\infty$.

2. The map

$$\begin{aligned} N :\ & \mathbb{R}^4 \longrightarrow \mathbb{R}_{\geq 0} \\ & \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto x_0^2 + x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

   is a polynomial map and therefore of class $\mathcal{C}^\infty$.

3. The map $\cdot^{-1}$ is a composition of functions of class $\mathcal{C}^\infty$ with the same argument as in proposition 1.1.11 and thus itself of class $\mathcal{C}^\infty$.

4. The function

$$\begin{aligned} \cdot :\ & \mathbb{R}^8 \longrightarrow \mathbb{R}^4 \\ (x_0, x_1, x_2, x_3, y_1, y_2, y_3)^T & \longmapsto \begin{pmatrix} x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \\ x_1y_0 + x_0y_1 - x_3y_2 + x_2y_3 \\ x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3 \\ x_2y_0 - x_2y_1 + x_1y_2 + x_0y_3 \end{pmatrix} \end{aligned}$$

   is in every component a polynomial function and thus of class $\mathcal{C}^\infty$.

$\square$

1.2 Basic Quaternionic Analysis

In this section we will work out the basic principles and theorems of quaternionic analysis in comparison to the well-known complex analysis. To be able to even talk about analysis we first have to define a suitable concept of differentiability for a distinguished class of quaternionic-valued functions on open subsets of the quaternions. Since the quaternions are inherently non-commutative, it is natural to consider two concepts of differentiability, one from the right and one from the left.

Furthermore it is true that the naive approach to differentiability via differential quotients as for real- and complex-valued functions does not lead to a satisfying class of differentiable functions, since it only leads us to affine $\mathbb{H}$-linear functions, as is shown for example in [4, Satz 5.8]. Instead we choose to define differentiability in analogy to complex holomorphy. For that we will use two four-dimensional analogs of the Cauchy-Riemann differential equations to distinguish the two suitable classes of quaternionic-valued functions that should be considered by quaternionic analysis. This leads to the following definition, which is taken directly from the paper [3]:

Definition 1.2.1 (Left-regularity and right-regularity)
Let $U \subset \mathbb{H}$ be open in $\mathbb{H}$.

1. We define
$$\nabla^+ . : \mathcal{C}^1(U, \mathbb{H}) \to \mathcal{C}^0(U, \mathbb{H}),\ f \mapsto \nabla^+ f := \sum_{\ell=0}^{3} e_\ell \cdot \partial_\ell f$$
and
$$.\nabla^+ : \mathcal{C}^1(U, \mathbb{H}) \to \mathcal{C}^0(U, \mathbb{H}),\ f \mapsto f\, \nabla^+ := \sum_{\ell=0}^{3} \partial_\ell f \cdot e_\ell$$
2. Let $f \in \mathcal{C}^1(U, \mathbb{H})$ then we call f *left-regular*, if and only if $\nabla^+ f = 0$. We call f *right-regular*, if and only if $f\, \nabla^+ = 0$.

Now we will introduce an important regular function, we will need later on. To make the proof of its regularity a bit easier, we introduce two new notions first:

Definition 1.2.2

1. We set
$$\nabla . : \mathcal{C}^1(U, \mathbb{H}) \to \mathcal{C}^0(U, \mathbb{H}),\ f \mapsto \nabla f := (\nabla^+ f)^+$$
and
$$.\nabla : \mathcal{C}^1(U, \mathbb{H}) \to \mathcal{C}^0(U, \mathbb{H}),\ f \mapsto f\, \nabla := (f\, \nabla^+)^+$$
2. We introduce the *four-dimensional Laplacian* as
$$\Box . : \mathcal{C}^2(U, \mathbb{H}) \to \mathcal{C}^0(U, \mathbb{H}),\ f \mapsto \Box f := \nabla \nabla^+ f = \nabla^+ \nabla f.$$
Explicitly we have for $f \in \mathcal{C}^2(U, \mathbb{H})$:
$$\Box f = \sum_{\ell=0}^{3} \partial_\ell \partial_\ell f$$
3. We call a function $f \in \mathcal{C}^2(U, \mathbb{H})$ *harmonic* if and only if $\Box f = 0$.

Remark 1.2.3. *1. If we consider the operator*

$$.\Box = \mathcal{C}^2(U,\mathbb{H}) \to \mathcal{C}^0(U,\mathbb{H}),\ f \mapsto \Box f := f.\nabla\nabla^+ = f.\nabla^+\nabla,$$

then we see that again we explicitly get for $f \in \mathcal{C}^2(U,\mathbb{H})$:

$$f\Box = \sum_{\ell=0}^{3} \partial_\ell\partial_\ell f.$$

Thus we see that the Laplacian formally commutes with twice differentiable functions on the quaternions, i.e. right-Laplacian and left-Laplacian are identic.

2. From the explicit formula for the Laplacian we see that it coincides with the standard Laplacian for subsets of $\mathbb{R}^4$, if we identify $\mathbb{R}^4$ with $\mathbb{H}$ using the distinguished basis of $\mathbb{H}$.

An important insight in complex analysis with many applications is that the function $k : \mathbb{C}\setminus\{0\} \to \mathbb{C}, z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C}\setminus\{0\}$. For the theory of quaternionic analysis we need an analog of this function, but we have to take into account that the quaternions are four-dimensional over $\mathbb{R}$. This leads to the following definition and proposition:

Proposition 1.2.4.
The function

$$k:\ \mathbb{H}\setminus\{0\} \to \mathbb{H},\ X \mapsto \frac{X^{-1}}{N(X)}$$

is both left- and right-regular on $\mathbb{H}\setminus\{0\}$.

Proof:
First we define the auxiliary function

$$k_0:\ \mathbb{H}\setminus\{0\} \to \mathbb{H},\ X \mapsto \frac{1}{N(X)}.$$

With elementary calculus one can check that this function is harmonic on $\mathbb{H}\setminus\{0\}$, i.e.

$$\Box k_0 = 0.$$

Furthermore we have by the well-known rules for partial differentials:

$$
\begin{aligned}
\nabla\, k_0(X) &= \nabla\, \frac{1}{N(X)} \\
&= \nabla\, (-N(X)) \cdot \frac{1}{N(X)^2} \\
&= -\frac{2X^+}{N(X)^2} \\
&= -2\, \frac{X^{-1}}{N(X)} \\
&= -2\, k(X)
\end{aligned}
$$

The same calculation works for an application of ∇ from the right, too. Hence we get the identity

$$k = -\frac{1}{2}(\nabla\, k_0) = -\frac{1}{2}(k_0\, \nabla).$$

In summary we get:

$$\nabla^+ k = -\frac{1}{2}(\nabla^+ \nabla\, k_0) = -\frac{1}{2}(\Box\, k_0) = 0$$

and

$$k\nabla^+ = -\frac{1}{2}(k_0\, \nabla\nabla^+) = -\frac{1}{2}(\Box\, k_0) = 0.$$

For the second equality we invoked the remark after the definition of $\Box$. This shows the claim. $\Box$

Although at the first glance the two notions of regularity look very similar to the notion of holomorphy, there is a crucial difference from an algebraic point of view. Sets of left- or right-regular functions in general can't be endowed with the algebraic structure one would expect from comparison with the ring of holomorphic functions in the complex case.

Proposition 1.2.5.

1. *The identity function* $\mathrm{id}_{\mathbb{H}} : \mathbb{H} \to \mathbb{H}$, $X \mapsto X$ *is neither right- nor left-regular.*

2. *Let* $U \subset \mathbb{H}$ *be open and non-empty, then the set of left-regular functions from* U *to* $\mathbb{H}$ *together with point-wise addition and multiplication does not form a ring. The same statement is true for right-regular functions.*

Proof:

1. Let $X \in \mathbb{H}$. Then we have:

$$\nabla^+ \operatorname{id}_{\mathbb{H}}(X) = \sum_{\ell=0}^{3} e_\ell \cdot \partial_\ell \operatorname{id}_{\mathbb{H}}(X) = \sum_{\ell=0}^{3} (e_\ell)^2 = 1 - 3 = -2 \neq 0$$

   Hence $\operatorname{id}_{\mathbb{H}}$ is not left-regular. A similar calculation shows that $\operatorname{id}_{\mathbb{H}}$ is also not right-regular.

2. Consider the maps

$$f : \mathbb{H} \to \mathbb{H}, \sum_{\ell=0}^{3} x_\ell e_\ell \mapsto x_0\, e_0 + x_1\, e_1$$

   and

$$g : \mathbb{H} \to \mathbb{H}, \sum_{\ell=0}^{3} x_\ell e_\ell \mapsto x_1\, e_1 - x_2\, e_2.$$

   One can check that then f and g are both left- and right-regular on $\mathbb{H}$, but for the product

$$f \cdot g : \mathbb{H} \to \mathbb{H}, \sum_{\ell=0}^{3} x_\ell e_\ell \mapsto -(x_1)^2\, e_0 + x_0 x_1\, e_1 + x_0 x_2\, e_2 + x_1 x_2\, e_1 e_2$$

   we find

$$(f \cdot g)\, \nabla^+ \left(\sum_{\ell=0}^{3} x_\ell e_\ell \right) = -2x_0 e_0 - 2x_1 e_1 + 2x_2 e_2.$$

   This is not identically zero and hence $f \cdot g$ is not right-regular. A similar calculation shows that $g \cdot f$ is not left-regular.

□

Although there seems to be a huge difference between complex holomorphy and the quaternionic regularity concepts, it is possible to transfer some very basic results of complex analysis to the quaternionic case. In the following we will show that two analogs of the Cauchy integral formula are true for the quaternionic setting. As we know from complex analysis, this leads to a rich amount of corollaries. We will start with some definitions we will need in the following:

Definition 1.2.6 (Quaternionic differential forms)

1. We define for $p \in \mathbb{N}$ now quaternionic-valued differential forms on $\mathbb{R}^4$ in the usual way as
$$\Omega^p(\mathbb{R}^4;\mathbb{H}) := \Gamma(\mathbb{H} \otimes \Lambda^p\, T^*\mathbb{R}^4)$$
Thereby we denote
$$X\omega := X \otimes \omega$$
for $X \in \mathbb{H}$, $\omega \in \Lambda^p T^*\mathbb{R}^4$.

2. The maps
$$\mathrm{d}_n : \Omega^n(\mathbb{R}^4;\mathbb{H}) \longrightarrow \Omega^{n+1}(\mathbb{R}^4;\mathbb{H})$$
are the standard de Rham differentials. In general we will omit the index.

3. $\Omega^1(\mathbb{R}^4,\mathbb{H}) \ni \mathrm{d}X := e_0\mathrm{d}x_0 + e_1\mathrm{d}x_1 + e_2\mathrm{d}x_2 + e_3\mathrm{d}x_3$

4. Since $\mathbb{R}^4$ with the standard scalar product is a finite-dimensional oriented euclidean vector space, we have for every $\omega \in \Omega^p(\mathbb{R}^4;\mathbb{H})$ with $p \in \{0,1,2,3,4\}$ a Hodge dual $*(\omega) \in \Omega^{4-p}(\mathbb{R}^4;\mathbb{H})$. In particular we set
$$\Omega^3(\mathbb{R}^4,\mathbb{H}) \ni \mathrm{D}x := *(\mathrm{d}X) =$$
$$e_0\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - e_1\mathrm{d}x_0 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + e_2\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 - e_3\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$$

5. $\Omega^4(\mathbb{R}^4,\mathbb{H}) \ni \mathrm{d}V := \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$

6. With $\mathrm{d}S$ we will denote the standard euclidean volume form on the three-dimensional Sphere $S^3_1 \subset \mathbb{H}$ with orientation as a boundary of the open ball B^4_1.

Now we have to collect some lemmas about these differential forms:

Lemma 1.2.7.
Let $f \in \mathcal{C}^1(U,\mathbb{H})$. Then $\mathrm{d}(\mathrm{D}x \cdot f) = (\nabla^+ f)\mathrm{d}V$ and $\mathrm{d}(f \cdot \mathrm{D}x) = (f\nabla^+)\mathrm{d}V$.

Proof:
We will prove the proposition by straightforward calculation. We see that for pairwise different $\ell, m, n, o \in \mathbb{N}$ the following holds:

$$\begin{aligned}
\mathrm{d}((e_\ell \mathrm{d}x_m \wedge \mathrm{d}x_n \wedge \mathrm{d}x_o) \cdot f) &= \mathrm{d}((e_\ell \cdot f)\mathrm{d}x_m \wedge \mathrm{d}x_n \wedge \mathrm{d}x_o) \\
&= \partial_\ell (e_\ell \cdot f)\mathrm{d}x_\ell \wedge \mathrm{d}x_m \wedge \mathrm{d}x_n \wedge \mathrm{d}x_o \\
&= e_\ell \cdot \partial_\ell f \mathrm{d}x_\ell \wedge \mathrm{d}x_m \wedge \mathrm{d}x_n \wedge \mathrm{d}x_o
\end{aligned}$$

Combining this with the definition of $\mathrm{D}x$ we get:

$$\begin{aligned}\mathrm{d}(\mathrm{D}x \cdot f) &= \left(\sum_{j=0}^{3} e_j \cdot \partial_j f\right) \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= (\nabla^+ f)\mathrm{d}V\end{aligned}$$

This shows the first part of the lemma. The second part follows analogously. □

Corollary 1.2.8.
For $f, g \in \mathcal{C}(U, \mathbb{H})$ we have:

$$\mathrm{d}(g \cdot \mathrm{D}x \cdot f) = ((g\nabla^+)f + g(\nabla^+ f))\mathrm{d}V$$

Proof:
Using the Leibniz rule for the de Rham differential and Lemma 1.2.7, we get

$$\begin{aligned}\mathrm{d}(g \cdot \mathrm{D}x \cdot f) &= \mathrm{d}(g \cdot \mathrm{D}x) \cdot f + g \cdot \mathrm{d}(\mathrm{D}x \cdot f) \\ &= ((g\nabla^+)f + g(\nabla^+ f))\mathrm{d}V.\end{aligned}$$

□

Proposition 1.2.9.
Let $U \subset \mathbb{H}$ be an open, bounded subset with piecewise $\mathcal{C}^1$-boundary ∂U and $f, g \in \mathcal{C}^1(O, \mathbb{H})$ with $O \supset \bar{U}$ open. Then the following equality holds:

$$\int_{\partial U} g \cdot \mathrm{D}x \cdot f = \int_U ((g\nabla^+)f + g(\nabla^+ f))\mathrm{d}V$$

Proof:
We just use a version of Stokes' Theorem together with corollary 1.2.8:

$$\int_{\partial U} g \cdot \mathrm{D}x \cdot f = \int_U \mathrm{d}(g \cdot \mathrm{D}x \cdot f) = \int_U ((g\nabla^+)f + g(\nabla^+ f))\mathrm{d}V$$

□

At this point we are ready to formulate and prove a quaternionic analog of the Cauchy integral theorem:

Corollary 1.2.10. *(Cauchy integral theorem)*
Let $U \subset \mathbb{H}$ be an open bounded subset with piecewise $\mathcal{C}^1$-boundary ∂U. Let $f \in \mathcal{C}^1(O, \mathbb{H})$ be left-regular and $g \in \mathcal{C}^1(O, \mathbb{H})$ right-regular with $O \supset \bar{U}$ open. Then we have:

$$\int_{\partial U} g \cdot \mathrm{D}x \cdot f = 0.$$

Proof:
This follows directly from the definition of left- and right-regularity and proposition 1.2.9. □

Now we are ready to prove the quaternionic analogs for the Cauchy integral formula in complex analysis. The following proof is inspired by the proof in [4, Satz 7.12].

Theorem 1.2.11. *(Cauchy-Fueter formulas)*
Let $U \subset \mathbb{H}$ be open and bounded with piecewise $\mathcal{C}^1$-boundary ∂U. Then we have for an open subset O with $\bar{U} \subset O$, $f \in \mathcal{C}(O, \mathbb{H})$ and $X_0 \in \mathbb{H} \setminus \partial U$:

1. *If f is right-regular on O, then*

$$\frac{1}{2\pi^2} \int_{\partial U} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) = \begin{cases} f(X_0), & X_0 \in U \\ 0, & X_0 \notin U. \end{cases}$$

2. *If f is left-regular on O, then*

$$\frac{1}{2\pi^2} \int_{\partial U} f(X) \cdot \mathrm{D}x \cdot k(X - X_0) = \begin{cases} f(X_0), & X_0 \in U \\ 0, & X_0 \notin U. \end{cases}$$

Proof:
Again we will only prove the first part of the proposition, since the second part follows in the same way.
First let $X_0 \notin U$. Then there is an open neighborhood $\tilde{O}$ of $\bar{U}$ such that the function $k' : \tilde{O} \to \mathbb{H}$, $X \mapsto k(X - X_0)$ is left-regular. Then by corollary 1.2.10 this part of the proposition is shown by restricting f and k' to $O \cap \tilde{O}$.
Let now $X_0 \in U$. Then we choose an $\epsilon \in \mathbb{R}_{>0}$ and $U_\epsilon = U \setminus \overline{B_\epsilon(X_0)} \neq \emptyset$ with boundary $\partial U_\epsilon \sqcup (-S_\epsilon(X_0))$, where we denote by S_ϵ the sphere with center X_0 and radius ϵ with outwards orientation and with $-S_\epsilon$ the same with inward orientation.
Then we have by proposition 1.2.9 and the regularity of $k(X - X_0)$ on U_ϵ:

$$\begin{aligned}
&\int_{\partial U} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) - \int_{S_\epsilon} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) \\
&= \int_{\partial U_\epsilon} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) \\
&= \int_{U_\epsilon} k(X - X_0) \cdot (\nabla^+ f)(X) \mathrm{d}V \\
&= 0
\end{aligned}$$

This shows that

$$\int_{\partial U} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) = \int_{S_\epsilon} k(X - X_0) \cdot \mathrm{D}x \cdot f(X).$$

Furthermore we have that $\mathrm{D}x\mid_{S_\epsilon(X_0)}= \frac{X-X_0}{\epsilon}\,\mathrm{d}S$ and thus

$$\begin{aligned}\int_{S_\epsilon(X_0)} k(X - X_0) \cdot \mathrm{D}x\mid_{S_\epsilon(X_0)} f(X) &= \int_{S_\epsilon(X_0)} \frac{(X-X_0)^+}{\epsilon^4} \cdot \frac{X-X_0}{\epsilon} f(X)\,\mathrm{d}S \\ &= \int_{S_\epsilon(X_0)} \epsilon^{-3} f(X)\,\mathrm{d}S.\end{aligned}$$

To solve this integral we consider the following transformation

$$\begin{aligned} T:\ S_\epsilon(X_0) &\longrightarrow S_1(0) \\ P &\longmapsto \frac{1}{\epsilon} \cdot (P - X_0)\end{aligned}$$

with inverse

$$\begin{aligned} T^{-1}:\ S_1(0) &\longrightarrow S_\epsilon(X_0) \\ P &\longmapsto \epsilon \cdot P + X_0\end{aligned}$$

This is a diffeomorphism and so we can apply the transformation formula to get:

$$\int_{S_\epsilon(X_0)} k(X - X_0) \cdot \mathrm{D}x\mid_{S_\epsilon} f(X) = \int_{S_1(0)} f(X_0 + \epsilon \cdot X)\,\mathrm{d}S$$

We know that $S_1(X_0)$ is compact and f continuous. So as long as ϵ is sufficiently small that for a given $X \in S_1(0)$ we have $X_0+\epsilon\cdot X \in S_1(X_0)$, there is a $c \in S_1(X_0)$ such that $|f(X_0 - \epsilon \cdot X)| \leq |f(c)|$. Then by Lebesgue's Theorem of majorized convergence we can calculate:

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{S_1(0)} f(X_0 + \epsilon \cdot X)\,\mathrm{d}S &= \int_{S_1(0)} \lim_{\epsilon\to 0} f(X_0 + \epsilon \cdot X)\,\mathrm{d}S \\ &= \int_{S_1(0)} f(X_0)\,\mathrm{d}S \\ &= 2\pi^2 \cdot f(X_0)\end{aligned}$$

So we finally get:

$$\frac{1}{2\pi^2} \int_{\partial U} k(X - X_0) \cdot \mathrm{D}x \cdot f(X) = f(X_0).$$

This was the claim. □

1.3 Quaternionic projective spaces and conformal transformations

In this section we want to introduce conformal transformations as the quaternionic analogue of the Möbius transformations known from complex analysis. Especially we want to show that conformal transformations similar to Möbius transformations preserve certain geometric structures in the one-point compactification of the quaternions. In order to motivate the definition of conformal transformations we will firstly consider quaternionic projective spaces. Because of the lack of multiplicative commutativity of the quaternions there are two projective spaces of the lowest dimension that will be introduced together with the one-point compactification of the quaternions in the next definitions. We should note here that the quaternions itself form a normed space with the norm introduced in section 1.

Definition 1.3.1 (Projective Spaces)
We define binary relations $\sim_\ell$ and $\sim_r$ on $\mathbb{H}^2$ in the following way for arbitrary $v, w \in \mathbb{H}^2$

$$v \sim_\ell w \;\Leftrightarrow\; \exists\, a \in \mathbb{H} \setminus \{0\} : \; w = v \cdot a,$$
$$v \sim_\ell w \;\Leftrightarrow\; \exists\, a \in \mathbb{H} \setminus \{0\} : \; w = a \cdot v,$$

where $\cdot$ denotes component-wise multiplication from the left, respectively from the right. Now we define the *quaternionic right-projective space* as the quotient space

$$P^1\mathbb{H}_\ell := (\mathbb{H}^2 \setminus \{0\}) / \sim_\ell$$

and the *quaternionic left-projective space* as the quotient space

$$P^1\mathbb{H}_r := (\mathbb{H}^2 \setminus \{0\}) / \sim_r \;.$$

Remark 1.3.2. *The relations $\sim_\ell$ and $\sim_r$ are indeed equivalence relations. Reflexivity and transitivity are directly visible and symmetry follows from the fact that $\mathbb{H}$ contains inverses.*
In the following we will denote for any $v \in \mathbb{H}^2 \setminus \{0\}$ its equivalence class in $P^1\mathbb{H}_\ell$ by $[v]_\ell$ and its equivalence class in $P^1\mathbb{H}_r$ by $[v]_r$.

Definition 1.3.3 (One-Point compactification of $\mathbb{H}$)
Consider the sets $\hat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$ and

$$\mathcal{O}_{\hat{\mathbb{H}}} := \{O \in \mathcal{P}(\hat{\mathbb{H}}) \;\mid\; O \text{ open in } \mathbb{H} \;\vee\; \hat{\mathbb{H}} \setminus O \text{ is compact in } \mathbb{H}\}.$$

We will prove in the next proposition that $(\mathbb{H}, \mathcal{O}_{\hat{\mathbb{H}}})$ is a topological space and call it the one-point compactification of $\mathbb{H}$. If we do not state it differently, $\hat{\mathbb{H}}$ will always carry the topology $\mathcal{O}_{\hat{\mathbb{H}}}$.

We go on with showing that the one-point compactification as just defined is indeed a compact topological space.

Proposition 1.3.4. *(Topological Properties of the One-Point compactification of $\mathbb{H}$)*

1. *The set $\mathcal{O}_{\hat{\mathbb{H}}}$ is a topology on $\hat{\mathbb{H}}$.*
2. *The topological space $\mathcal{O}_{\hat{\mathbb{H}}}$ is compact.*

Proof:

1. We know that $\emptyset$ is open in $\mathbb{H}$ and therefore $\emptyset \in \mathcal{O}_{\hat{\mathbb{H}}}$.
   Furthermore $\emptyset$ is compact in $\mathbb{H}$ and therefore $\hat{\mathbb{H}} \setminus \emptyset = \hat{\mathbb{H}} \in \mathcal{O}_{\hat{\mathbb{H}}}$.
   We now consider binary intersections of elements of $\mathcal{O}_{\hat{\mathbb{H}}}$ and conduct a case-by-case analysis. At first we see that the intersection of two open subsets of $\mathbb{H}$ is again an open subset of $\mathbb{H}$. Let now $O, U \in \mathcal{O}_{\hat{\mathbb{H}}}$ be such that O is an open subset of $\mathbb{H}$ and $\hat{\mathbb{H}} \setminus U$ is compact in $\mathbb{H}$. Since a compact subset of $\mathbb{H}$ is also closed, we see that $\mathbb{H} \setminus (\hat{\mathbb{H}} \setminus U) = U \setminus \{\infty\}$ is open in $\mathbb{H}$. We also know that $\infty \notin O$ and therefore
   $$O \cap U = O \cap (U \setminus \{\infty\})$$
   is open in $\mathbb{H}$ and consequently $O \cap U \in \mathcal{O}_{\hat{\mathbb{H}}}$. The third possible case is that $U_1, U_2 \in \mathcal{O}_{\hat{\mathbb{H}}}$ are both complements of compact sets. Then we have:
   $$\hat{\mathbb{H}} \setminus (U_1 \cap U_2) = (\hat{\mathbb{H}} \setminus U_1) \cup (\hat{\mathbb{H}} \setminus U_2).$$
   The union of two compact sets is again compact and therefore we have that $U_1 \cap U_2 \in \mathcal{O}_{\hat{\mathbb{H}}}$. By complete induction we can now deduce that $\mathcal{O}_{\hat{\mathbb{H}}}$ is closed under finite intersections.
   We now consider arbitrary unions of elements of $\mathcal{O}_{\hat{\mathbb{H}}}$. Set union is associative so for every such union V we can find two index sets I, J and families of sets $(O_i)_{i \in I}$, $(U_j)_{j \in J}$, where every O_i is open in $\mathbb{H}$ and every U_j is a complement of a compact set, such that
   $$V = \bigcup_{i \in I} O_i \cup \bigcup_{j \in J} U_j.$$
   We see that $\bigcup\limits_{i \in I} O_i$ is as a union of open subsets of $\mathbb{H}$ itself an open subset of $\mathbb{H}$. For the second union we see:
   $$\hat{\mathbb{H}} \setminus \left(\bigcup_{j \in J} U_j \right) = \bigcap_{j \in J} (\hat{\mathbb{H}} \setminus U_j).$$

We know that an arbitrary intersection of compact subsets of $\mathbb{H}$ is again compact.
Therefore it is only left to show that the binary union of any two $O, U \in \mathcal{O}_{\hat{\mathbb{H}}}$ with O open in $\mathbb{H}$ and $\hat{\mathbb{H}} \setminus U$ compact in $\mathbb{H}$ is again an element of $\mathcal{O}_{\hat{\mathbb{H}}}$. We have that

$$\hat{\mathbb{H}} \setminus (O \cup U) = (\hat{\mathbb{H}} \setminus O) \cap (\hat{\mathbb{H}} \setminus U) = (\mathbb{H} \setminus O) \cap (\hat{\mathbb{H}} \setminus U).$$

The set $\mathbb{H} \setminus O$ is a closed subset of $\mathbb{H}$ and the intersection of a closed subset with a compact subset of $\mathbb{H}$ is again compact. Therefore we get that

$$\hat{\mathbb{H}} \setminus (O \cup U) \in \mathcal{O}_{\hat{\mathbb{H}}}.$$

By that we have proved that $\mathcal{O}_{\hat{\mathbb{H}}}$ is a topology on $\hat{\mathbb{H}}$.

2. Let $\mathcal{U}$ be an arbitrary open covering of $\hat{\mathbb{H}}$. There is a set $U \in \mathcal{U}$ such that $\infty \in U$. By the definition of $\mathcal{O}_{\hat{\mathbb{H}}}$ we have that $\hat{\mathbb{H}} \setminus U$ is compact in $\mathbb{H}$. Furthermore $\mathcal{U} \setminus \{U\}$ is an open covering of $\hat{\mathbb{H}} \setminus U$. By compactness we can find a finite open covering $\{O_1, \dots, O_n\} \subseteq \mathcal{U} \setminus \{U\}$ of $\hat{\mathbb{H}} \setminus U$ and therefore the set $\{U, O_1, \dots, O_n\} \subseteq \mathcal{U}$ is a finite open sub-covering of $\hat{\mathbb{H}}$. Hence $\hat{\mathbb{H}}$ is compact.

□

Since this thesis is about quaternionic analysis, we want to consider differential structures on the spaces just defined. For this reason we will go on with showing that both the one-point compactification and the projective spaces are manifolds of class $\mathcal{C}^\infty$.
In the following two propositions we will silently identify $\mathbb{H}$ with $\mathbb{R}^4$ as a vector space using the distinguished basis of $\mathbb{H}$.

Proposition 1.3.5. *(One-Point compactification as smooth manifold)*
Consider the maps

$$\begin{aligned} \varphi_1 :\ & \mathbb{H} \longrightarrow \mathbb{H}, \\ & X \longmapsto X \end{aligned}$$

and

$$\begin{aligned} \varphi_2 :\ & \hat{\mathbb{H}} \setminus \{0\} \longrightarrow \mathbb{H}, \\ & X \longmapsto \begin{cases} X^{-1} & for\, X \neq \infty, \\ 0 & else. \end{cases} \end{aligned}$$

Then the set $\{\varphi_1, \varphi_2\}$ is an atlas for $\hat{\mathbb{H}}$ that endows $\hat{\mathbb{H}}$ with the structure of a 4-manifold of class $\mathcal{C}^\infty$.

Proof:
Firstly we state that $\hat{\mathbb{H}}$ is second-countable and Hausdorff. This follows directly from the definition of its topology.
Now we have to check that φ_1 and φ_2 are charts. We see that $\mathbb{H}$ is open in $\mathbb{H}$ and therefore also in $\hat{\mathbb{H}}$. Furthermore φ_1 is just the identity on $\mathbb{H}$ and therefore a homeomorphism onto $\mathbb{R}^4$. Therefore φ_1 is a chart. For φ_2 we see that the set $\{0\}$ is compact in $\mathbb{H}$ and therefore the set $\hat{\mathbb{H}} \setminus \{0\}$ is open in $\hat{\mathbb{H}}$ as its complement. Furthermore φ_2 is bijective. On the domain $\mathbb{H} \setminus \{0\}$ it is inverse to itself and continuous as was shown in proposition 1.1.11. Therefore the restriction of φ_2 on $\mathbb{H} \setminus \{0\}$ is a homeomorphism onto $\mathbb{H} \setminus \{0\}$. Next we consider for an arbitrary $\varepsilon \in \mathbb{R}_{>0}$ the open ball $\mathring{B}^4_\varepsilon$ and get

$$\begin{aligned}\varphi_2^{-1}\left(\mathring{B}^4_\varepsilon\right) &= \{\infty\} \cup \left\{Y \in \mathbb{H} \setminus \{0\} \;\middle|\; \left|Y^{-1}\right| < \varepsilon\right\} \\ &= \{\infty\} \cup \left\{Y \in \mathbb{H} \;\middle|\; |Y| > \frac{1}{\varepsilon}\right\},\end{aligned}$$

where we used that $|Y| = \left|Y^{-1}\right|^{-1}$ for all $Y \in \mathbb{H} \setminus \{0\}$ as we noted in remark 1.1.7. We see now that

$$\hat{\mathbb{H}} \setminus \varphi_2^{-1}\left(\mathring{B}^4_\varepsilon\right) = \left\{X \in \mathbb{H} \;\middle|\; |X| \leq \frac{1}{\varepsilon}\right\} = B^4_{1/\varepsilon}$$

is a compact subset of $\mathbb{H}$ and therefore $\varphi_2^{-1}\left(\mathring{B}^4_\varepsilon\right)$ is open in $\hat{\mathbb{H}}$. This shows the continuity of φ_2. For the openness we consider an arbitrary compact subset $C \subset \mathbb{H}$ such that $0 \in C$ and get by using the bijectivity of φ_2

$$\begin{aligned}\varphi_2\left(\hat{\mathbb{H}} \setminus C\right) &= \varphi_2\left((\hat{\mathbb{H}} \setminus \{0\}) \setminus C\right) \\ &= \varphi_2\left(\hat{\mathbb{H}} \setminus \{0\}\right) \setminus \varphi_2(C) \\ &= \mathbb{H} \setminus \varphi_2(C).\end{aligned}$$

Since p_2 is continuous we see that $p_2(C)$ is compact and therefore also closed in $\mathbb{H}$. This implies that $\varphi_2\left(\hat{\mathbb{H}} \setminus C\right)$ is open in $\mathbb{H}$ as the complement of a closed set. This shows the openness of φ_2. Therefore both maps are charts and the fact that $\mathbb{H} \cup (\hat{\mathbb{H}} \setminus \{0\}) = \hat{\mathbb{H}}$ shows that the set $\{\varphi_1, \varphi_2\}$ is an atlas.
It remains to be shown that the one transition map is of class $\mathcal{C}^\infty$. For that we have to consider the map

$$\begin{aligned}\varphi_1 \circ \varphi_2^{-1} : \mathbb{H} \setminus \{0\} &\longrightarrow \mathbb{H} \setminus \{0\} \\ X &\longmapsto X^{-1}.\end{aligned}$$

We saw in lemma 1.1.13 that this map is of class $\mathcal{C}^\infty$ and thus the proposition is shown. $\square$

Proposition 1.3.6. *(Projective Spaces as smooth manifolds)*

1. *Consider the maps*

$$\begin{aligned} \psi_1 : \ P^1\mathbb{H}_\ell \setminus \left\{[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]_\ell\right\} &\longrightarrow \mathbb{H}, \\ \left[\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right)\right]_\ell &\longmapsto X_1X_2^{-1} \end{aligned}$$

   and

$$\begin{aligned} \psi_2 : \ P^1\mathbb{H}_\ell \setminus \left\{[\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)]_\ell\right\} &\longrightarrow \mathbb{H}, \\ \left[\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right)\right]_\ell &\longmapsto X_2X_1^{-1} \end{aligned}$$

   Then the set $\{\psi_1, \psi_2\}$ is an atlas for $P^1\mathbb{H}_\ell$ that endows $P^1\mathbb{H}_\ell$ with the structure of a 4-manifold of class $\mathcal{C}^\infty$.

2. *Consider the maps*

$$\begin{aligned} \chi_1 : \ P^1\mathbb{H}_r \setminus \left\{[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]_r\right\} &\longrightarrow \mathbb{H}, \\ \left[\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right)\right]_r &\longmapsto X_2^{-1}X_1 \end{aligned}$$

   and

$$\begin{aligned} \chi_2 : \ P^1\mathbb{H}_\ell \setminus \left\{[\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)]_r\right\} &\longrightarrow \mathbb{H}, \\ \left[\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right)\right]_r &\longmapsto X_1^{-1}X_2 \end{aligned}$$

   Then the set $\{\chi_1, \chi_2\}$ is an atlas for $P^1\mathbb{H}_r$ that endows $P^1\mathbb{H}_r$ with the structure of a 4-manifold of class $\mathcal{C}^\infty$.

Proof:
The proof of this proposition is very similar to the proof of proposition 1.3.5 with the addition that we have to show well-definedness of the charts. We will only show the first part of the proposition, since one can get the proof of the second part by reversion of the multiplication orders. Again we simply state, but do not show that $P^1\mathbb{H}_\ell$ and $P^1\mathbb{H}_\ell$ are second-countable Hausdorff spaces.
Firstly we will show that the maps ψ_1 and ψ_2 are well-defined. Let $\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right), \left(\begin{smallmatrix}Y_1\\Y_2\end{smallmatrix}\right) \in \mathbb{H}^2$ such that $X_2 \neq 0$, $Y_2 \neq 0$ and

$$\left(\begin{smallmatrix}X_1\\X_2\end{smallmatrix}\right) \sim_\ell \left(\begin{smallmatrix}Y_1\\Y_2\end{smallmatrix}\right).$$

Then there exists an $a \in \mathbb{H} \setminus \{0\}$ such that

$$\left(\begin{smallmatrix}Y_1\\Y_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}X_1 \cdot a\\X_2 \cdot a\end{smallmatrix}\right).$$

But then we also have the equality:

$$Y_1 \cdot Y_2^{-1} = (X_1 \cdot a) \cdot (X_2 \cdot a)^{-1} = X_1 \cdot a \cdot a^{-1} X_2^{-1} = X_1 \cdot X_2^{-1}.$$

Therefore we have shown the implication

$$\forall\, v, w \in \mathbb{H}^2 :\ [v]_\ell = [w]_\ell \ \Rightarrow \psi_1([v]_\ell) = \psi_1([w]_\ell)$$

and thus the well-definedness of ψ_1. The well-definedness of ψ_2 is shown in the same way.
We see that ψ_1 and ψ_2 are bijections onto $\mathbb{H}$ as we can see by giving its both-sided inverses as

$$\begin{aligned} \psi_1^{-1} :\ \mathbb{H} &\longrightarrow P^1\mathbb{H}_\ell \setminus \left\{ [\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)]_\ell \right\} \\ X &\longmapsto \left[\left(\begin{smallmatrix} X \\ 1 \end{smallmatrix}\right) \right]_\ell \end{aligned}$$

and

$$\begin{aligned} \psi_2^{-1} :\ \mathbb{H} &\longrightarrow P^1\mathbb{H}_\ell \setminus \left\{ [\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)]_\ell \right\} \\ X &\longmapsto \left[\left(\begin{smallmatrix} 1 \\ X \end{smallmatrix}\right) \right]_\ell . \end{aligned}$$

Furthermore let $pr_\ell : \mathbb{H}^2 \setminus \{0\} \to P^1\mathbb{H}_\ell$ be the canonical projection, then we see that $\psi_1 \circ pr_\ell$ and $\psi_2 \circ pr_\ell$ are continuous as composition of continuous maps and thus by the universal property of the quotient we get that ψ_1 and ψ_2 are continuous. Furthermore also the maps ψ_1^{-1} and ψ_2^{-1} are continuous and thus they are homeomorphisms. Now we consider the transition map between ψ_1 and ψ_2:

$$\begin{aligned} \varphi_1 \circ \varphi_2^{-1} :\ \mathbb{H} \setminus \{0\} &\longrightarrow \mathbb{H} \setminus \{0\} \\ X &\longmapsto X^{-1}. \end{aligned}$$

This is a map of class $\mathcal{C}^\infty$ as stated in lemma 1.1.13 and together with the fact that

$$P^1\mathbb{H}_\ell = P^1\mathbb{H}_\ell \setminus \left\{ [\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)]_\ell \right\} \cup P^1\mathbb{H}_\ell \setminus \left\{ [\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)]_\ell \right\}$$

we get that the set $\{\psi_1, \psi_2\}$ is indeed an atlas of $P^1\mathbb{H}_\ell$. □

Conformal transformations on $\hat{\mathbb{H}}$ will be defined as an action of $GL(2, \mathbb{H})$ on $\hat{\mathbb{H}}$. Therefore we will give a characterization of the group $GL(2, \mathbb{H})$ now. After that we will give explicit forms for inverses of elements of $GL(2, \mathbb{H})$.

Lemma 1.3.7. *(Invertible quaternionic matrices)*
Let $A, B, C, D \in \mathbb{H}$. Then the following equivalence holds:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2, \mathbb{H})$$

$$\Leftrightarrow (C = 0 \Rightarrow AD \neq 0) \wedge (C \neq 0 \Rightarrow AC^{-1}D - B \neq 0)$$

Proof:
We begin with showing the forward direction. Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(2, \mathbb{H}) \in GL(2, \mathbb{H})$. Then the set $\left\{\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right), \left(\begin{smallmatrix} C \\ D \end{smallmatrix}\right)\right\} \subset \mathbb{H}^2$ is linearly independent over $\mathbb{H}$. We distinguish two cases now:
Firstly let $C = 0$. Then we consider the subcase that $D = 0$. Then $\left(\begin{smallmatrix} C \\ D \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and thus trivially the set $\left\{\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right), \left(\begin{smallmatrix} C \\ D \end{smallmatrix}\right)\right\}$ is linearly dependent over $\mathbb{H}$, which would lead to a contradiction. Now we consider the subcase that $D \neq 0$ but $A = 0$. Then we have that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ B \end{pmatrix} = BD^{-1} \cdot \begin{pmatrix} 0 \\ D \end{pmatrix} = BD^{-1} \cdot \begin{pmatrix} C \\ D \end{pmatrix}.$$

This proves linear dependence over $\mathbb{H}$ and again leads to a contradiction.
Now we consider the case that $C \neq 0$. Furthermore we assume that $AC^{-1}D - B = 0$. Then we get the following equality:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ AC^{-1}D \end{pmatrix} = AC^{-1} \cdot \begin{pmatrix} C \\ D \end{pmatrix}.$$

This is again a contradiction. In summary the forward direction of the lemma is shown.
For the backward direction we see first that if $C = 0$ and both $A \neq 0$ and $D \neq 0$ then the two vectors $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right), \left(\begin{smallmatrix} C \\ D \end{smallmatrix}\right)$ cannot be linearly dependent. Secondly if $C \neq 0$ and $AC^{-1}D \neq B$ and the elements $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right), \left(\begin{smallmatrix} C \\ D \end{smallmatrix}\right) \in \mathbb{H}^2$ are linearly dependent over $\mathbb{H}$, then there is a $\lambda \in \mathbb{H}$ such that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \lambda \cdot \begin{pmatrix} C \\ D \end{pmatrix}.$$

From this equation we get that $\lambda = AC^{-1}$ and $\lambda \cdot D = B$ and thus $AC^{-1}D = B$. This is a contradiction and thus the backward direction of the lemma is shown. $\square$

Lemma 1.3.8. *(Inverses of quaternionic matrices)*
Let $M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2, \mathbb{H})$.
If $C = 0$ *we have*

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.$$

If $C \neq 0$ *we have with the abbreviation* $H := CAC^{-1}D - CB$

$$M^{-1} = \begin{pmatrix} C^{-1}DH^{-1}C & C^{-1} - C^{-1}DH^{-1}CAC^{-1} \\ -H^{-1}C & H^{-1}CAC^{-1} \end{pmatrix}$$

Proof:
First let $C = 0$. Then we get

$$\begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix} \cdot \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A^{-1}A & A^{-1}B - A^{-1}B \\ 0 & D^{-1}D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This shows the first part of the lemma.
Now let $C \neq 0$. We note first that lemma 1.3.7 ensures that H is well-defined. Then we can calculate:

$$\begin{aligned}
&\begin{pmatrix} C^{-1}DH^{-1}C & C^{-1}-C^{-1}DH^{-1}CAC^{-1} \\ -H^{-1}C & H^{-1}CAC^{-1} \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}DH^{-1}CA+(C^{-1}-C^{-1}DH^{-1}CAC^{-1})C & C^{-1}DH^{-1}CB+(C^{-1}-C^{-1}DH^{-1}CAC^{-1})D \\ -H^{-1}CA+H^{-1}CAC^{-1}C & -H^{-1}CB+H^{-1}CAC^{-1}D \end{pmatrix} \\
&= \begin{pmatrix} C^{-1}DH^{-1}CA+1-C^{-1}DH^{-1}CA & C^{-1}D(H^{-1}CB+1-H^{-1}CAC^{-1}D) \\ -H^{-1}CA+H^{-1}CA & -H^{-1}(CB-CAC^{-1}D) \end{pmatrix} \\
&= \begin{pmatrix} 1 & C^{-1}D(1+H^{-1}(CB-CAC^{-1}D)) \\ 0 & H^{-1}H \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

This finishes the proof of the proposition. □

Now we are nearly ready to define conformal transformations. Before we consider conformal transformations on $\hat{\mathbb{H}}$ as we will need them later on, we will first consider natural actions of $GL(2, \mathbb{H})$ on the projective spaces, which will later naturally induce the conformal transformations. We will construct this actions in a way such that the action on the quaternionic left-projective space and the action on the quaternionic right-projective space induce actions on $\hat{\mathbb{H}}$ that coincide and thus give us exactly one group of conformal transformations. In order to do that we have to modify the action on the quaternionic right-projective space. In the next definition we introduce the map we will use for this purpose and in the lemma after that we will prove that it is indeed a group homomorphism.

Definition 1.3.9 (Pinch map)
We refer to the map

$$\begin{aligned}
\tau : GL(2, \mathbb{H}) &\longrightarrow GL(2, \mathbb{H}) \\
\begin{pmatrix} A & B \\ C & D \end{pmatrix} &\longmapsto \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}
\end{aligned}$$

as the *pinch map* on $GL(2, \mathbb{H})$.

Remark 1.3.10. *The pinch map is indeed well-defined. We see this by checking the equations from lemma 1.3.7. Let* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(2,\mathbb{H})$.
In the case that $C = 0$ *we have that* $A \cdot D \neq 0$ *and thus also* $\left(\begin{smallmatrix} A & -B \\ 0 & D \end{smallmatrix}\right) \in GL(2,\mathbb{H})$.
In the case that $C \neq 0$ *we have that* $AC^{-1}D - B \neq 0$. *This implies that*

$$A(-C)^{-1}D - (-B) = -(AC^{-1}D - B) \neq 0$$

and thus that $\tau(M) \in GL(2,\mathbb{H})$.

Lemma 1.3.11. *(Properties of the pinch map)*
The pinch map is a group automorphism of $GL(2,\mathbb{H})$.

Proof:
Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) \in GL(2,\mathbb{H})$. Then we have

$$\begin{aligned}
\tau\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} &= \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \cdot \begin{pmatrix} A' & -B' \\ -C' & D' \end{pmatrix} \\
&= \begin{pmatrix} AA' + BC' & -AB' - BD' \\ -CA' - DC' & CB' + DD' \end{pmatrix} \\
&= \tau\begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix} \\
&= \tau\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\right)
\end{aligned}$$

The consideration that τ is an involution finishes the proof. □

Now we introduce the actions of $GL(2,\mathbb{H})$ on the projective spaces we mentioned before.

Lemma 1.3.12. *(Actions of* $GL(2,\mathbb{H})$ *on projective spaces)*

1. *The map*

$$\bar{\pi}_\ell :\ GL(2,\mathbb{H}) \longrightarrow \mathrm{Diff}(P^1\mathbb{H}_\ell, P^1\mathbb{H}_\ell)$$

   defined by

$$\bar{\pi}_\ell(M).[v]_\ell := [M^{-1}v]_\ell$$

   for $M \in GL(2,\mathbb{H})$ *and* $v \in \mathbb{H}^2 \setminus \{0\}$ *is a group homomorphism.*

2. *Furthermore the map*

$$\bar{\pi}_r :\ GL(2,\mathbb{H}) \longrightarrow \mathrm{Diff}(P^1\mathbb{H}_r, P^1\mathbb{H}_r)$$

defined by

$$\bar{\pi}_r(M).[v]_r := [(v^T\tau(M))^T]_r$$

for $M \in GL(2,\mathbb{H})$ *and* $v \in \mathbb{H}^2 \setminus \{0\}$ *is a group homomorphism.*

Proof:
Let $M = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right) \in GL(2,\mathbb{H})$ with inverse $M^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(2,\mathbb{H})$ and $v = \left(\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right) \in \mathbb{H}^2 \setminus \{0\}$.

1. We have the following equality:

$$M^{-1}v = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} AX_1 + BX_2 \\ CX_1 + DX_2 \end{pmatrix}$$

Furthermore we have for $a \in \mathbb{H} \setminus \{0\}$:

$$\begin{aligned} M^{-1}(v \cdot a) &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} X_1 a \\ X_2 a \end{pmatrix} = \begin{pmatrix} AX_1 a + BX_2 a \\ CX_1 a + DX_2 a \end{pmatrix} \\ &= \begin{pmatrix} AX_1 + BX_2 \\ CX_1 + DX_2 \end{pmatrix} \cdot a = (M^{-1}v) \cdot a. \end{aligned}$$

Therefore we get that for all $v, w \in \mathbb{H} \setminus \{0\}, M \in GL(2,\mathbb{H})$:

$$[v]_\ell = [w]_\ell \;\Rightarrow\; [M^{-1}v]_\ell = [M^{-1}w]_\ell$$

Thus the map $\bar{\pi}_\ell(M) :\; P^1\mathbb{H}_\ell \to P_1\mathbb{H}_\ell$ is well-defined. Now we have to show that this map is a diffeomorphism of class $\mathcal{C}^\infty$. Since $M^{-1} \in GL(2,\mathbb{H})$, we see directly that $\bar{\pi}_\ell(M)$ is bijective and its inverse is given by the map

$$\bar{\pi}_\ell(M^{-1}) :\; P^1\mathbb{H}_\ell \to P^1\mathbb{H}_\ell,\; [v]_\ell \mapsto [Mv]_\ell.$$

The map $\bar{\pi}_\ell(M)$ is continuous for every $M \in GL(2,\mathbb{H})$ and thus a homeomorphism.
Now we are going to show that $\bar{\pi}_\ell(M)$ is of class $\mathcal{C}^\infty$. For this we see that for any open subset $U \subseteq \mathbb{H} \setminus \{0\}$ we have that with the notation from proposition 1.3.6 the maps

$$\begin{aligned} \psi_1 \circ \bar{\pi}_\ell(M) \circ \psi_1^{-1} &: U \to \mathbb{H},\; u \mapsto (Au + C) \cdot (Bu + D)^{-1}, \\ \psi_1 \circ \bar{\pi}_\ell(M) \circ \psi_2^{-1} &: U \to \mathbb{H},\; u \mapsto (Bu + D) \cdot (Au + C)^{-1}, \\ \psi_2 \circ \bar{\pi}_\ell(M) \circ \psi_1^{-1} &: U \to \mathbb{H},\; u \mapsto (A + Cu) \cdot (B + Du)^{-1}, \\ \psi_2 \circ \bar{\pi}_\ell(M) \circ \psi_2^{-1} &: U \to \mathbb{H},\; u \mapsto (B + Du) \cdot (A + Cu)^{-1} \end{aligned}$$

are of class $\mathcal{C}^\infty$ as concatenations of smooth maps regarding to lemma 1.1.13. Together with the fact that $\bar{\pi}_\ell(M)$ is continuous this shows that

$\bar{\pi}_\ell(M)$ is of class $\mathcal{C}^\infty$. Since $M \in GL(2,\mathbb{H})$ is arbitrary and $\bar{\pi}_\ell(M)^{-1} = \bar{\pi}_\ell(M^{-1})$ this shows that $\bar{\pi}_\ell(M)$ is a diffeomorphism of class $\mathcal{C}^\infty$. Thus the map $\bar{\pi}_\ell$ is well-defined.
The last point that remains to be shown is that $\bar{\pi}_\ell$ is a group homomorphism. Consider $M, N \in GL(2,\mathbb{H})$. Then we have for $v \in \mathbb{H}^2 \setminus \{0\}$:

$$\begin{aligned}\bar{\pi}_\ell(M \cdot N).[v]_\ell &= [(M \cdot N)^{-1} v]_\ell = [N^{-1} \cdot M^{-1} v]_\ell \\ &= \bar{\pi}_\ell(N) \circ \bar{\pi}_\ell(M).[v]_\ell.\end{aligned}$$

Taking into account that the group multiplication in $\mathrm{Diff}(P^1\mathbb{H}_\ell)$ is given as

$$\cdot : \mathrm{Diff}(P^1\mathbb{H}_\ell) \times \mathrm{Diff}(P^1\mathbb{H}_\ell) \to \mathrm{Diff}(P^1\mathbb{H}_\ell),\ (f,g) \mapsto g \circ f$$

we thus see that $\bar{\pi}_\ell$ is a group homomorphism and have finished the proof for $\bar{\pi}_\ell$.

2. The proof for $\bar{\pi}_r$ is very similar to the proof for $\bar{\pi}_\ell$. For this reason we will present it in less detail.
Firstly we note the following equality:

$$(v^T \tau(M))^T = \left[\begin{pmatrix} X_1 & X_2 \end{pmatrix} \cdot \begin{pmatrix} A' & -B' \\ -C' & D' \end{pmatrix}\right]^T = \begin{pmatrix} X_1 A' - X_2 C' \\ -X_1 B' + X_2 D' \end{pmatrix}$$

Furthermore we have for $a \in \mathbb{H} \setminus \{0\}$:

$$\begin{aligned}((a \cdot v)^T \tau(M))^T &= \left[\begin{pmatrix} aX_1 & aX_2 \end{pmatrix} \cdot \begin{pmatrix} A' & -B' \\ -C' & D' \end{pmatrix}\right]^T \\ &= \begin{pmatrix} aX_1 A' - aX_2 C' \\ -aX_1 B' + aX_2 D' \end{pmatrix} = a \cdot (v^T \tau(M))^T\end{aligned}$$

Therefore we get that for all $v, w \in \mathbb{H} \setminus \{0\}, M \in GL(2,\mathbb{H})$:

$$[v]_r = [w]_r \ \Rightarrow\ [(v^T \tau(M))^T]_r = [(w^T \tau(M))^T]_r$$

Thus the map $\bar{\pi}_r(M) : P^1\mathbb{H}_r \to P_1\mathbb{H}_r$ is well-defined. Again we see that for every $M \in GL(2,\mathbb{H})$ the map $\bar{\pi}_r(M)$ is bijective by seeing that the map $\bar{\pi}_r(M^{-1})$ is its both-sided inverse. We show that $\bar{\pi}_r(M)$ is of class $\mathcal{C}^\infty$ in the same way we showed that $\bar{\pi}_\ell(M)$ is of class $\mathcal{C}^\infty$ by showing continuity and using the atlas of $P_r^{\mathbb{H}}$ given in proposition 1.3.6. Lastly we will check that $\bar{\pi}_r$ is a group homomorphism. We consider $M, N \in GL(2,\mathbb{H})$ then we have for $v \in \mathbb{H}^2 \setminus \{0\}$:

$$\begin{aligned}\bar{\pi}_r(M \cdot N).[v]_r &= [(v^T \tau(M \cdot N))^T] = [(v^T \tau(M) \cdot \tau(N))^T] \\ &= [(((v^T \tau(M))^T)^T \cdot \tau(N))^T] = \bar{\pi}_r(N) \circ \bar{\psi}_r(M).[v]_r.\end{aligned}$$

With this equality we see that $\bar{\pi}_r$ is a group homomorphism and thus we have finished the proof.

□

Now we prove that the space $\hat{\mathbb{H}}$ and the two quaternionic projective spaces are diffeomorphic by giving suitable diffeomorphisms. These diffeomorphism can then be used to define the actions of $GL(2,\mathbb{H})$ on $\hat{\mathbb{H}}$ which give us the conformal transformations.

Lemma 1.3.13. *(Diffeomorphisms between $\hat{\mathbb{H}}$ and projective spaces)*
The maps

$$\begin{aligned} I_\ell :\ & P^1\mathbb{H}_\ell \longrightarrow \hat{\mathbb{H}}, \\ & \left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right] \longmapsto \begin{cases} X_1 \cdot X_2^{-1} & \textit{for } X_2 \neq 0 \\ \infty & \textit{else} \end{cases} \end{aligned}$$

and

$$\begin{aligned} I_r :\ & P^1\mathbb{H}_r \longrightarrow \hat{\mathbb{H}}, \\ & \left[\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right] \longmapsto \begin{cases} X_1^{-1} \cdot X_2 & \textit{for } X_1 \neq 0 \\ \infty & \textit{else} \end{cases} \end{aligned}$$

are diffeomorphisms of class $\mathcal{C}^\infty$ with inverses

$$\begin{aligned} I_\ell^{-1} :\ \hat{\mathbb{H}} &\longrightarrow P^1\mathbb{H}_\ell, \\ \mathbb{H} \ni X &\longmapsto \left[\begin{pmatrix} X \\ 1 \end{pmatrix}\right]_\ell, \\ \infty &\longmapsto \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]_\ell. \end{aligned}$$

and

$$\begin{aligned} I_r^{-1} :\ \hat{\mathbb{H}} &\longrightarrow P^1\mathbb{H}_r, \\ \mathbb{H} \ni X &\longmapsto \left[\begin{pmatrix} 1 \\ X \end{pmatrix}\right]_r, \\ \infty &\longmapsto \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]_r. \end{aligned}$$

Proof:
We will just show that I_ℓ is a diffeomorphism. The case of I_r follows analogously. At first we see that I_ℓ is a bijection by noting that its both-sided inverse is indeed given by I_ℓ^{-1}. To show that I_ℓ is of class $\mathcal{C}^\infty$ we again consider the atlases given in proposition 1.3.5 and proposition 1.3.6. Now we note first that by definition $I_\ell(P^1\mathbb{H}_\ell \setminus \{[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]\}) = \mathbb{H}$ and that furthermore for any $X \in \mathbb{H} \setminus \{0\}$:

$$\varphi_1 \circ I_\ell \circ \psi_1^{-1}(X) = \varphi_1 \circ I_\ell\left(\left[\begin{pmatrix} X \\ 1 \end{pmatrix}\right]_\ell\right) = \varphi_1(X) = X$$

and thus

$$\psi_1 \circ I_\ell \circ \varphi_1^{-1} = \mathrm{id}_{\mathbb{H}\setminus\{0\}}.$$

The identity on an open subset of a Banach space is of class $\mathcal{C}^\infty$. Furthermore we see that again by definition $I_\ell(P^1\mathbb{H}_\ell \setminus \{[\binom{1}{0}]_\ell\}) = \hat{\mathbb{H}} \setminus \{0\}$ and that for any $X \in \mathbb{H} \setminus \{0\}$:

$$\varphi_2 \circ I_\ell \circ \psi_2^{-1}(X) = \varphi_2 \circ I_\ell \left(\left[\binom{1}{X}\right]_\ell\right) = \varphi_2(X^{-1}) = X$$

and thus

$$\psi_2 \circ I_\ell \circ \varphi_2^{-1} = \mathrm{id}_{\mathbb{H}\setminus\{0\}}.$$

Again this is a $\mathcal{C}^\infty$ function and since $P^1\mathbb{H}_\ell = P^1\mathbb{H}_\ell \setminus \{[\binom{1}{0}]\} \cup P^1\mathbb{H}_\ell \setminus \{[\binom{0}{1}]\}$ it is shown that I_ℓ is of class $\mathcal{C}^\infty$. The same calculation for I_ℓ^{-1} also leads to identities on the atlases and thus I_ℓ is a diffeomorphism of class $\mathcal{C}^\infty$. $\square$

Definition 1.3.14 (Conformal Transformations)
With the notation of lemma 1.3.13 we define two group homomorphism from $GL(2,\mathbb{H})$ to $\mathrm{Diff}(\hat{\mathbb{H}},\hat{\mathbb{H}})$ as

$$\begin{aligned} \pi_\ell :\ GL(2,\mathbb{H}) &\longrightarrow \mathrm{Diff}(\hat{\mathbb{H}},\hat{\mathbb{H}}) \\ M &\longmapsto I_\ell \circ \bar{\pi}_\ell(M) \circ I_\ell^{-1} \end{aligned}$$

and

$$\begin{aligned} \pi_r :\ GL(2,\mathbb{H}) &\longrightarrow \mathrm{Diff}(\hat{\mathbb{H}},\hat{\mathbb{H}}) \\ M &\longmapsto I_r \circ \bar{\pi}_r(M) \circ I_r^{-1} \end{aligned}$$

We call the group $\pi_\ell(GL(2,\mathbb{H}))$ the *group of conformal transformations.*

In the next proposition we give the explicit form of the conformal transformations on $\hat{\mathbb{H}}$. In order to shorten our notation we define on $\hat{\mathbb{H}}$ that from now on $0^{-1} = \infty$ and that $Y \cdot \infty = \infty \cdot Y = \infty$ for any $Y \in \hat{\mathbb{H}} \setminus \{0\}$.

Proposition 1.3.15. *(Evaluation of the conformal transformations)*
Let $M = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in GL(2,\mathbb{H})$ with inverse $M^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2,\mathbb{H})$. Then we have for $X \in \mathbb{H}$:

$$\begin{aligned} \pi_\ell(M).X &= (AX + B)(CX + D)^{-1}, \\ \pi_r(M).X &= (A' - XC')^{-1}(-B' + XD'). \end{aligned}$$

Furthermore we get

$$\begin{aligned} \pi_\ell(M).\infty &= AC^{-1}, \\ \pi_r(M).\infty &= -(C')^{-1}D'. \end{aligned}$$

Proof:
Let $X \in \mathbb{H}$. Then we get

$$\begin{aligned}\pi_\ell(M).X &= I_\ell \circ \bar{\pi}_\ell(M) \circ I_\ell^{-1}(X) = I_\ell \circ \bar{\pi}_\ell(M).\left[\left(\begin{smallmatrix} X \\ 1 \end{smallmatrix}\right)\right]_\ell = I_\ell\left(\left[\left(\begin{smallmatrix} AX+B \\ CX+D \end{smallmatrix}\right)\right]_\ell\right) \\ &= (AX+B)(CX+D)^{-1}.\end{aligned}$$

Furthermore we get

$$\begin{aligned}\pi_r(M).X &= I_r \circ \bar{\pi}_r(M) \circ I_r^{-1}(X) = I_r \circ \bar{\pi}_r(M).\left[\left(\begin{smallmatrix} 1 \\ X \end{smallmatrix}\right)\right]_r = I_r\left(\left[\left(\begin{smallmatrix} A'-XC' \\ -B'+XD' \end{smallmatrix}\right)\right]_r\right) \\ &= (A'-XC')^{-1}(-B'+XD').\end{aligned}$$

For the remaining two cases we consider $\infty \in \hat{\mathbb{H}}$. Then we get

$$\begin{aligned}\pi_\ell(M).\infty &= I_\ell \circ \bar{\pi}_\ell(M) \circ I_\ell^{-1}(\infty) = I_\ell \circ \bar{\pi}_\ell(M).\left[\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right]_\ell = I_\ell\left(\left[\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)\right]_\ell\right) \\ &= AC^{-1}\end{aligned}$$

and

$$\begin{aligned}\pi_r(M).\infty &= I_r \circ \bar{\pi}_r(M) \circ I_r^{-1}(\infty) = I_r \circ \bar{\pi}_r(M).\left[\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right]_r = I_r\left(\left[\left(\begin{smallmatrix} -C' \\ D' \end{smallmatrix}\right)\right]_r\right) \\ &= -(C')^{-1}D'\end{aligned}$$

This finishes the proof. $\square$

Next we are going to prove that in fact we have not defined two different actions π_ℓ and π_r on $\hat{\mathbb{H}}$, but that these maps actually are just two presentations of the same map and therefore that we can use them indiscriminately. The following proof is taken from the book [4].

Lemma 1.3.16. *(Coincidence of π_ℓ and π_r)*
For every $M \in GL(2, \mathbb{H})$ it is true that

$$\pi_\ell(M) = \pi_r(M).$$

Thus in particular the groups $\pi_\ell(GL(2, \mathbb{H}))$ and $\pi_r(GL(2, \mathbb{H}))$ are equal.

Proof:
At first, let $M \in GL(2, \mathbb{H})$ be given with $M^{-1} = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$. Then we have that $A \neq 0$, since else $M \neq GL(2, \mathbb{H})$ by lemma 1.3.7. In this case we thus get for every $X \in \mathbb{H}$:

$$\begin{aligned}\pi_\ell(M).X &= (AX+B)D^{-1} = AXD^{-1} + BD^{-1} \\ &= (A^{-1})^{-1}(A^{-1}BD^{-1} + XD^{-1}) = \pi_r\begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}.X.\end{aligned}$$

Furthermore we get

$$\pi_\ell(M).\infty = \infty = \pi_r \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix} .\infty.$$

From lemma 1.3.8 we know that

$$M = (M^{-1})^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

and thus we have $\pi_\ell(M) = \pi_r(M)$ for M as above.

Now we consider $M \in GL(2, \mathbb{H})$ with $M^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $C \neq 0$. Then we calculate for $X \in \mathbb{H}$:

$$\begin{aligned}
\pi_\ell(M).X &= (AX + B)(CX + D)^{-1} \\
&= (A(X + C^{-1}D) + B - AC^{-1}D)(X + C^{-1}D)^{-1}C^{-1} \\
&= A(X + C^{-1}D)(X + C^{-1}D)^{-1}C^{-1} + (B - AC^{-1}D)(X + C^{-1}D)^{-1}C^{-1} \\
&= AC^{-1} - C^{-1}(CAC^{-1}D - CB)(X + C^{-1}D)^{-1}C^{-1}
\end{aligned}$$

With the abbreviation $H := CAC^{-1}D - CB$ we get from this by using the fact that $H \neq 0$ by lemma 1.3.7:

$$\begin{aligned}
\pi_\ell(M).X &= AC^{-1} - C^{-1}H(X + C^{-1}D)^{-1}C^{-1} \\
&= AC^{-1} - [(X + C^{-1}D)H^{-1}C]^{-1}C^{-1} \\
&= (XH^{-1}C + C^{-1}DH^{-1}C)^{-1}(XH^{-1}CAC^{-1} + C^{-1}DH^{-1}CAC^{-1} - C^{-1}) \\
&= (C^{-1}DH^{-1}C + XH^{-1}C)^{-1}(C^{-1}DH^{-1}CAC^{-1} - C^{-1} + XH^{-1}CAC^{-1}) \\
&= \pi_r \begin{pmatrix} C^{-1}DH^{-1}C & C^{-1} - C^{-1}DH^{-1}CAC^{-1} \\ -H^{-1}C & H^{-1}CAC^{-1} \end{pmatrix} .X.
\end{aligned}$$

Furthermore we get

$$\begin{aligned}
\pi_\ell(M).\infty &= AC^{-1} \\
&= (H^{-1}C)^{-1}(H^{-1}CAC^{-1}) \\
&= \pi_r \begin{pmatrix} C^{-1}DH^{-1}C & C^{-1} - C^{-1}DH^{-1}CAC^{-1} \\ -H^{-1}C & H^{-1}CAC^{-1} \end{pmatrix} .\infty.
\end{aligned}$$

From lemma 1.3.8 we know that

$$M = (M^{-1})^{-1} = \begin{pmatrix} C^{-1}DH^{-1}C & C^{-1} - C^{-1}DH^{-1}CAC^{-1} \\ -H^{-1}C & H^{-1}CAC^{-1} \end{pmatrix}$$

and thus we have $\pi_\ell(M) = \pi_r(M)$ for M as above. This finishes the proof. $\square$

Our main goal for the rest of this chapter will be to show that conformal transformations on $\hat{\mathbb{H}}$ in analogy to Möbius transformations on the one-point compactification $\hat{\mathbb{C}}$ of the complex numbers preserve hyper-surfaces and spheres. For this reason we will define first these geometric structures in $\hat{\mathbb{H}}$ and then introduce a standard form to describe them:

Definition 1.3.17 (Hyper-surfaces and Spheres in $\mathbb{H}$ and $\hat{\mathbb{H}}$)
We call a subset $S \subset \hat{\mathbb{H}}$ a *sphere in* $\hat{\mathbb{H}}$, if and only if there are $R \in \mathbb{R}_{>0}$ and $X \in \mathbb{H}$ such that $S = S^3_R(X)$, i.e. spheres in $\hat{\mathbb{H}}$ are exactly spheres in $\mathbb{H}$.
We call a subset $S \subset \mathbb{H}$ a *hyper-surface*, if and only if there exist $c \in \mathbb{R}$ and $B \in \mathbb{H} \setminus \{0\}$ such that

$$S = \{X \in \mathbb{H} \mid \langle X, B\rangle = c\}.$$

We call a subset $S \subset \mathbb{H}$ a *hyper-surface in* $\hat{\mathbb{H}}$, if and only if there exists a hyper-surface $S' \subset \mathbb{H}$ such that $S = S' \cup \{\infty\}$.

Proposition 1.3.18. *(Hyper-surfaces and Spheres in $\mathbb{H}$)*
A subset $S \subseteq \mathbb{H}$ is a hyper surface or a sphere, if and only if there are $a, c \in \mathbb{R}$, $B \in \mathbb{H}$ with $N(B) - ac > 0$ such that

$$S := \{X \in \mathbb{H} \mid aXX^+ + XB^+ + X^+B + c = 0\}.$$

Furthermore S is a sphere, if and only if $a \neq 0$ and S is a hyperspace if and only if $a = 0$.

Proof:
We firstly define for $B \in \mathbb{H}\setminus\{0\}$ and $c \in \mathbb{R}$ the set $H_{B,c} := \{s \in \mathbb{R}^4 \mid \langle s, B\rangle = c\}$. We know by the definition that every hyper-surface is of this form. By using the algebra structure of the quaternions we can characterize hyper-surfaces and spheres in the following way with $B \in \mathbb{H}$ and $k \in \mathbb{R}$

$$\begin{aligned}
H_{B,k} &= \{X \in \mathbb{H} \mid \Re(X^+B) = k\} = \{X \in \mathbb{H} \mid X^+B + XB^+ = 2k\} \text{ and}\\
S^3_r(-B) &= \{X \in \mathbb{H} \mid \Re((X+B)(X+B)^+) = r^2\}\\
&= \{X \in \mathbb{H} \mid XX^+ + BX^+ + XB^+ + N(B) = r^2\}\\
&= \{X \in \mathbb{H} \mid XX^+ + BX^+ + XB^+ = r^2 - N(B)\}\\
&= \{X \in \mathbb{H} \mid XX^+ + 2 \cdot \Re(XB^+) = r^2 - N(B)\}\\
&= \{X \in \mathbb{H} \mid XX^+ + 2 \cdot \Re(X^+B) = r^2 - N(B)\}\\
&= \{X \in \mathbb{H} \mid XX^+ + X^+B + XB^+ = r^2 - N(B)\}
\end{aligned}$$

We see that $H_{B,c}$ is of the form given in the proposition by setting $a = 0$ and $c = -2k$. Similarly $S^3_r(-B)$ can be written in this form by setting $a = 1$ and

$c = -r^2 + N(B)$. Furthermore we see that in both cases the proposed inequality $N(B) - ac > 0$ holds.
In the other direction let $a, c \in \mathbb{R}$, $B \in \mathbb{H}$ with $N(B) - ac > 0$ be given and consider the set

$$S = \{X \in \mathbb{H} \mid aXX^+ + XB^+ + X^+B + c = 0\}.$$

If $a = 0$ we see in the notation from above that

$$\begin{aligned} S &= \{X \in \mathbb{H} \mid XB^+ + X^+B + c = 0\} \\ &= \{X \in \mathbb{H} \mid \langle X, B\rangle = -\frac{c}{2}\} \\ &= H_{B,-\frac{c}{2}}. \end{aligned}$$

This is valid because by assumption $N(B) > 0$ and therefore $B \in \mathbb{H} \setminus \{0\}$.
If $a \neq 0$ we get

$$\begin{aligned} S &= \{X \in \mathbb{H} \mid XX^+ + a^{-1}XB^+ + a^{-1}X^+B = -a^{-1}c\} \\ &= \{X \in \mathbb{H} \mid XX^+ + a^{-1}XB^+ + a^{-1}X^+B + a^{-2} \cdot N(B) = -a^{-1}c + a^{-2} \cdot N(B)\} \\ &= \{X \in \mathbb{H} \mid (X + a^{-1}B)(X + a^{-1}B)^+ = \frac{N(B) - ac}{a^2}\} \\ &= \{X \in \mathbb{H} \mid \left\langle X + a^{-1}B, X + a^{-1}B\right\rangle = \frac{N(B) - ac}{a^2}\} \\ &= S^3_{\frac{N(B)-ac}{a^2}}(-a^{-1}B). \end{aligned}$$

This is valid, because by assumption $(N(B) - AC) \in \mathbb{R}_{>0}$ and thus $\frac{N(B)-ac}{a^2} \in \mathbb{R}_{>0}$. Hence we have shown the proposition. $\square$

Corollary 1.3.19.
Let $S, S' \subset \mathbb{H}$ be hyper-surfaces or spheres in $\mathbb{H}$. If there exist six points s_i with $i \in \{1, 2, 3, 4, 5, 6\}$ such that $s_i \in S$ and $s_i \in S'$, then we have $S = S'$.

Proof:
Let $a, c \in \mathbb{R}$ and $B = \sum\limits_{n=0}^{3} b_n e_n \in \mathbb{H}$. Then we have in components

$$aXX^+ + XB^+ + X^+B + c = a \cdot N(X) + 2 \cdot \sum_{n=0}^{3} b_n x_n + c.$$

Fixing six different values for (x_0, x_1, x_2, x_3) induces a system of six inhomogeneous real linear equations, which determines the values for a, B and c, if it is solvable. The characterisation given in proposition 1.3.18 for spheres and hyperplanes in $\mathbb{H}$ now implies the corollary. $\square$

Now we are going to show that conformal transformations map any sphere or hyper-surface in $\hat{\mathbb{H}}$ to either a sphere or a hyper-surface in $\hat{\mathbb{H}}$. At first we give generators of the group of conformal transformations in order to be able to split up the proof for this preservation into small pieces.

Definition 1.3.20 (Translation, rotation scaling and inversion)

1. Let $A \in \mathbb{H}$. Then we call a conformal transformation of the form

$$\begin{aligned} T_A : \hat{\mathbb{H}} &\longrightarrow \hat{\mathbb{H}} \\ \mathbb{H} \ni X &\longmapsto X + A \\ \infty &\longmapsto \infty \end{aligned}$$

   a *translation map.*

2. Let $A, B \in \mathbb{H} \setminus \{0\}$. Then we call a map of the form

$$\begin{aligned} RS_{(A,B)} : \hat{\mathbb{H}} &\longrightarrow \hat{\mathbb{H}} \\ X &\longmapsto AXB \\ \infty &\longmapsto \infty \end{aligned}$$

   a rotational scaling.

3. We call the map

$$\begin{aligned} I : \hat{\mathbb{H}} &\longrightarrow \hat{\mathbb{H}} \\ X &\longmapsto \begin{cases} \frac{1}{X} & \text{for } X \in \mathbb{H} \setminus \{0\} \\ 0 & \text{for } X = \infty \\ \infty & \text{for } X = 0 \end{cases} \end{aligned}$$

   the inversion map.

Remark 1.3.21. *The maps given in definition 1.3.20 are indeed conformal transformations. Explicitly we have for $A \in \mathbb{H}$:*

$$T_A = \pi_\ell \begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}.$$

For $A, B \in \mathbb{H} \setminus \{0\}$ we have:

$$RS_{(A,B)} = \pi_\ell \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix}$$

Finally we have

$$I = \pi_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Lemma 1.3.22. *(Generators of conformal transformations)*
The group $\pi_\ell(GL(2,\mathbb{H}))$ of conformal transformations is generated by the set

$$\{T_A \mid A \in \mathbb{H}\} \cup \{RS_{(A,B)} \mid A, B \in \mathbb{H} \setminus \{0\}\} \cup \{I\} \subseteq \pi_\ell(GL(2,\mathbb{H})).$$

Proof:
Let $M \in GL(2,\mathbb{H})$ with $M^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in GL(2,\mathbb{H})$. In the case that $C \neq 0$ we have the following equality that was shown in the proof of lemma 1.3.16 for $X \in \mathbb{H}$ (let again be $H := CAC^{-1} - CB$):

$$\begin{aligned}\pi_\ell(M).X &= AC^{-1} - C^{-1}H(X + C^{-1}D)^{-1}C^{-1} \\ &= T_{AC^{-1}}(-C^{-1}H(X + C^{-1}D)^{-1}C^{-1}) \\ &= T_{AC^{-1}} \circ RS_{(-C^{-1}H,C^{-1})} \circ I(X + C^{-1}D) \\ &= T_{AC^{-1}} \circ RS_{(-C^{-1}H,C^{-1})} \circ I \circ T_{C^{-1}D}(X)\end{aligned}$$

For $\infty \in \hat{\mathbb{H}}$ we also have:

$$\begin{aligned}\pi_\ell(M).\infty &= AC^{-1} = T_{AC^{-1}}(0) \\ &= T_{AC^{-1}} \circ RS_{(-C^{-1}H,C^{-1})}(0) \\ &= T_{AC^{-1}} \circ RS_{(-C^{-1}H,C^{-1})} \circ I(\infty) \\ &= T_{AC^{-1}} \circ RS_{(-C^{-1}H,C^{-1})} \circ I \circ T_{C^{-1}D}(\infty)\end{aligned}$$

Therefore we have:

$$\pi_\ell(M) = T_{AC^{-1}} \circ RS_{-C^{-1}H,C^{-1}} \circ I \circ T_{C^{-1}D}.$$

In the case that $C = 0$ we have for all $X \in \mathbb{H}$:

$$\pi_\ell(M).X = A^{-1}(-B + XD) = RS_{(A^{-1},1)} \circ T_{-B} \circ RS_{(1,D)}(X)$$

and for $\infty \in \hat{\mathbb{H}}$:

$$\pi_\ell(M).\infty = \infty = RS_{(A^{-1},1)} \circ T_{-B} \circ RS_{(1,D)}(\infty)$$

and thus

$$\pi_\ell(M) = RS_{(A^{-1},1)} \circ T_{-B} \circ RS_{(1,D)}.$$

This shows that every element of $\pi_\ell(GL(2,\mathbb{H}))$ can be written as composition of translations, rotation scalings and inversions. This finishes the proof. $\square$

Now we are nearly ready to present the proof that conformal transformations preserve spheres and hyper-surfaces, but we will first show a small lemma that we will need.

Lemma 1.3.23. *(Orthogonality of rotations)*
The maps $RS_{(A,B)}$ with $A, B \in SU(2) \subset \mathbb{H}$ are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}$.

Proof:
For $X, Y \in \mathbb{H}$ we have

$$
\begin{aligned}
& \left\langle RS_{(A,B)}(X), RS_{(A,B)}(Y) \right\rangle \\
&= \langle AXB, AYB \rangle \\
&= \Re((AXB)^+(AYB)) \\
&= \frac{1}{2}\left((AXB)^+(AYB) + (AYB)^+(AXB)\right) \\
&= \frac{1}{2}\left(B^+X^+A^+AYB + B^+Y^+A^+AXB\right) \\
&= \frac{1}{2}\left(B^+X^+N(A)YB + B^+Y^+N(A)XB\right) \\
&= \frac{1}{2}\left(B^+X^+YB + B^+Y^+XB\right) && \mid N(a) = 1 \\
&= \frac{1}{2}B^+\left(X^+Y + Y^+X\right)B \\
&= B^+ \langle X, Y \rangle B \\
&= \langle X, Y \rangle \cdot N(b) && \mid \langle X, Y \rangle \in \mathbb{R} \\
&= \langle X, Y \rangle && \mid N(b) = 1
\end{aligned}
$$

This finishes the proof. □

Proposition 1.3.24. *(Preservation of spheres and hyper-surfaces)*
Let $S \subset \hat{\mathbb{H}}$ be a hyper-surface or a sphere in $\hat{\mathbb{H}}$. Then for every $M \in \mathrm{GL}(2, \mathbb{H})$ the image $\pi_\ell(M)(S) = \pi_r(M)(S)$ is a hyper-surface or a sphere in $\hat{\mathbb{H}}$.

Proof:
By lemma 1.3.22 we only have to show that translations, rotation scalings and inversions preserve the set of hyper-surfaces and spheres. Let thus $S' \subseteq \mathbb{H}$ be a hyper-surface or a sphere. By proposition 1.3.18 we know that there are $a, c \in \mathbb{R}$, $B \in \mathbb{H}$ with $N(B) - ac > 0$ such that

$$S' := \{s \in \mathbb{H} \mid aXX^+ + XB^+ + X^+B + c = 0\}$$

and furthermore S' is also a sphere in $\hat{\mathbb{H}}$, if $a \neq 0$ and the set $S = S' \cup \{\infty\}$ is a hypersurface in $\hat{\mathbb{H}}$, if $a = 0$. Taking this into account we see that for arbitrary translations with $A \in \mathbb{H}$ we have

$$\begin{aligned}
T_A(S') &= \{X + A \in \mathbb{H} \mid aXX^+ + XB^+ + X^+B + c = 0\} \\
&= \{X \in \mathbb{H} \mid a(X-A)(X-A)^+ + (X-A)B^+ + (X-A)^+B + c = 0\} \\
&= \{X \in \mathbb{H} \mid \\
&\qquad a(XX^+ - AX^+ - XA^+ + A\overline{A}) + X\overline{B} - A\overline{B} + \overline{X}B - \overline{A}B + c = 0\} \\
&= \{X \in \mathbb{H} \mid a(XX^+ - 2\langle A, X\rangle + A\overline{A}) + 2\langle X, B\rangle - 2\langle A, B\rangle + c = 0\} \\
&= \{X \in \mathbb{H} \mid aXX^+ - 2\langle X, aA - B\rangle + (a \cdot N(A) - 2\langle A, B\rangle + c) = 0\} \\
&= \{X \in \mathbb{H} \mid \\
&\qquad aXX^+ + X(B - aA)^+ + X^+(B - aA) + (a \cdot N(A) - 2\langle A, B\rangle + c) = 0\}
\end{aligned}$$

We have $(B - aA) \in \mathbb{H}$ and $(aN(A) - 2\langle A, B\rangle + c) \in \mathbb{R}$ and furthermore

$$\begin{aligned}
&N(B - aA) - a \cdot (aN(A) - 2\langle A, B\rangle + c) \\
&= \langle B - aA, B - aA\rangle - a^2N(A) + 2\langle aA, B\rangle - ac \\
&= N(B) - 2\langle aA, B\rangle + a^2N(A) - a^2N(A) + 2\langle aA, B\rangle - ac \\
&= N(B) - ac > 0
\end{aligned}$$

Considering the coefficient of $X\overline{X}$ in the expression for $T_A(S')$ we thus see that in $\mathbb{H}$ spheres get mapped to spheres and hypersurfaces to hypersurfaces under the map T_A. Considering the fact that $T_A(\infty) = \infty$ this also holds true for spheres and hypersurfaces in $\hat{\mathbb{H}}$.
For rotation scaling with arbitrary $A, C \in \mathbb{H} \setminus \{0\}$ we calculate

$$\begin{aligned}
&RS_{(A,C)}(S') \\
&= \{AXC \in \mathbb{H} \mid aXX^+ + XB^+ + X^+B + c = 0\} \\
&= \{X \in \mathbb{H} \mid \\
&\qquad a(A^{-1}XC^{-1})(A^{-1}XC^{-1})^+ + (A^{-1}XC^{-1})B^+ + (A^{-1}XC^{-1})^+B + c = 0\} \\
&= \{X \in \mathbb{H} \mid a(A^{-1}XC^{-1})(A^{-1}XC^{-1})^+ + 2\left\langle A^{-1}XC^{-1}, B\right\rangle + c = 0\} \\
&= \left\{X \in \mathbb{H} \mid aN(A^{-1}XC^{-1}) + 2\left|A^{-1}\right|\left|C^{-1}\right| \cdot \left\langle \frac{A^{-1}}{|A^{-1}|} X \frac{C^{-1}}{|C^{-1}|}, B\right\rangle + c = 0\right\} \\
&= \left\{X \in \mathbb{H} \mid aN(A^{-1}XC^{-1}) + 2\left\langle RS_{(|A|\cdot A^{-1}, |C|\cdot C^{-1})(X)}, \frac{B}{|A \cdot C|}\right\rangle + c = 0\right\} \\
&= \left\{X \in \mathbb{H} \mid aN(A^{-1}XC^{-1}) + 2\left\langle X, RS_{(|A|\cdot A^{-1}, |C|\cdot C^{-1})}\left(\frac{ABC}{|A \cdot C|}\right)\right\rangle + c = 0\right\} \\
&= \{X \in \mathbb{H} \mid aN(A^{-1})N(C^{-1})N(X) + 2\left\langle X, \frac{ABC}{N(A) \cdot N(C)}\right\rangle + c = 0\} \\
&= \{X \in \mathbb{H} \mid \\
&\qquad \frac{a}{N(A)N(C)}XX^+ + X \cdot \left(\frac{ABC}{N(A) \cdot N(C)}\right)^+ + X^+\left(\frac{ABC}{N(A) \cdot N(C)}\right) + c = 0\}
\end{aligned}$$

Here we used the orthogonality of the map $RS_{(A,B)}$ as shown in lemma 1.3.23. By calculating

$$\begin{aligned} & N\left(\frac{ABC}{N(A)\cdot N(C)}\right) - \frac{a}{N(A)N(C)}\cdot c \\ =& \frac{N(A)N(B)N(C)}{N(A)^2N(C)^2} - \frac{ac}{N(A)N(C)} \\ =& \frac{1}{N(A)N(C)}\cdot (N(B)-ac) > 0 \qquad\qquad | \; N(A)N(B) > 0 \end{aligned}$$

we see that $RS_{(A,C)}(S')$ is again a hypersurface or a sphere. More precisely we see again that, since $a = 0$ if and only if $\frac{a}{N(A)N(C)}$ hypersurfaces in $\mathbb{H}$ get mapped to hypersurfaces in $\mathbb{H}$ and spheres in $\mathbb{H}$ to spheres in $\mathbb{H}$ by the map $RS_{(A,C)}$. Together with the fact that $RS_{(A,C)}(\infty) = \infty$ we see that rotation scaling maps indeed preserve spheres and surfaces in $\hat{\mathbb{H}}$.
Next we consider the inversion map. In this case more care is necessary than in the cases before, because $I(\mathbb{H}) \neq \mathbb{H}$, since $I(0) = \infty$. But instead we can use the fact that $I(\mathbb{H}\setminus\{0\}) = \mathbb{H}\setminus\{0\}$. Thus we consider first the set $S'\setminus\{0\}$ and calculate:

$$\begin{aligned} I(S'\setminus\{0\}) &= \left\{X^{-1}\in\mathbb{H}\setminus\{0\} \mid aXX^+ + XB^+ + X^+B + c = 0\right\} \\ &= \left\{X\in\mathbb{H}\setminus\{0\} \mid aX^{-1}(X^{-1})^+ + X^{-1}B^+ + (X^{-1})^+B + c = 0\right\} \\ &= \left\{X\in\mathbb{H}\setminus\{0\} \mid \frac{a}{N(X)} + \frac{X^+}{N(X)}B^+ + \frac{X}{N(X)}B + c = 0\right\} \\ &= \left\{X\in\mathbb{H}\setminus\{0\} \mid a + X^+B^+ + XB + cN(X) = 0\right\} \\ &= \left\{X\in\mathbb{H}\setminus\{0\} \mid cXX^+ + XB + X^+B^+ + a = 0\right\} \end{aligned}$$

Furthermore we see that

$$N(B) - ca = N(B) - ac > 0$$

and thus the set $I(S'\setminus\{0\})$ is a subset of a sphere or hypersurface in $\mathbb{H}$.
Furthermore it is $I(\infty) = 0$ and thus by the bijectivity of I for any $S\subset\hat{\mathbb{H}}$ we have that $0\in I(S)$ if and only if $\infty\in S$. If S is a sphere or a hypersurface in $\hat{\mathbb{H}}$ this assumption implies that S is a hypersurface in $\hat{\mathbb{H}}$, i.e. $a = 0$ in the definition of $S\setminus\{\infty\}$. But in this case 0 is indeed a solution of the equation

$$cXX^+ + XB + X^+B^+ + a = 0$$

and therefore we have

$$I(S\setminus\{0\}) = \left\{X\in\mathbb{H} \mid cXX^+ + XB + X^+B^+ = 0\right\}.$$

This is a hypersurface or a sphere.
Next we see that $I(0) = \infty$ and thus for any $S'\subset\hat{\mathbb{H}}$ we have that $\infty\in I(S)$ if

and only if $0 \in S$. If S is a sphere or a hypersurface in $\hat{\mathbb{H}}$ this implies that 0 has to be a solution of the equation

$$aXX^+ + XB^+ + X^+B + c = 0.$$

This is true if and only if $c = 0$, but in this case we have that the equation

$$cXX^+ + XB + X^+B^+ + aN(X) = 0$$

for $X \in \mathbb{H}$ defines a hypersurface in $\mathbb{H}$. Using the result for the case that $0 \in S$ we can thus deduce that if S is a sphere or hypersurface in $\hat{\mathbb{H}}$ with $0 \in S$, the image $I(S)$ is a hypersurface in $\hat{\mathbb{H}}$.

This finishes the proof. $\square$

Chapter 2

Applications of Quaternionic Analysis

2.1 The quaternionic Poisson formula

In this section we want to apply methods of quaternionic analysis and representation theory of a certain subgroup of the group $GL(2,\mathbb{H})$ in order to prove a four dimensional Poisson formula for complex-valued harmonic functions that are defined on $B_1^4 \subset \mathbb{H}$. Before we can state it, we have to give a definition:

Definition 2.1.1 (Degree operator)
Let $S \subseteq \mathbb{H}$ be a connected subset. For $\varphi \in \hat{\mathcal{H}}(S)$ we define its *shifted degree* on $\mathring{S}$ by

$$\begin{aligned} \widetilde{\deg}\,\varphi :\ & \mathring{S} \longrightarrow \mathbb{C} \\ X = \sum_{j=0}^{3} x_n e_n \longmapsto & \ \varphi\,(X) + \sum_{j=0}^{3} x_j \cdot \partial_j \varphi(X) \end{aligned}$$

and extend $\widetilde{\deg}\,\varphi$ to a function on S by considering the continuous extension of the above definition to ∂S, when this is defined.

Before we can state the quaternionic Poisson formula we will first introduce a suitable space of harmonic functions for the formulation of the formula as well as a linear operator on this space that will represent the integral on one side of the Poisson formula:

Definition 2.1.2

1. Let $S \subset \mathbb{H}$ be a connected subset. Then we define

$$\hat{\mathcal{H}}(S) := \Big\{ \varphi \in \mathcal{C}(S, \mathbb{C}) \mid \varphi \,\text{harmonic on } \mathring{S} \,\wedge \sup_{X \in \mathring{B}_1^4} \left| \sum_{i=0}^{3} D_i \varphi(X) \right| + \sup_{X \in \mathring{B}_1^4} \left| \sum_{i=0}^{3} \sum_{j=0}^{3} D_i D_j \varphi(X) \right| < \infty \Big\}.$$

2. To simplify our notation we set $\hat{\mathcal{H}} := \hat{\mathcal{H}}(B_1^4)$.

3. We define the map

$$\mathrm{Harm}: \ \hat{\mathcal{H}} \to \hat{\mathcal{H}}$$

   by setting

$$\mathrm{Harm}(\varphi)(Y) = \frac{1}{2\pi^2} \cdot \int_{X \in S_1^3} \frac{1 - N(Y)}{N(X-Y)^2} \cdot \varphi(X) \, \mathrm{d}S(X)$$

   for all $Y \in \mathring{B}_1^4$ and

$$\mathrm{Harm}(\varphi)(Y) = \varphi(Y)$$

   for all $Y \in S_1^3$.

Now we will collect some results from harmonic analysis, which will be necessary in the following.

Remark 2.1.3. *(Harmonic Analysis)*
Let $S \subseteq \mathbb{H}$ be a connected subset with connected interior.

1. *The map* Harm *is well-defined by Theorem II.1.10 in [7], i.e. for all $\varphi \in$* Harm *we have that for any $Y \in \mathring{B}_1^4$:*

$$\lim_{X \to Y} \mathrm{Harm}(\varphi)(X) = \mathrm{Harm}(\varphi)(Y).$$

2. *Every harmonic function $\varphi : S \to \mathbb{C}$ is of class $\mathcal{C}^\infty$ (cf. [7, Theorems II.1.1 and II.1.7]) and real analytic (cf. [7, Result IV.5.5]).*

With the differential operator from definition 2.1.1 we can now give the quaternionic Poisson formula in the following way:

Theorem 2.1.4. *(Quaternionic Poisson formula)*
Let $R \in \mathbb{R}_{>0}$ and $\varphi : \ B_R^4 \to \mathbb{C}$ a continuous function that is harmonic on $\mathring{B}_R^4$. Then for all $Y \in \mathring{B}_R^4$ the chain of equalities

$$\begin{aligned}\varphi(Y) &= \frac{1}{2\pi^2} \int_{S_R^3} \frac{R^2 - N(Y)}{N(X-Y)^2} \cdot \frac{\varphi(X)}{R} \, \mathrm{d}S(X) \\ &= -\frac{1}{2\pi^2} \int_{S_R^3} \left(\widetilde{\deg}_X \frac{1}{N(X-Y)} \right) \cdot \frac{\varphi(X)}{R} \, \mathrm{d}S(X) \\ &= \frac{1}{2\pi^2} \int_{S_R^3} \frac{1}{N(X-Y)} \cdot \frac{1}{R} \widetilde{\deg} \varphi(X) \, \mathrm{d}S(X)\end{aligned}$$

holds.

For this proof of theorem 2.1.4 we will mainly follow [3, section 2.8]. Unfortunately I was not able to prove the irreducibility of a certain representation that is crucial in that proof. Furthermore I will not be able to show that the linear endomorphism Harm is equivariant with respect to this representation. Nevertheless I will give the main steps of the proof as far as possible and then give a reference for the validity of the Poisson formula, since we will rely on this result in the next section.

For the proof of the Poisson formula we rely on the representation theory of Lie groups and thus we will fix our notation for the treatment of Lie theory in the next definition. Note that with the term Lie group we will always refer to finite-dimensional real Lie groups.

Definition 2.1.5 (Lie groups and Lie algebras)

1. Let G be a Lie group. Then we denote the *Lie algebra associated to* G by *Lie*(G).

2. Let G be a Lie group and H a subgroup of G. Then we call H a Lie subgroup of G, if and only if H is a submanifold of class $\mathcal{C}^\infty$ that is embedded into G.

3. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Then we will denote by $\mathfrak{gl}(n, \mathbb{K})$ the Lie algebra of $n \times n$-matrices over $\mathbb{K}$ together with the commutator as Lie bracket.

4. We will denote by $\mathfrak{su}(2)$ the Lie algebra

$$\begin{aligned}&\left\{ \begin{pmatrix} -\mathrm{i}x_3 & -\mathrm{i}x_1 - x_2 \\ -\mathrm{i}x_1 - x_2 & \mathrm{i}x_3 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{C}) \ \middle| \ x_1, x_2, x_3 \in \mathbb{R} \right\} \\ &= \{ X \in \mathbb{H} \mid \Re(X) = 0 \}\end{aligned}$$

together with the commutator as Lie bracket.

5. On $\mathfrak{gl}(2,\mathbb{C})$ we define the exponential map

$$\exp: \mathfrak{gl}(2,\mathbb{C}) \to GL(2,\mathbb{H}),\ X \mapsto \sum_{j=0}^{\infty} \frac{1}{j!} \cdot X^j$$

In the next proposition we sum up the results about Lie groups and Lie algebras and their interrelation that we will need for the arguments that will follow.

Proposition 2.1.6. *(Lie theory)*

1. *The group $GL(2,\mathbb{C})$ is a Lie group with $\mathfrak{gl}(2,\mathbb{C})$ as associated Lie algebra.*
2. *The group $SU(2)$ is a Lie group with $\mathfrak{su}(2,\mathbb{C})$ as associated Lie algebra.*
3. *Let G and H be Lie groups. Then $G \times H$ with the induced group structure is also a Lie group and its Lie algebra is given as*
$$Lie(G \times H) = Lie(G) \oplus Lie(H).$$
4. *Let G be a Lie group and $\mathfrak{h} \subseteq Lie(G)$ a Lie subalgebra. Then there is a unique connected Lie subgroup of $H \subseteq G$ such that $Lie(H) = \mathfrak{h}$.*
5. *Let $\mathfrak{h} \subseteq \mathfrak{gl}(n,\mathbb{C})$ be a one-dimensional Lie subalgebra. Then the group given by*
$$H = \left\{ e^h \mid h \in \mathfrak{h} \right\} \subseteq GL(n,\mathbb{C})$$
together with matrix multiplication is connected and has the property that $Lie(H) = \mathfrak{h}$.

Proof:
The first two statements can be found in [5, Chapter 1, 1]. The next statement can be found in [5, (A.121), Appendix A]. The last two statements follow from Theorem 20.13 and Example 20.5 in [6]. □

Now we will start by showing that the group $GL(2,\mathbb{H})$ that we encountered in section 1.3 is a Lie group.

Lemma 2.1.7. *($GL(2,\mathbb{H})$ as a Lie group)*
The group $GL(2,\mathbb{H})$ can be endowed with the structure of an 8-dimensional Lie group and we have

$$Lie(GL(2,\mathbb{H})) = \mathfrak{gl}(2,\mathbb{H})$$

Proof:
First we consider the map

$$\begin{aligned} \det_{\mathbb{H}}: \ & M(2,\mathbb{H}) \longrightarrow \mathbb{H} \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{cases} A \cdot D & \text{for}\, C = 0 \\ CAC^{-1}D - CB & \text{for}\, C \neq 0 \end{cases} \end{aligned}$$

We see that this map is continuous and by lemma 1.3.7 in chapter 1 we have that for all $M \in M(2,\mathbb{H})$:

$$M \in GL(2,\mathbb{H}) \quad \Leftrightarrow \quad \det_{\mathbb{H}}(M) \in \mathbb{H} \setminus \{0\}.$$

Since $\mathbb{H}\setminus\{0\}$ is open in $\mathbb{H}$ we see that $GL(2,\mathbb{H})$ is open in $M(2,\mathbb{H})$ and is therefore endowed with the structure of a submanifold of class $\mathcal{C}^\infty$ by the identity chart. By recalling that $\mathbb{H}$ together with its multiplication is a subgroup of $GL(2,\mathbb{C})$ we see that $GL(2,\mathbb{H})$ is a subgroup of $GL(4,\mathbb{C})$. We can now use the isomorphy between the Lie algebra corresponding to a Lie group and the tangent space of the same Lie group to construct a Lie algebra isomorphism $\alpha: \ \mathfrak{gl}(2,\mathbb{H}) \to \mathfrak{gl}(4,\mathbb{C})$. Since this needs a better understanding of Lie algebras and Lie groups than given in this thesis until now, we refer to the proof of [6, Proposition 8.40]. One gets the explicit argument for our case by just replacing $\mathbb{C}$ by $\mathbb{H}$ and $\mathbb{R}$ by $\mathbb{C}$ in this proof. $\square$

For the purpose of this section the group $GL(2,\mathbb{H})$ is too big. In the next lemma we will treat a Lie subgroup of $GL(2,\mathbb{H})$ that preserves the unit sphere, when acting on $\hat{\mathbb{H}}$ via conformal transformations (cf. section 3 in chapter 1 of this thesis).

Lemma 2.1.8. *(Group of sphere-preserving matrices)*
Let $G \subseteq GL(2,\mathbb{H})$ be the maximal subgroup of $GL(2,\mathbb{H})$ such that for all $g \in G$ we have:

$$\pi_\ell(g).S_1^3 = S_1^3.$$

Then

$$Lie(G) = \left\{ \begin{pmatrix} A & B \\ B^+ & D \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \ | \ \Re(A) = \Re(D) \right\}.$$

Proof:
Let $M \in \mathfrak{gl}(2,\mathbb{H})$. In the proof for lemma 2.1.7 we showed that $GL(2,\mathbb{H})$ is open in $\mathfrak{gl}(2,\mathbb{H})$. This implies that there is an interval $(-\varepsilon,\varepsilon) \in \mathbb{R}$ such that the curve

$$\begin{aligned} \gamma_M: \ & (-\varepsilon,\varepsilon) \longrightarrow GL(2,\mathbb{H}) \\ & t \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \cdot M \end{aligned}$$

is well-defined. Using this map we can now define a curve in $\hat{\mathbb{H}}$ in the following way for $M \in \mathfrak{gl}(2, \mathbb{H})$ and $X \in \mathbb{H}$:

$$\begin{aligned} \omega_{M,X} : \ (-\varepsilon, \varepsilon) &\longrightarrow \hat{\mathbb{H}} \\ t &\longmapsto \pi_\ell(\gamma(t)).X \ . \end{aligned}$$

We see that $\omega_{M,X}(0) = X$. And thus by differentiating the curve at $t = 0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_{M,X}(t) \mid_{t=0} \in T_X \hat{\mathbb{H}}.$$

Now we note that by the definition of G we have $M \in Lie(G)$ if and only if it preserves the tangential space of the sphere, i.e.

$$\forall\, X \in S_1^3 : \ \frac{\mathrm{d}}{\mathrm{d}t}\omega_{M,X}(t) \mid_{t=0} \in T_X(S_1^3).$$

Evaluating this expression we get for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{gl}(2, \mathbb{H})$ and $X \in S_1^3$ by using that $\varphi_\ell = \varphi_r$ as maps on $GL(2, \mathbb{H})$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\, \omega_{M,X}(t) \mid_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, ((1 + tA) - tXC)^{-1} \cdot (-tB + X(1 + tD)) \mid_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, \frac{X + t \cdot (D - B + (A^+ - C^+X^+)X) + t^2 \cdot (A^+ - C^+X^+)(XD - B)}{N(1 + t(A - XC))} \mid_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\, N(1 + t(A - XC))^{-1} \mid_{t=0} \cdot X \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\, (X + t \cdot (D - B + (A^+ - C^+X^+)X) + t^2 \cdot (A^+ - C^+X^+)(XD - B)) \mid_{t=0} \\
&= -2\Re(A - XC) \cdot X + D - B + A^+X - C^+X^+X \\
&= (XC - A + C^+X^+ - A^+) \cdot X + D - B + A^+X - C^+X^+X \\
&= XCX - AX + XD - B.
\end{aligned}$$

From this equation we get for $X \in S_1^3$ the equivalence

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\, \omega_{M,X}(t) \mid_{t=0} \in T_x(S_1^3) \\ \Leftrightarrow\ &\langle XCX - AX + XD - B, X\rangle = 0. \end{aligned}$$

But for $X \in S_1^3$ we have

$$\begin{aligned}
&\langle XCX - AX + XD - B, X\rangle \\
&= \langle XCX, X\rangle - \langle AX, X\rangle + \langle XD, X\rangle - \langle B, X\rangle \\
&= \Re(XCXX^+) - \Re(AXX^+) + \Re(X^+XD) - \Re(BX^+) \\
&= \Re((D - A) + (BX^+ - XC)).
\end{aligned}$$

Here we used that for $X \in S^3_1$ the equality $N(X) = 1$ holds. By successively inserting $e_0, -e_0, e_1, e_2$ and e_3 for X into the above equation we can determine that

$$\langle XCX - AX + XD - B, X\rangle = 0 \;\Leftrightarrow\; \Re(A) = \Re(D) \wedge C = B^+.$$

This implies that

$$Lie(G) = \left\{ \begin{pmatrix} A & B \\ B^+ & D \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \mid \Re(A) = \Re(D) \right\}.$$

□

Now we are ready to define the Lie group that will be of highest significance in this section and then single out its generators.

Definition 2.1.9 (The Lie group G_0)
Let $G_0 \subseteq GL(2,\mathbb{H})$ be the unique connected Lie-subgroup of $GL(2,\mathbb{H})$ such that

$$\mathfrak{g}_0 := Lie(G_0) = \left\{ \begin{pmatrix} A & B \\ B^+ & D \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \mid \Re(A) = \Re(D) = 0 \right\}.$$

Lemma 2.1.10. *(Generators of G_0)*
The group G_0 is generated by the subgroup $SU(2) \times SU(2) \subset GL(2,\mathbb{H})$ realized as the group of diagonal matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in \mathfrak{gl}(2,\mathbb{H})$ with $|A| = |B| = 1$, and the one-parameter group

$$G_0' := \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \mid t \in \mathbb{R} \right\}.$$

Proof:
We show that the Lie algebra $\mathfrak{g}_0$ is generated by the lie algebras

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \mid \Re(A) = \Re(D) = 0 \right\}$$

and

$$\mathfrak{g}_0' := \left\{ \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) \mid t \in \mathbb{R} \right\}.$$

To see that consider an arbitrary $\left(\begin{smallmatrix} A & B \\ B^+ & D \end{smallmatrix}\right) \in \mathfrak{g}_0$. Then we see that $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\left(\begin{smallmatrix} \Im(B) & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\left(\begin{smallmatrix} 0 & \Re(B) \\ \Re(B) & 0 \end{smallmatrix}\right) \in \mathfrak{g}_0'$. Using these matrices we get:

$$\begin{aligned}
&\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \left[\begin{pmatrix} \Im(B) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + \begin{pmatrix} 0 & \Re(B) \\ \Re(B) & 0 \end{pmatrix} \\
&= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & \Im(B) \\ -\Im(B) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Re(B) \\ \Re(B) & 0 \end{pmatrix} \\
&= \begin{pmatrix} A & B \\ B^{+} & D \end{pmatrix}
\end{aligned}$$

Since $\mathfrak{g}_0'$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ are Lie sub-algebras of $\mathfrak{g}_0$, this already implies that $\mathfrak{g}_0$ is indeed generated by the Lie algebras $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{g}_0'$.
Next we see that $SU(2) \times SU(2)$ as given in the statement of the lemma is the connected Lie subgroup of $GL(2, \mathbb{H})$ with $Lie(SU(2) \times SU(2)) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, where $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is embedded into $\mathfrak{gl}(2, \mathbb{H})$ in the way given above. Furthermore we get the one-parameter subgroup generated by g_0' by applying the matrix exponential to it. This is possible since g_0' is a subalgebra of $\mathfrak{gl}(4, \mathbb{C})$ and we calculate:

$$\begin{aligned}
&\left\{ \exp\left(\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \right) \mid t \in \mathbb{R} \right\} \\
&= \left\{ \exp\left(\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \right) \mid t \in \mathbb{R} \right\} \\
&= \left\{ \sum_{n=0}^{\infty} t^n \cdot \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^n \mid t \in \mathbb{R} \right\} \\
&= \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mid t \in \mathbb{R} \right\} \qquad \mid (*) \\
&= G_0'
\end{aligned}$$

For the equality $(*)$ we used that $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the Taylor series expansion of cosh and sinh around the point $0 \in \mathbb{R}$. Since G_0' is a one-parameter group, it is connected.
Therefore G_0 is the smallest connected subgroup of $GL(2, \mathbb{H})$ that contains the connected Lie subgroups $SU(2) \times SU(2)$ and G_0' and hence the lemma is shown. □

Corollary 2.1.11.
The subgroup $\pi_\ell(G_0)$ of the conformal group $\pi_\ell(GL(2, \mathbb{H}))$ preserves the unit sphere S_1^3 the unit ball $\mathring{B}_1^4$ and its complement $\hat{\mathbb{H}} \setminus \mathring{B}_1^4$.

Proof:
Since G_0 is by definition a subgroup of G, lemma 2.1.8 directly implies that $\pi_\ell(G_0)$ preserves S_1^3.
Now consider any $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in SU(2) \times SU(2)$. Then we get

$$\begin{aligned} \pi_\ell \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} .0 &= \pi_r \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} .0 \\ &= A \cdot 0 \cdot B = 0. \end{aligned}$$

Since $\pi_\ell\left(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)\right)$ is a homeomorphism and $0 \in \mathring{B}_1^4$ we have shown that $\pi_\ell(SU(2) \times SU(2))$ preserves $\mathring{B}_1^4$.
Analogously we get for every $t \in \mathbb{R}$:

$$\begin{aligned} \pi_\ell \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} .0 &= \pi_r \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} .0 \\ &= (\cosh(t) - \sinh(t) \cdot 0)^{-1} \cdot (-\sinh(t) + \cosh(t) \cdot 0) \\ &= -\frac{\sinh(t)}{\cosh(t)} \\ &= -\tanh(t). \end{aligned}$$

But we know that $|\tanh(t)| < 1$ for every $t \in \mathbb{R}$ and thus $(-\tanh(t)) \in \mathring{B}_1^4$ for all $t \in \mathbb{R}$. By the same argument as for $SU(2) \times SU(2)$ the one-parameter subgroup $\left\{\left(\begin{smallmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{smallmatrix}\right) \mid t \in \mathbb{R}\right\}$ preserves $\mathring{B}_1^4$.
To finish the proof we now invoke that by lemma 2.1.10 G_0 is generated by the two Lie groups just considered and that π_ℓ is a group homomorphism. $\square$

Next we will recall some basic notions of representation theory.

Definition 2.1.12 (Representation Theory)
Let V be an arbitrary complex vector space and H a group.

1. We denote by $\mathrm{Aut}(V)$ the space of linear automorphisms of V. This space is endowed with the structure of a group by compositon of maps.

2. A *representation of H on V* is a group homomorphism

$$\rho: \; H \to \mathrm{Aut}(V).$$

3. Let ρ be a representation of H on V. Then we call V *simple as representation of H via ρ* if and only if there is no linear subspace W of V such that

$$\forall\, h \in H: \; \forall\, w \in W: \; \rho(h).w \in W.$$

4. Let ρ be a representation of H on V and $L : V \to V$ a linear map. Then we call L equivariant with respect to ρ if and only if

$$\forall\, h \in H : \ \forall\, v \in V : \ \rho(h).L(v) = L(\rho(h).v).$$

Now we define an action of G_0 on the space $\hat{\mathcal{H}}$, defined in definition 2.1.2, that is induced by the action of G_0 on $\hat{\mathbb{H}}$ via conformal transformation. The well-definedness of the map we are about to define will be shown as part of the next two propositions.

Definition 2.1.13
For $\varphi \in \hat{\mathcal{H}}$ and $M \in G_0$ with $M^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G_0$ we define

$$\begin{aligned} \pi^0_\ell(g) :\ & \hat{\mathcal{H}} \longrightarrow \hat{\mathcal{H}} \\ & \varphi \longmapsto \left[X \mapsto \frac{1}{N(CX+D)} \cdot \varphi(\pi_\ell(g).X) \right] \end{aligned}$$

Lemma 2.1.14.
Let $g \in GL(2, \mathbb{H})$ with $g^{-1} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and with this notation

$$F : GL(2, \mathbb{H}) \times B^4_1 \to \mathbb{R}_{\geq 0}, \ \ (g, X) \mapsto N(CX + D).$$

Then we have for all $g, g' \in G$ and $X \in B^4_1$:

$$F(g \cdot g', X) = F(g, X) \cdot F(g', \pi_\ell(g).X).$$

Furthermore we have for all $g \in G_0$ and $X \in B^4_1$:

$$F(g, X) \neq 0.$$

Proof:
Let first $g, g' \in G_0$ with $g^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $(g')^{-1} = \left(\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right)$. Then we have

$$(g \cdot g')^{-1} = (g')^{-1} \cdot g^{-1} = \begin{pmatrix} A'A + B'C & A'B + B'D \\ C'A + D'C & C'B + D'D \end{pmatrix}$$

and thus we get for any $X \in B^4_1$:

$$F(g \cdot g', X) = N((C'A + D'C)X + (C'B + D'D)).$$

On the other hand we also get for any $X \in B_1^4$:

$$\begin{aligned} F(g,X)\cdot F(g',\pi_\ell(g).X) &= N(CX+D)\cdot N(C'\cdot \pi_\ell(g).X+D') \\ &= N(CX+D)\cdot N(C'(AX+B)(CX+D)^{-1}+D') \\ &= N(C'(AX+B)(CX+D)^{-1}+D')\cdot N(CX+D) \\ &= N(C'(AX+B)+D'(CX+D)) \\ &= N((C'A+D'C)X+(C'B+D'D)) \end{aligned}$$

This shows the first part of the lemma.
For the second part we note that by lemma 2.1.10 the group G_0 is generated by its subgroups G_0' and $SU(2)\times SU(2)$. Thus we can show the claim by induction over products of elements of G_0' and $SU(2)\times SU(2)$. First we note that for $e := \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) \in G_0$ and for any $X \in B_1^4$ we get:

$$F(e,X) = N(1) = 1.$$

Now we consider a $g \in G_0$ with $(g)^{-1} = \left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ such that $F(g,X) \neq 0$ for all $X \in B_1^4$, i.e.

$$\forall\, X \in B_4^1:\ CX+D \neq 0 \qquad\qquad (*).$$

Now we consider an arbitrary $g' \in SU(2)\times SU(2)$. Then there are $A', D' \in \mathbb{H}$ with $|A| = |B| = 1$ such that $g' = \begin{pmatrix}A' & 0\\ 0 & D'\end{pmatrix}$ and therefore $(g')^{-1} = \begin{pmatrix}(A')^{-1} & 0\\ 0 & (D')^{-1}\end{pmatrix}$.
Using this we get for any $X \in B_4^1$:

$$\begin{aligned} F(g\cdot g',X) &= N((D')^{-1}CX + D'^{-1}D) \\ &= \frac{N(CX+D)}{N(D)} \\ &= N(CX+D). \end{aligned}$$

Thus $F(g\cdot g',X) = 0$ if and only if $CX+D = 0$. This is not possible because of $(*)$ and therefore

$$\forall\, X \in B_1^4:\ f(g\cdot g',X) \neq 0.$$

Now let $g \in G_0$ be given as before and consider an arbitrary $g' \in G_0'$. Then there is a $t \in \mathbb{R}$ such that $g' = \left(\begin{smallmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{smallmatrix}\right)$ and therefore $(g')^{-1} = \left(\begin{smallmatrix}\cosh(t) & -\sinh(t)\\ -\sinh(t) & \cosh(t)\end{smallmatrix}\right)$.
Now we get for any $X \in B_4^1$:

$$F(g\cdot g',X) = F(g,X)\cdot F(g',\pi_\ell(g).X) \qquad\qquad (**)$$

We know from $(*)$, that $F(g,X) \neq 0$ for all $X \in B_4^1$. Furthermore we know from lemma 2.1.8 that for $X \in B_4^1$ we also have $\pi_\ell(g).X \in B_4^1$. Now we calculate for arbitrary $X \in B_4^1$:

$$F(g',X) = N(-\sinh(t)\cdot X + \cosh(t)).$$

This term is zero if and only if $\sinh(t) \neq 0$ and $X = \frac{1}{\tanh(t)}$. But since $|\tanh(t)| < 1$ for any $t \in \mathbb{R}$ we have that $X = \frac{1}{\tanh(t)} \notin B_1^4$ for any $t \in \mathbb{R}$. This shows that

$$\forall\, X \in B_1^4 : \; F(g', X) \neq 0.$$

Inserting this into $(**)$ we finally get:

$$\forall\, X \in B_1^4 : \; F(g \cdot g', X) \neq 0.$$

Since every element of G_0 can be written as product of elements of $SU(2) \times SU(2)$ and G_0', we have thus shown by induction that

$$\forall\, g \in G_0 : \; \forall\, X \in B_1^4 : \; F(g, X) \neq 0.$$

□

Lemma 2.1.15. *($\hat{\mathcal{H}}$ as representation of G_0)*
The map

$$\begin{aligned} \pi_\ell^0 : \; G_0 &\longrightarrow \mathrm{Aut}(\hat{\mathcal{H}}) \\ g &\longmapsto \pi_\ell^0(g) \end{aligned}$$

is a representation of G_0 on the vector space $\hat{\mathcal{H}}$.

Proof:
We have to show that for every $g \in G_0$ we have $\pi_\ell^0(g) \in \mathrm{Aut}(\hat{\mathcal{H}})$ and that $\pi_\ell^0(g)$ is a group homomorphism. First we note that for $X \in B_0^4$ we have by lemma 2.1.8 that also $\pi_\ell(g).X \in B_0^4$, which implies that for every $f \in \mathrm{Map}(B_0^4, \mathbb{C})$ we also have that $f \circ \pi_\ell(g) \in \mathrm{Map}(B_0^4, \mathbb{C})$.
Now we show that for $\varphi \in \hat{\mathcal{H}}$ the map $\pi_\ell^0(g).\varphi$ does not have any singularity in B_0^4. With the notation of lemma 2.1.14 we have for any $X \in B_0^4$:

$$\pi_\ell^0(g).\varphi(X) = \frac{1}{F(g, X)} \cdot \varphi(\pi_\ell(g).X).$$

Since by lemma 2.1.14 for any $g \in G_0$ and $X \in B_1^4$ we have $F(g, X) \neq 0$, it is thus clear, that $\pi_\ell^0(g).\varphi$ is defined on B_1^4 for all $g \in G_0$ and all $\varphi \in \hat{\mathcal{H}}$. We see now that π_ℓ^0 is a well-defined map from G_0 to the space $\mathrm{End}(\hat{\mathcal{H}})$ of linear endomorphisms of $\hat{\mathcal{H}}$.
Now it remains to be shown that π_ℓ^0 is a group homomorphism. By invoking

lemma 2.1.14 we see that for any two $g, g' \in G_0$ and for $\varphi \in \hat{\mathcal{H}}$, $X \in B_1^4$:

$$\begin{aligned}
\pi_\ell^0(gg').\varphi(X) &= \frac{1}{F(gg', X)} \cdot \varphi(\pi_\ell(gg').X) \\
&= \frac{1}{F(g, X)} \cdot \frac{1}{F(g', \pi_\ell(g).X)} \cdot \varphi(\pi_\ell(g') \circ \pi_\ell(g).X) \\
&= \frac{1}{F(g', \pi_\ell(g).X)} \cdot \frac{1}{F(g, X)} \cdot \varphi(\pi_\ell(g') \circ \pi_\ell(g).X) \\
&= \pi_\ell^0(g').\left[\frac{1}{F(g, X)} \cdot \varphi(\pi_\ell(g).X)\right] \\
&= \pi_\ell^0(g') \circ \pi_\ell^0(g).\varphi(X).
\end{aligned}$$

This shows that π_ℓ^0 is a group homomorphism and therefore its image is contained in $\mathrm{Aut}(\hat{\mathcal{H}})$. This finishes the proof. □

The next result we want to obtain is that Harm is equivariant with respect to the action of G_0 on $\hat{\mathcal{H}}$ we just defined. The idea of the proof for this proposition given in the paper [3] is to check equivariance with respect to the actions of the Lie subgroups $SU(2) \times SU(2) \subseteq G_0$ and $G_0' \subseteq G_0$ as given in lemma 2.1.10. This suffices since by the same lemma these two groups generate G_0. The equivariance of Harm with respect to the $SU(2)\times SU(2)$-action on $\hat{\mathcal{H}}$ can be calculated directly, but for the proof of the equivariance of Harm with respect to the G_0'-action on $\hat{\mathcal{H}}$ we consider a corresponding action of the Lie algebra $\mathrm{Lie}(G_0)$ on $\hat{\mathcal{H}}$ and deduce from the equivariance of this Lie algebra action the equivariance of the original Lie group action. To make this last step precise we need methods from infinite-dimensional representation theory of Lie groups and Lie algebras, which I am not going to present here. For this reason we just assume the result without any proof.

Assumption 2.1.16. *(Equivariance of Harm)*
The map Harm is equivariant with respect to the π_ℓ^0-action of G_0 on $\hat{\mathcal{H}}$.

The proof of the Poisson formula via representation theory will rely on Schur's Lemma, which we will recall now:

Lemma 2.1.17. *(Schur)*
Let H be a group, V a complex vector space and

$$\rho : H \to \mathrm{Aut}(V)$$

a representation of H on V such that V is simple as representation of H. Then every linear map $L : V \to V$ that is equivariant with respect to ρ is of the form

$$\mu_z : V \to V, \; v \mapsto z \cdot v$$

for a $z \in \mathbb{C}$.

Proof:
The Proof of this lemma is given in [2, Theorem 1.10.(ii)]. □

The next result we are going to state is the simplicity of $\hat{\mathcal{H}}$ as a representation of the non-compact Lie group G_0. In [3] it this is simply stated as true and this statement is significant, since we are about to use Schur's Lemma to prove that Harm is just the identity map on $\hat{\mathcal{H}}$, but I was not able to prove the result in the following paragraph. For this reason we will simply assume it as true at this point in order to follow the rest of the argument. Of course we note that because of the lack of proof for the assumptions 2.1.16 and 2.1.18 our argument for the validity of theorem 2.1.4 in this thesis is not a complete proof.

Assumption 2.1.18. *(Simplicity of $\hat{\mathcal{H}}$)*
The space $\hat{\mathcal{H}}$, viewed as a representation of G_0 via π^0_ℓ is simple.

Now we are ready to follow the argument for a version of the Poisson formula on the unit ball.

Proposition 2.1.19. *(Poisson formula on S^1_3)*
The map Harm *is equal to the identity on $\hat{\mathcal{H}}$, i.e. for all $Y \in \mathring{B}^4_1$:*

$$\varphi(Y) = \frac{1}{2\pi^2} \cdot \int_{S^3_1} \frac{1 - N(Y)}{N(X-Y)^2} \cdot \varphi(X)\,\mathrm{d}S(X)$$

Proof:
As stated in assumption 2.1.18, we assume that $\hat{\mathcal{H}}$ is irreducible viewed as a representation of G_0 via π^0_ℓ. Furthermore we assume by assumption 2.1.16 that the linear map $\mathrm{Harm} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is equivariant with respect to π^0_ℓ. Since $\hat{\mathcal{H}}$ is a complex vector space we can apply lemma 2.1.17 and thus there exists a $z \in \mathbb{C}$ such that

$$\forall\, \varphi \in \hat{\mathcal{H}} : \ \mathrm{Harm}(\varphi) = z \cdot \varphi.$$

But we can see that $z = 1$, since we defined Harm in a way that for every $X \in S^3_1$ we have

$$\forall\, \varphi \in \hat{\mathcal{H}} : \ \mathrm{Harm}(\varphi)(X) = \varphi(X).$$

□

Now we are going to prove that the shifted degree operator is a injective endomorphism of $\hat{\mathcal{H}}$.

Lemma 2.1.20. *(Injectivity of the Degree operator)*
The map

$$\widetilde{\deg} : \hat{\mathcal{H}} \to \hat{\mathcal{H}},\ \varphi \mapsto \widetilde{\deg}\,\varphi$$

is well-defined, $\mathbb{C}$-linear and injective.

Proof:
We know from remark 2.1.3 that every element of $\hat{\mathcal{H}}$ is real-analytic. This implies that linear combinations of its partial derivatives are also real-analytic and thus continuous. Now we just have to check that for a $\varphi \in \hat{\mathcal{H}}$ the map $\widetilde{\deg}\,\varphi$ is harmonic on $\mathring{B}_1^4$. For this we calculate for $j \in \{0,1,2,3\}$ and $X = \sum\limits_{n=0}^{3} x_n e_n \in \mathring{B}_1^4$:

$$\begin{aligned}
\partial_j\partial_j\widetilde{\deg}\varphi(X) &= \partial_j\partial_j\varphi(X) + \partial_j\partial_j\left(\sum_{n=0}^{3} x_n\partial_n\varphi(X)\right)\\
&= \partial_j\partial_j\varphi(X) + \partial_j\partial_j(x_j\partial_j\varphi(X)) + \sum_{n\in\{0,1,2,3\}\setminus\{j\}} x_n\cdot\partial_j\partial_j\partial_n\varphi(X)\\
&= \partial_j\partial_j\varphi(X) + \partial_j(\partial_j\varphi(X) + x_j\partial_j\partial_j\varphi(X))\\
&\quad + \sum_{n\in\{0,1,2,3\}\setminus\{j\}} x_n\cdot\partial_j\partial_j\partial_n\varphi(X)\\
&= \partial_j\partial_j\varphi(X) + \partial_j\partial_j\varphi(X) + \partial_j\partial_j\varphi(X) + x_j\cdot\partial_j\partial_j\partial_j\varphi(X)\\
&\quad + \sum_{n\in\{0,1,2,3\}\setminus\{j\}} x_n\cdot\partial_j\partial_j\partial_n\varphi(X)\\
&= 3\cdot\partial_j\partial_j\varphi(X) + x_j\cdot\partial_j\partial_j\partial_j\varphi(X) + \sum_{n\in\{0,1,2,3\}\setminus\{j\}} x_n\cdot\partial_n\partial_j\partial_j\varphi(X)\\
&= 3\cdot\partial_j\partial_j\varphi(X) + \sum_{n=1}^{3} x_n\cdot\partial_n\partial_j\partial_j\varphi(X)
\end{aligned}$$

Therefore we get:

$$\Box\,\widetilde{\deg}\varphi(X) = 3\cdot\Box\varphi(X) + \sum_{n=0}^{3} x_n\cdot\partial_n\Box\varphi(X) = 0$$

Therefore $\widetilde{\deg}$ is well-defined. From the linearity of the involved differential operators we can also readily see that $\widetilde{\deg}$ is $\mathbb{C}$-linear. To show the injectivity of $\widetilde{\deg}$ we characterize its kernel. To do this we consider solutions $\varphi \in \hat{\mathcal{H}}$ of the partial differential equation

$$\begin{aligned}
&\forall\, X \in \mathring{B}_1^4 :\ \widetilde{\deg}\varphi(X) = 0\\
\Leftrightarrow\ &\forall\, X \in \mathring{B}_1^4 :\ \varphi(X) = -\sum_{n=0}^{3} x_n\partial_n\varphi(X). \qquad (*)
\end{aligned}$$

Now we want to give a general form for solutions of this partial differential equation. For this we first consider $x_j \in (-1,1) \subset \mathbb{R}$ and get by equation $(*)$ for fixed $j \in \{0,1,2,3\}$:

$$\forall\, x_j \in (-1,1):\ \varphi(x_j e_j) = -x_j \partial_j \varphi(x_j e_j).$$

We know that a simultaneous solution of these equations for all $j \in \{0,1,2,3\}$ is given by a function of the form

$$\varphi(X) = c_0 \cdot \exp(-x_0) + c_1 \cdot \exp(-x_1) + c_2 \cdot \exp(-x_2) + c_3 \cdot \exp(-x_3) + d$$

with $c_0, c_1, c_2, c_3, d \in \mathbb{C}$. Therefore every solution of the partial differential equation $(*)$ has to be of this form. But now we see that:

$$\begin{aligned} &\Box\, \varphi(x_0 e_0 + x_1 e_1) = 0 \\ \Leftrightarrow\ & c_0 = -c_1 \end{aligned}$$

Analogous arguments show that $c_0 = -c_1$ and $c_1 = -c_2$. But this implies that $c_0 = 0$. Proceeding in this way, we get that $c_0 = c_1 = c_2 = c_3 = 0$. Furthermore we see that the equality

$$\forall\, X \in \mathring{B}_1^4:\ \varphi(X) = -\sum_{n=0}^{3} x_n \partial_n \varphi(X)$$

already implies that $\varphi(0) = 0$ and thus we get that for all $\varphi \in \hat{\mathcal{H}}$:

$$\widetilde{\deg}\varphi = 0 \ \Leftrightarrow\ \varphi = 0$$

and this finishes the proof for injectivity of $\widetilde{\deg}$. $\Box$

Now we are going to complete the argument for theorem 2.1.4:

Proof of Theorem 2.1.4:
We proof the equalities in the theorem separately.
First equality:
For fixed $R \in \mathbb{R}_{>0}$ we define a function

$$\varphi':\ B_1^4 \to \mathbb{C},\ X \mapsto \varphi(R \cdot X).$$

By the chain rule we can deduce that this function is harmonic on $\mathring{B}_1^4$, because φ is harmonic on $\mathring{B}_R^4$. Furthermore φ is continuous as a composition of continuous maps. This implies that $\varphi' \in \hat{\mathcal{H}}$ and thus we can apply proposition 2.1.19 to get the following equality for all $Y' \in \mathring{B}_1^4$:

$$\begin{aligned}
\varphi(R \cdot Y') &= \varphi'(Y') \\
&= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1 - N(Y')}{N(X' - Y')^2} \cdot \varphi'(X') \, \mathrm{d}S(X') \\
&= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1 - N(Y')}{N(X' - Y')^2} \cdot \varphi(R \cdot X') \, \mathrm{d}S(X') \\
&= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1 - N(Y')}{N\left(\frac{X}{R} - Y'\right)^2} \cdot \frac{\varphi(X)}{R^3} \, \mathrm{d}S(X)
\end{aligned}$$

Here we used the transformation theorem to make the substitution $X' = \frac{X}{R}$. From this we can see the following equality for all $Y \in \mathring{B}_1^4$:

$$\begin{aligned}
\varphi(Y) &= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1 - N\left(\frac{Y}{R}\right)}{N\left(\frac{X}{R} - \frac{Y}{R}\right)^2} \cdot \frac{\varphi(X)}{R^3} \, \mathrm{d}S(X) \\
&= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{R^2 - N(Y)}{N(X - Y)^2} \cdot \frac{\varphi(X)}{R} \, \mathrm{d}S(X).
\end{aligned}$$

Second equality:
Let $Y = \sum\limits_{n=0}^{3} y_n e_n \in \mathring{B}_R^4$ and consider the function

$$\begin{aligned}
k_Y : \ B_R^4 \setminus \{Y\} &\longrightarrow \mathbb{R} \\
X &\longmapsto \frac{1}{N(X - Y)}.
\end{aligned}$$

Then we have for $X = \sum\limits_{n=0}^{3} x_n e_n \in B_R^4 \setminus \{Y\}$ and $j \in \{0, 1, 2, 3\}$:

$$\begin{aligned}
\partial_j k_Y(X) &= \partial_j \left(\left(\sum_{n=0}^{3} (x_n - y_n)^2 \right)^{-1} \right) \\
&= -1 \cdot \left(\sum_{n=0}^{3} (x_n - y_n)^2 \right)^{-2} \cdot 2(x_j - y_j) \\
&= -\frac{2(x_j - y_j)}{N(X - Y)^2}
\end{aligned}$$

And so we get:

$$\begin{aligned}\widetilde{\deg} k_Y(X) &= k_Y(X) + \sum_{j=0}^{3} x_j \cdot \partial_j k_Y(X) \\ &= \frac{1}{N(X-Y)} - \frac{2 \cdot N(X) - 2\langle X, Y\rangle}{N(X-Y)^2} \\ &= \frac{N(X-Y) - 2 \cdot N(X) + 2\langle X, Y\rangle}{N(X-Y)^2} \\ &= \frac{N(X) - 2\langle X, Y\rangle + N(Y) - 2 \cdot N(X) + 2\langle X, Y\rangle}{N(X-Y)^2} \\ &= -\frac{N(X) - N(Y)}{N(X-Y)^2}\end{aligned}$$

In particular for $X \in S_R^3$ we get:

$$\widetilde{\deg} k_Y(X) = -\frac{R^2 - N(Y)}{N(X-Y)^2}$$

This shows the second equality.

Third equality:

First we consider the linear operator $\mathrm{Harm}' : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ that is defined in the following way for $\varphi \in \hat{\mathcal{H}}$ and $Y \in \mathring{B}_1^4$:

$$\mathrm{Harm}'\varphi(Y) := \int_{S_1^3} \frac{1}{N(X-Y)} \cdot \varphi(X) \cdot \mathrm{d}S$$

On S_1^3 we define $\mathrm{Harm}'\varphi$ by its continuous extension. By differentiation under the integral we can show that Harm' is well-defined. Now we see that for all $X, Y \in \mathring{B}_0^4$ with $X \neq Y$ we have

$$\widetilde{\deg} k_Y(X) = -\widetilde{\deg} k_X(Y).$$

And with this equality and proposition 2.1.19 we get for any $Y \in \mathring{B}_1^4$:

$$\begin{aligned}\varphi(Y) &= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1 - N(Y)}{N(X-Y)^2} \cdot \varphi(X)\, \mathrm{d}S(X) \\ &= -\frac{1}{2\pi^2} \cdot \int_{S_1^3} \widetilde{\deg}\, k_Y(X) \cdot \varphi(X)\, \mathrm{d}S(X) \\ &= \frac{1}{2\pi^2} \cdot \int_{S_1^3} \widetilde{\deg}\, k_X(Y) \cdot \varphi(X)\, \mathrm{d}S(X) \\ &= \widetilde{\deg}_Y \left(\frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1}{N(X-Y)} \cdot \varphi(X)\, \mathrm{d}S(X) \right).\end{aligned}$$

This implies the equality

$$id_{\hat{\mathcal{H}}} = \mathrm{Harm} = \widetilde{\deg} \circ \mathrm{Harm}'.$$

But from lemma 2.1.20 we know that $\widetilde{\deg} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$ is injective and therefore we see now that it is bijective and its two-sided inverse is given by Harm′. Thus we have

$$\mathrm{Harm}' \circ \widetilde{\deg} = \widetilde{\deg} \circ \mathrm{Harm}'.$$

From which we get for any $Y \in \mathring{B}_1^4$:

$$\begin{aligned}
&\widetilde{\deg}_Y \left(\frac{1}{2\pi^2} \cdot \int_{S_1^3} \frac{1}{N(X-Y)} \cdot \varphi(X)\,\mathrm{d}S(X) \right) \\
&= -\frac{1}{2\pi^2} \cdot \int_{S_1^3} \widetilde{\deg}\, k_Y(X) \cdot \varphi(X)\,\mathrm{d}S(X) \\
&= \frac{1}{2\pi^2} \cdot \int_{S_1^3} k_Y(X) \cdot \widetilde{\deg}\,\varphi(X)\,\mathrm{d}S(X)
\end{aligned}$$

This nearly proves the last equality except for the fact that we have only treated the special case for $\varphi \in \hat{\mathcal{H}}$ here. By passing to functions $\varphi' \in \hat{\mathcal{H}}(B_R^4)$ as defined in the proof for the first equality, we can infer the last equality in the generality in which it is stated in theorem 2.1.4. □

As already mentioned, we did not prove assumptions 2.1.16 and 2.1.18 which are essential for the proof of proposition 2.1.19. But since we will need theorem 2.1.4, which we deduced as a consequence of proposition 2.1.19, in the next section, we will give a reference for its validity:

Remark 2.1.21. *(Reference for the Poisson formula)*
The Poisson formula in proposition 2.1.19 follows from Corollary II.1.11. in combination with Corollary II.1.4 in the book [7].

2.2 The discrete spectrum of the Hydrogen Atom

In this section we follow section 2.9 of [3], in which the authors calculate the discrete spectrum of the Hydrogen atom using the quaternionic Poisson formula in order to give a physical application of quaternionic analysis. Their calculation contains a little mistake, which we will repair in this section.
But first we will discuss the physical concepts behind the result and how we can formalize them mathematically.

The Hydrogen atom is a quantum mechanical analogue of the classical Coloumb system. Our main goal in this chapter is to calculate possible energy levels that an electron in the electromagnetic potential of the Hydrogen nucleus, which we model as a point charge, can occupy. The energy of a system is governed by its Hamiltonian operator. In order to be able to analytically calculate the energy levels for the Hydrogen atom, we choose to represent the quantum mechanical states on a space of square-integrable functions and consider a linear operator of the following form:

Definition 2.2.1
We consider $\mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \cap \mathcal{L}_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$ as subspace of $\mathcal{L}_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$ and define for arbitrary $\kappa \in \mathbb{R}_{>0}$ the linear operator

$$\begin{aligned}\mathcal{H}_\kappa :\ \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \cap \mathcal{L}_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \cap \mathcal{L}_2(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \\ \psi &\longmapsto \mathcal{H}_\kappa.\psi,\end{aligned}$$

which is for any $\xi \in \mathbb{R}^3 \setminus \{0\}$ defined by

$$\mathcal{H}_\kappa.\psi(\xi) = -\frac{1}{2}\Delta\psi(\xi) + \frac{\kappa}{|\xi|}\psi(\xi),$$

where Δ denotes the Laplace operator in three dimensions.

This is indeed the Hamiltonian for the Hydrogen atom, when we set the mass of the electron to 1 and choose the fine-structure constant for κ. The discrete spectrum of the Hydrogen atom is given by the discrete spectrum of $\mathcal{H}_\kappa$ in a functional analytic sense, i.e. by its eigenvalues. The main theorem in this section gives a characterization of some eigenvalues of this operator:

Theorem 2.2.2. *(Discrete spectrum of the hydrogen atom)*
Let $\kappa \in \mathbb{R}_{>0}$ and let $\sigma_d(\mathcal{H}_\kappa)$ denote the set of eigenvalues of H_κ. Then we have that for all $k \in \mathbb{N}$

$$\frac{\kappa^2}{2(k+1)^2} \in \sigma_d(\mathcal{H}_\kappa).$$

In fact it is true that $\sigma_d(\mathcal{H}_\kappa) = \left\{ \frac{\kappa^2}{2(k+1)^2} \mid k \in \mathbb{N} \right\}$ for all $\kappa \in \mathbb{R}_{>0}$ as is shown for example in [8, Section 5.1], but we will only show the weaker statement of theorem 2.2.2. The idea of the proof will be to use tools of quaternionic analysis to explicitly calculate eigenfunctions for $\mathcal{H}_\kappa$. The main tool will be the Poisson formula we encountered in section 2.1.

To be able to follow the argument given in [3] for the validity of theorem 2.2.2, we have to prove some apparently unrelated lemmas. We begin with the definition of a class of conformal transformations, the Cayley transformations, which interrelate the unit sphere in $\mathbb{H}$ and the space of quaternions with a vanishing real part. Before we define the Cayley transformations itself, we first define geometric structures that will play a significant role in the following.

Definition 2.2.3 (Upper half-spaces)
Let $t \in \mathbb{R}_{\geq 0}$ then we define

$$H_t^4 := \{X \in \mathbb{H} \mid \Re(X) \geq t\}$$

Remarks 2.2.4. *1. From the definition above it is evident that for $t \in \mathbb{R}_{\geq 0}$:*

$$\partial H_t^4 = \{X \in \mathbb{H} \mid \Re(X) = t\}$$

and

$$\mathring{H}_t^4 = \{X \in \mathbb{H} \mid \Re(X) > t\}.$$

2. Consider the map

$$\lambda : \partial H_t^4 \to \mathbb{R}^3, \ (te_0 + x_1e_1 + x_2e_2 + x_3e_3) \mapsto \left(\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right).$$

Then the set $\{\lambda\}$ is an atlas that endows ∂H_t^4 with the structure of a manifold of class $\mathcal{C}^\infty$. We will refer to λ as the standard chart for ∂H_t^4. Furthermore we see that ∂H_t^4 is a hyper-plane.

Now we are ready do define the Cayley transformation and to state and prove the geometric properties we are going to use.

Definition 2.2.5 (Cayley transformations)
Let $\rho \in \mathbb{R}_{>0}$ then we call the map

$$C_\rho := \pi_\ell \begin{pmatrix} \rho & \rho \\ -1 & 1 \end{pmatrix} : \hat{\mathbb{H}} \to \hat{\mathbb{H}}$$

the *Cayley transform with coefficient ρ.*

Remark 2.2.6. *(Presentation of the Cayley transformations)*
Let $\rho \in \mathbb{R}_{>0}$. Then we have

$$\begin{pmatrix} \rho & \rho \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2\rho} \cdot \begin{pmatrix} 1 & -\rho \\ 1 & \rho \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho} & -\frac{1}{2} \\ \frac{1}{2\rho} & \frac{1}{2} \end{pmatrix}.$$

and thus we get for every $X \in \mathbb{H}$:

$$C_\rho(X) = \pi_\ell \begin{pmatrix} \rho & \rho \\ -1 & 1 \end{pmatrix}.X = \left(\frac{1}{2\rho}X - \frac{1}{2}\right) \cdot \left(\frac{1}{2\rho}X + \frac{1}{2}\right)^{-1} = (X - \rho) \cdot (X + \rho)^{-1}.$$

On the other hand because $\pi_\ell \left(\begin{smallmatrix} \rho & \rho \\ -1 & 1 \end{smallmatrix}\right)$ and $\pi_r \left(\begin{smallmatrix} \rho & \rho \\ -1 & 1 \end{smallmatrix}\right)$ agree on $\mathbb{H}$ we also have

$$C_\rho(X) = \pi_r \begin{pmatrix} \rho & \rho \\ -1 & 1 \end{pmatrix}.X = (\rho + X)^{-1}(-\rho + X) = (X + \rho)^{-1} \cdot (X - \rho)$$

Thus we get the following equality for all $X \in \mathbb{H}$:

$$(X - \rho) \cdot (X + \rho)^{-1} = (X + \rho)^{-1} \cdot (X - \rho).$$

This non-trivial result justifies the notation

$$C_\rho(X) = \frac{X-\rho}{X+\rho}$$

for $X \in \mathbb{H}$ *that is used in section 2.9 of [3].*

Lemma 2.2.7. *(Geometric property of the Cayley transformations)*
For every $\rho \in \mathbb{R}_{>0}$ *we have*

$$C_\rho(\partial H_0^4 \cup \{\infty\}) = S_1^3$$

and

$$C_\rho(H_0^4 \cup \{\infty\}) = B_1^4.$$

Proof:
From proposition 1.3.24 we know that the set $C_\rho(\partial H_0^4 \cup \{\infty\})$ is either a sphere or a hyper-surface in $\hat{\mathbb{H}}$. Furthermore we saw in corollary 1.3.19 in chapter 1 that a sphere or hyper-surface in $\mathbb{H}$ is determined by six points. Thus we consider now the images of elements of $\partial H_0^4 \cup \{\infty\}$ under C_ρ. For $i \in \{1,2,3\}$ we have

$$\begin{aligned} N(C_\rho(e_i)) &= N\left[\frac{(e_i-\rho)(-e_i+\rho)}{N(e_i+\rho)}\right] = \frac{N((1-\rho^2)+2\rho e_1)}{N(e_i+\rho)^2} \\ &= \frac{1-2\rho^2+\rho^4+4\rho^2}{1+2\rho^2+\rho^4} = 1 \end{aligned}$$

and

$$\begin{aligned} N(C_\rho(-e_i)) &= N\left[\frac{(-e_i-\rho)(e_i+\rho)}{N(-e_i+\rho)}\right] = \frac{N((1-\rho^2)-2\rho e_1)}{N(-e_i+\rho)^2} \\ &= \frac{1-2\rho^2+\rho^4+4\rho^2}{1+2\rho^2+\rho^4} = 1. \end{aligned}$$

This proves that $C_\rho(\partial H_0^4 \cup \{\infty\}) = S_1^3$, since $S_1^3 = \{X \in \mathbb{H} \mid N(X) = 1\}$. To prove the second part of the lemma we first infer that because of the fact that C_ρ is a homeomorphism from $\hat{\mathbb{H}}$ onto $\hat{\mathbb{H}}$ we see that one of the following cases must be true:

$$C_\rho(H_0^4 \cup \{\infty\}) = B_1^4 \quad \text{or} \quad C_\rho(H_0^4 \cup \{\infty\}) = \hat{\mathbb{H}} \setminus \mathring{B}_1^4.$$

Now we note that for the element $\rho \in H_0^4$ we have

$$C_\rho(\rho) = 0 \in B_1^4.$$

Thus the first alternative is true and the lemma is shown. □

Next we will prove a proposition about the integrability of a class of composed functions we are going to use:

Lemma 2.2.8.
Let $\varphi \in \mathcal{C}^0(B_1^4, \mathbb{C})$, $c \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}_{>0}$. Then the function

$$\begin{aligned}\psi :\ H_0^4 &\longrightarrow \mathbb{C}\\ Z &\longmapsto \frac{1}{(N(Z)+c)^2} \cdot \varphi \circ C_\rho(Z)\end{aligned}$$

is absolutely integrable and square-integrable on H_0^4.

Proof:
By lemma 2.2.7 we have

$$C_\rho(H_0^4) \subset C_\rho(H_0^4 \cup \{\infty\}) = B_1^4.$$

Furthermore we know that B_1^4 is compact in $\mathbb{H}$ and because of the continuity of φ therefore there exists an $M \in \mathbb{R}_{\geq 0}$ such that $|\varphi(X)| \leq M$ for every $X \in B_1^4$. In particular this implies

$$\forall\, Z \in H_0^4 :\ |\varphi \circ C_\rho(Z)| \leq M.$$

This in turn implies for all $Z \in H_0^4$:

$$|\psi(Z)| = \left| \frac{1}{(N(Z)+c)^2} \cdot \varphi \circ C_\rho(Z) \right| \leq \frac{M}{(N(Z)+c)^2}$$

and

$$\left|\psi(Z)^2\right| = |\psi(Z)|^2 \leq \frac{M}{(N(Z)+c)^4}.$$

Now we note that the functions

$$\begin{aligned}\chi_1 :\ H_0^4 \to \mathbb{R}_{>0},\ (N(Z)+c)^{-2}\\ \chi_2 :\ H_0^4 \to \mathbb{R}_{>0},\ (N(Z)+c)^{-4}\end{aligned}$$

are integrable on H_0^4 and thus we have found majorizing integrable functions for ψ and ψ^2, which finishes the proof. □

To continue the argument for 2.2.2 we have to collect some results about the Fourier transformation and the Fourier cotransformation. First we will define the following two notions:

Definition 2.2.9 (Fourier)

1. Let $\psi \in \mathcal{L}_1(\mathbb{R}^n, \mathbb{C})$. Then we call the complex-valued function

$$\begin{aligned} \mathcal{F}.\psi :\ \mathbb{R}^n &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \frac{1}{(2\pi)^n} \cdot \int_{\mathbb{R}^n} \psi(x) \cdot e^{-\mathrm{i}\langle \xi, x\rangle}\, \mathrm{d}x \end{aligned}$$

the *Fourier transform of* ψ.
Furthermore we call the complex-valued function

$$\begin{aligned} \widetilde{\mathcal{F}}.\psi :\ \mathbb{R}^n &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \int_{\mathbb{R}^n} \psi(x) \cdot e^{\mathrm{i}\langle \xi, x\rangle}\, \mathrm{d}x \end{aligned}$$

the *Fourier cotransform of* ψ.

2. We call the map

$$\mathcal{F} :\ \mathcal{L}_1(\mathbb{R}^n, \mathbb{C}) \to \mathrm{Map}(\mathbb{R}^n, \mathbb{C}),\ \psi \mapsto \mathcal{F}.\psi$$

the *Fourier transformation.*
Furthermore we call the map

$$\widetilde{\mathcal{F}} :\ \mathcal{L}_1(\mathbb{R}^n, \mathbb{C}) \to \mathrm{Map}(\mathbb{R}^n, \mathbb{C}),\ \psi \mapsto \widetilde{\mathcal{F}}.\psi$$

the *Fourier cotransformation.*

Proposition 2.2.10. *(Properties of Fourier transformation and cotransformation)*

1. *Fourier transformation and Fourier cotransformation are linear operators.*

2. *There is a unique unitary extension of* $\mathcal{F}\mid_{\mathcal{L}_1(\mathbb{R}^n,\mathbb{C})\cap\mathcal{L}_2(\mathbb{R}^n,\mathbb{C})}$ *to* $\mathcal{L}_2(\mathbb{R}^n, \mathbb{C})$. *We will also denote it by* $\mathcal{F}$.
Furthermore there is a unique unitary extension of $\widetilde{\mathcal{F}}\mid_{\mathcal{L}_1(\mathbb{R}^n,\mathbb{C})\cap\mathcal{L}_2(\mathbb{R}^n,\mathbb{C})}$ *to* $\mathcal{L}_2(\mathbb{R}^n, \mathbb{C})$. *We will also denote it by* $\widetilde{\mathcal{F}}$.

3. *For* $\psi \in \mathcal{L}_2(\mathbb{R}^n, \mathbb{C})$ *we have for all* $\xi \in \mathbb{R}^n$*:*

$$\begin{aligned} \mathcal{F}.\psi(\xi) &= \lim_{R\to\infty} \frac{1}{(2\pi)^n} \int_{B_R^n} \psi(x) \cdot e^{\mathrm{i}\langle \xi, x\rangle}\, \mathrm{d}x, \\ \widetilde{\mathcal{F}}.\psi(\xi) &= \lim_{R\to\infty} \int_{B_R^n} \psi(x) \cdot e^{-\mathrm{i}\langle \xi, x\rangle}\, \mathrm{d}x. \end{aligned}$$

Furthermore for all $\psi \in \mathcal{L}_2(\mathbb{R}^n, \mathbb{C})$ *we have*

$$\widetilde{\mathcal{F}}.\mathcal{F}.\psi = \psi = \mathcal{F}.\widetilde{\mathcal{F}}.\psi$$

4. For $\psi \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{L}_2(\mathbb{R}^n, \mathbb{C})$ we have for every $\xi \in \mathbb{R}^n$:

$$\mathcal{F}.\left(\sum_{j=0}^{n} \partial_j \partial_j \psi\right)(\xi) = -\sum_{j=0}^{n} \xi_j^2 \cdot \mathcal{F}.\psi(\xi),$$

$$\widetilde{\mathcal{F}}.\left(\sum_{j=0}^{n} \partial_j \partial_j \psi\right)(\xi) = -\sum_{j=0}^{n} \xi_j^2 \cdot \widetilde{\mathcal{F}}.\psi(\xi).$$

Proof:

1. This follows directly from the definition of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ as integral operators.
2. This is Plancherel's Theorem as it is stated in [1, Theorem X.9.23].
3. The first part is stated in [1, Satz X.9.24] and the second part follows from [9, Satz V.2.8] and the density of the Schwartz space in $\mathcal{L}_2(\mathbb{R})$.
4. The first part is a special case of [9, Lemma V.2.11] and the second part is proven analogously.

□

In the following we will prove a lemma about the relationship between Fourier transformations and Fourier cotransformations that are defined for functions on hyper-surfaces.

Lemma 2.2.11. *(Inverse Fourier transformation)*
Let $\psi \in \mathcal{L}_1(H_0^4, \mathbb{C}) \cap \mathcal{L}_2(H_0^4, \mathbb{C})$ and let for $t \in \mathbb{R}_{\geq 0}$:

$$\begin{aligned} \hat{\psi}_t : \mathbb{H} &\longrightarrow \mathbb{C} \\ \xi &\longmapsto \frac{1}{(2\pi)^3} \cdot \int_{\partial H_t^4} \psi(Z) \cdot e^{-\mathrm{i}\langle \xi, Z\rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z). \end{aligned}$$

Then we have for all $W \in H_0^4$ the equality:

$$\psi(W) = \lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi).$$

Proof:
First we note that for every $W = W_0 e_0 + W_1 e_1 + W_2 e_2 + W_3 e_3 \in H_0^4$ we have

$$\begin{aligned} &\lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi) \\ = &\lim_{R\to\infty} \int_{B_R^3} \hat{\psi}_{\Re(W)}\left(\begin{pmatrix} 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\right) \cdot e^{\mathrm{i}(\xi_1 W_1 + \xi_2 W_2 + \xi_3 W_3)} \mathrm{d}(\xi_1, \xi_2, \xi_3) \end{aligned}$$

Furthermore we have for every $\xi = \begin{pmatrix} 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{H}_0^4$:

$$\begin{aligned}
\hat{\psi}_{\Re(W)}\left(\begin{pmatrix} 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\right) &= \frac{1}{(2\pi)^3} \cdot \int_{\partial H^4_{\Re(W)}} \psi(Z) \cdot e^{-\mathrm{i}\langle \xi, Z\rangle}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z) \\
&= \frac{1}{(2\pi)^3} \cdot \int_{\mathbb{R}^3} \psi\left(\begin{pmatrix} \Re(W) \\ Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}\right) \cdot e^{-\mathrm{i}(\xi_1 Z_1 + \xi_2 Z_2 + \xi_3 Z_3)}\, \mathrm{d}(Z_1, Z_2, Z_3)
\end{aligned}$$

Let $\lambda^{-1} : \mathbb{R}^3 \to \partial H_{\Re(W)}$ denote the inverse of the standard chart as defined in remark 2.2.4. Then we get form the equalities above:

$$\begin{aligned}
&\lim_{R\to\infty} \int_{B^4_R \cap \partial H^4_0} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi) \\
&= \lim_{R\to\infty} \int_{B^3_R} \widetilde{\mathcal{F}}.\psi \circ \lambda^{-1}\left(\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}\right) \cdot e^{\mathrm{i}(\xi_1 W_1 + \xi_2 W_2 + \xi_3 W_3)} \mathrm{d}(\xi_1, \xi_2, \xi_3) \\
&= \mathcal{F}.\widetilde{\mathcal{F}}.\psi \circ \lambda^{-1}\left(\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}\right) \\
&= \mathcal{F}.\widetilde{\mathcal{F}}.\psi\left(\Re(W) e_0 + W_1 e_2 + W_2 e_2 + W_3 e_3\right) \\
&= \psi(W)
\end{aligned}$$

Here we used the forms of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ for $\mathbb{R}^3$ as given in proposition 2.2.10. In the last step we used the fact that the Fourier transformation is inverse to the Fourier cotransformation for an element of $\mathcal{L}_1(\mathbb{H}, \mathbb{C}) \cap \mathcal{L}_2(\mathbb{H}, \mathbb{C})$. □

Before we can start the actual proof of theorem 2.2.2 we need to show one last result which relies on two definitions we will give now:

Definition 2.2.12 (Homogeneous functions)
For $n \in \mathbb{N}$ we call a function $\varphi : \mathbb{H} \to \mathbb{C}$ *homogeneous of degree* n if and only if for every $r \in \mathbb{R}$ and $X \in \mathbb{H}$ we have

$$\varphi(r \cdot X) = r^n \cdot \varphi(X).$$

Lemma 2.2.13. *(Existence and Properties of homogeneous functions)*

1. *For every $n \in \mathbb{N}$ there exist a function $\varphi \in \mathcal{C}^2(\mathbb{H}, \mathbb{C})$ that is harmonic and homogeneous of degree n.*

2. *For a function $\varphi \in \mathcal{C}^1(\mathbb{H}, \mathbb{C})$ that is homogeneous of degree n we have*

$$\widetilde{\deg}\, \varphi = (n+1)\varphi$$

Proof:

1. Consider the function

$$\begin{aligned} \varphi :\ \mathbb{H} &\longrightarrow \mathbb{C} \\ \sum_{j=0}^{n} x_j e_j &\longmapsto \Re((x_0 + x_1 \cdot \mathrm{i})^n). \end{aligned}$$

   Now we use that we know from complex analysis that the real part of every holomorphic function is harmonic in 2 dimensions and that in particular complex polynomials are holomorphic. Since for the function φ the second differentials $\partial_2\partial_2\varphi$ and $\partial_3\partial_3\varphi$ vanish this proves that the function is also harmonic as a function on $\mathbb{H}$. Furthermore we see directly from the definition that φ is harmonic of degree n.

2. By the definition of homogeneous functions we have for all $X \in \mathbb{H}$ and $c \in \mathbb{R}$:

$$\varphi(c \cdot X) = c^n \cdot \varphi(X). \qquad (*)$$

   Differentiating the right hand side of $(*)$ by c we get

$$\frac{\mathrm{d}}{\mathrm{d}c} \left(c^n \cdot \varphi(X)\right) |_{c=1} = n \cdot \varphi(X).$$

   Differentiating the left hand side of $(*)$ by c we get by considering φ as a complex-valued map on $\mathbb{R}^4$:

$$\frac{\mathrm{d}}{\mathrm{d}c} \varphi(c \cdot X) |_{c=1} = \sum_{j=0}^{3} x_j \cdot \partial_j \varphi(X).$$

   Thus by differentiating both sides of $(*)$ we get the equality

$$\sum_{j=0}^{3} x_j \cdot \partial_j \varphi(X) = n \cdot \varphi(X).$$

   Using this we now calculate:

$$\begin{aligned} \widetilde{\deg}\, \varphi(X) &= \varphi(X) + \sum_{j=0}^{3} x_j \cdot \partial_j \varphi(X) \\ &= \varphi(X) + n \cdot \varphi(X) \\ &= (n+1) \cdot \varphi(X). \end{aligned}$$

□

Now we give three last calculations regarding quaternions and functions on the quaternions. The first lemma we are going to present now looks slightly different from the result presented in [3, Lemma 40]. The reason for this is that the authors of [3] made a minor mistake in their statement, which we will present in the correct form here.

Lemma 2.2.14. *(Cayley transformation and squared norm)*
Let $Z \in \partial H_0^4$ and $W \in \mathring{H}_0^4$. Then the following equality holds for every $\rho \in \mathbb{R}_{>0}$:

$$N(C_\rho(Z) - C_\rho(W)) = \frac{4\rho^2 \cdot N(Z-W)}{(N(Z)+\rho^2)(N(W)+2\rho\cdot\Re(W)+\rho^2)}.$$

Proof:
We calculate (for the order of multiplication compare remark 2.2.6):

$$\begin{aligned}
N(C_\rho(Z) - C_\rho(W)) &= N\left((Z+\rho)^{-1}\cdot(Z-\rho) - (W-\rho)(W+\rho)^{-1}\right) \\
&= N((Z+\rho)^{-1}\cdot((Z-\rho)(W+\rho) \\
&\qquad\qquad - (Z+\rho)(W-\rho))\cdot(W+\rho)^{-1}) \\
&= \frac{N(ZW - \rho W + \rho Z - \rho^2 - ZW - \rho W + \rho Z + \rho^2)}{N(Z+\rho)N(W+\rho)} \\
&= \frac{N(2\rho\cdot(Z-W))}{N(Z+\rho)N(W+\rho)} \\
&= \frac{4\rho^2 N(Z-W)}{N(Z+\rho)N(W+\rho)}
\end{aligned}$$

Now we note that

$$\begin{aligned}
N(W+\rho) &= (W+\rho)(W+\rho)^+ = (W+\rho)(W^+ + \rho) \\
&= N(W) + \rho(W+W^+) + \rho^2 = N(W) + 2\rho\Re(W) + \rho^2.
\end{aligned}$$

Furthermore we get by considering the fact that $\Re(Z) = 0$:

$$N(Z+\rho) = N(Z) + \rho^2$$

□

Lemma 2.2.15. *(Pull-back of the surface form of S_1^3 along Cayley transformations)*
Let $\rho \in \mathbb{R}_{>0}$ and let $\mathrm{d}S$ denote the surface form of $S_1^3 \subset \mathbb{H}$ with outward orientation. Then we have for $Z \in H_0^4$:

$$(C_\rho)^*(\mathrm{d}S)(Z) = -\frac{8\rho^3}{(N(Z)+\rho^2)^3}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \,|_{\partial H_0^4}\,(Z)$$

Proof:
We have to explicitly calculate the pull-back of the surface form $\mathrm{d}S$ of S_1^3 considered as a submanifold of $\mathbb{H} = \mathbb{R}^4$ with outward orientation along C_ρ. This surface form is given as

$$\begin{aligned}\mathrm{d}S = (\ & x_0 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ & - x_1 \mathrm{d}x_0 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ & + x_2 \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 \\ & - x_3 \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2) \,|_{S_1^3} \,.\end{aligned}$$

Since this calculation is very lengthy, we will give a Maple 13 script in appendix A, with which one can calculate the pull-back. □

Remark 2.2.16.
We have to note that the authors of [3] only get the same result up to a minus sign for the pull-back we just calculated. This again leads to a sign error in the eigenvalue equation for the discrete spectrum of the Hydrogen atom in this paper. Nevertheless an execution of the Maple script will affirm that the result we state in this thesis is correct.

Lemma 2.2.17.
The following equality is true for every $W := \sum_{i=0}^{3} w_i e_i \in \mathring{H}_0^4$ *and* $\xi := \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \in \partial H_0^4 \setminus \{0\}$*:*

$$\int_{\partial H_0^4} \frac{e^{\mathrm{i}\langle \xi, Z\rangle}}{N(Z-W)} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z) = 2\pi^2 \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \cdot \frac{e^{\mathrm{i}\langle \xi, \Re(W)\rangle}}{|\xi|}.$$

Proof:
First we note that the function

$$\chi :\ \partial H_4^0 \to \mathbb{C},\ Z \mapsto \frac{1}{N(Z-W)}$$

is integrable on H_4^0 as can be checked by power counting and thus the integral on the right hand side of the proposed equality is well-defined. Pulling the integral back along the standard chart for ∂H_4^0 we get

$$\int_{\partial H_0^4} \frac{e^{\mathrm{i}\langle \xi, Z\rangle}}{N(Z-W)} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z) = \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}(\xi_1 z_1 + \xi_2 z_2 + \xi_3 z_3)}}{\Re(W)^2 + \sum_{j=1}^{3} (z_j - w_j)^2} \mathrm{d}(z_1, z_2, z_3).$$

Next we see that the map

$$T:\,\mathbb{R}^3\to\mathbb{R}^3,\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}\mapsto\begin{pmatrix}z_1+w_1\\z_2+w_2\\z_3+w_3\end{pmatrix}$$

is a diffeomorphism of class $\mathcal{C}^\infty$ and thus by invoking the transformation theorem we get

$$\begin{aligned}&\int_{\partial H_0^4}\frac{e^{\mathrm{i}\langle\xi,Z\rangle}}{N(Z-W)}\,\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3(Z)\\&=\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}(\xi_1(z_1+w_1)+\xi_2(z_2+w_2)+\xi_3(z_3+w_3))}}{\Re(W)^2+\sum\limits_{j=1}^{3}z_j^2}\,\mathrm{d}(z_1,z_2,z_3)\\&=e^{\mathrm{i}\langle\xi,\Im(W)\rangle}\cdot\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}(\xi_1z_1+\xi_2z_2+\xi_3z_3)}}{\Re(W)^2+\sum\limits_{j=1}^{3}z_j^2}\,\mathrm{d}(z_1,z_2,z_3).\end{aligned}$$

We see that the last integral in the equality above is invariant under rotations of ξ in $\mathbb{R}^3$ and thus we get:

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}(\xi_1z_1+\xi_2z_2+\xi_3z_3)}}{\Re(W)^2+\sum\limits_{j=1}^{3}z_j^2}\,\mathrm{d}(z_1,z_2,z_3)\\&=\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}|\xi|\cdot z_1}}{\Re(W)^2+\sum\limits_{j=1}^{3}z_j^2}\,\mathrm{d}(z_1,z_2,z_3)\end{aligned}$$

Now we set $\alpha(z_2, z_3) := \sqrt{\Re(W)^2 + z_2^2 + z_3^2}$ as an abbreviation and calculate:

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{e^{\mathrm{i}|\xi|\cdot z_1}}{\Re(W)^2 + \sum\limits_{j=1}^{3} z_j^2} \,\mathrm{d}(z_1, z_2, z_3) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{\mathrm{i}|\xi|\cdot z_1}}{z_1^2 + \alpha(z_2, z_3)^2} \,\mathrm{d}z_1 \,\mathrm{d}z_2 \,\mathrm{d}z_3 && \mid (1) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\pi}{\alpha(z_2, z_3)} e^{-|\xi|\alpha(z_2,z_3)} \,\mathrm{d}z_2 \,\mathrm{d}z_3 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\pi}{\sqrt{\Re(W)^2 + z_2^2 + z_3^2}} e^{-|\xi|\cdot\sqrt{\Re(W)^2+z_2^2+z_3^2}} \,\mathrm{d}z_2 \,\mathrm{d}z_3 \\
&= \int_{(0,\infty)} \int_{(0,2\pi)} \frac{\pi}{\sqrt{\Re(W)^2 + r^2}} e^{-|\xi|\cdot\sqrt{\Re(W)^2+r^2}} \cdot r \,\mathrm{d}\phi \,\mathrm{d}r && \mid (2) \\
&= 2\pi^2 \int_{(0,\infty)} \frac{e^{-|\xi|\cdot\sqrt{\Re(W)^2+r^2}}}{\sqrt{\Re(W)^2 + r^2}} \cdot r \,\mathrm{d}r \\
&= 2\pi^2 \int_{(\Re(W),\infty)} e^{-|\xi|s} \,\mathrm{d}s && \mid (3) \\
&= 2\pi^2 \cdot \frac{e^{-|\xi|\cdot\Re(W)}}{|\xi|}
\end{aligned}
$$

For the equality (1) we used Fubini's theorem, in (2) we made a transformation into polar coordinates in $\mathbb{R}^2$ and in (3) we made the substitution $s := \sqrt{\Re(W)^2 + r^2}$. In summary we get:

$$
\begin{aligned}
&\int_{\partial H_0^4} \frac{e^{\mathrm{i}\langle\xi,Z\rangle}}{N(Z-W)} \,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z) \\
&= e^{\mathrm{i}\langle\xi,\Im(W)\rangle} \cdot \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}(\xi_1 z_1+\xi_2 z_2+\xi_3 z_3)}}{\Re(W)^2 + \sum\limits_{j=1}^{3} z_j^2} \,\mathrm{d}(z_1, z_2, z_3) \\
&= 2\pi^2 \cdot e^{\mathrm{i}\langle\xi,\Im(W)\rangle} \cdot \frac{e^{-|\xi|\cdot\Re(W)}}{|\xi|}
\end{aligned}
$$

□

Now we have collected all the results we need to follow the main proof of this section.

Proof of Theorem 2.2.2:
For $k \in \mathbb{N}$ we consider a function $\varphi \in \mathcal{C}^2(\mathbb{H}, \mathbb{C})$ that is harmonic and homogeneous of degree k. Its existence is ensured by lemma 2.2.13. Then also by lemma 2.2.13 we have the following equality for every $X \in \mathbb{H}$:

$$(\widetilde{\deg}\varphi)(X) = (k+1)\varphi(X).$$

By the Poisson formula from theorem 2.1.4 we therefore get for every $Y \in \mathring{B}_1^4$:

$$\varphi(Y) = \frac{k+1}{2\pi^2} \int_{S_1^3} \frac{\varphi(X)}{N(X-Y)} \, \mathrm{d}S(X).$$

Now we consider an arbitrary $\rho \in \mathbb{R}_{\geq 0}$ and by invoking the second part of lemma 2.2.7 we get for every $W \in \mathring{H}_1^4$:

$$\varphi \circ C_\rho(W) = \frac{k+1}{2\pi^2} \int_{S_1^3} \frac{\varphi(X)}{N(X - C_\rho(W))} \, \mathrm{d}S(X).$$

Now we use the first part of lemma 2.2.7, the fact that Cayley transformations are conformal transformations and therefore diffeomorphisms of class $\mathcal{C}^\infty$ and the fact that the set $C_\rho(\{\infty\}) = \{1\}$ is a null set in S_1^3 to invoke the transformation theorem. With the pull-back of $\mathrm{d}S$ we have calculated in lemma 2.2.15 we therefore get for every $W \in \mathring{H}_1^4$:

$$\varphi \circ C_\rho(W) = -\frac{k+1}{2\pi^2} \int_{\partial H_0^4} \frac{\varphi \circ C_\rho(Z)}{N(C_\rho(Z) - C_\rho(W))} \cdot \frac{8\rho^3}{(N(Z)+\rho^2)^3} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z)$$

Now we take into account lemma 2.2.14 in order to simplify this equation to

$$\begin{aligned} &\varphi \circ C_\rho(W) \\ &= -\frac{(k+1)\rho}{\pi^2} \int_{\partial H_0^4} \varphi \circ C_\rho(Z) \cdot \frac{N(W)+\rho^2}{(N(Z)+\rho^2)^2} \cdot \frac{1}{N(Z-W)} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z). \end{aligned} \tag{1}$$

Now we introduce the auxiliary function

$$\begin{aligned} \psi :\ & H_0^4 \longrightarrow \mathbb{C} \\ & Z \longmapsto \frac{1}{(N(Z) + 2\rho\Re(Z) + \rho^2)^2} \cdot \varphi \circ C_\rho(Z). \end{aligned}$$

Then we get for every $W \in \mathring{H}_0^4$ by inserting ψ into equation (1):

$$\begin{aligned} &(N(W) + 2\rho\Re(W) + \rho^2) \cdot \psi(W) \\ &= -\frac{(k+1)\rho}{\pi^2} \int_{\partial H_0^4} \frac{\psi(Z)}{N(Z-W)} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z). \end{aligned} \tag{2}$$

At this point we note that $\varphi \in \mathcal{C}^2(\mathbb{H}, \mathbb{C})$ is a continuous function and therefore by lemma 2.2.8 we can deduce that $\psi \in \mathcal{L}_1(H_0^4, \mathbb{C}) \cap \mathcal{L}_2(H_0^4, \mathbb{C})$. From lemma 2.2.11 we see that if we define functions $\hat{\psi}_t$ for $t \in \mathbb{R}_{\geq 0}$ as in this lemma we get for every $W \in H_0^4$:

$$\psi(W) = \lim_{R \to \infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W) \rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi).$$

We can now insert this into equation (2) and get for every $W \in \mathring{H}_0^4$:

$$(N(W) + 2\rho\Re(W) + \rho^2) \cdot \lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi)$$
$$= -\frac{(k+1)\rho}{\pi^2} \lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \left(\int_{\partial H_0^4} \frac{\hat{\psi}_0(\xi) \cdot e^{\mathrm{i}\langle \xi, Z\rangle}}{N(Z-W)} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(Z) \right) \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi)$$

Here we invoked Fubini's theorem to interchange the order of integration on the right hand side. Now we can use the formula in lemma 2.2.17 to get for every $W \in \mathring{H}_0^4$:

$$(N(W) + 2\rho\Re(W) + \rho^2) \cdot \lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi)$$
$$= -2(k+1)\rho \lim_{R\to\infty} \int_{B_R^4 \cap \partial H_0^4} \hat{\psi}_0(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \cdot \frac{e^{-|\xi|\cdot\Re(W)}}{|\xi|} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3(\xi).$$

Now we pull back the two integrals along the standard chart of ∂H_0^4 and identify elements of ∂H_0^4 with elements of $\mathbb{R}^3$, suppressing the standard chart in our notation. Then we get for every $W \in \mathring{H}_0^4$:

$$(N(W) + 2\rho\Re(W) + \rho^2) \cdot \lim_{R\to\infty} \int_{B_R^3 \setminus \{0\}} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}\xi$$
$$= -2(k+1)\rho \lim_{R\to\infty} \int_{B_R^3 \setminus \{0\}} \hat{\psi}_0(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \cdot \frac{e^{-|\xi|\cdot\Re(W)}}{|\xi|} \, \mathrm{d}\xi. \tag{3}$$

Now we use that by the last part of proposition 2.2.10 we have for every $W \in \mathring{H}_0^4$:

$$(N(W) + 2\rho\Re(W) + \rho^2) \cdot \lim_{R\to\infty} \int_{B_R^3 \setminus \{0\}} \hat{\psi}_{\Re(W)}(\xi) \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}\xi$$
$$= (\Re(W)^2 + W_1^2 + W_2^2 + W_3^2 + 2\rho\Re(W) + \rho^2)\widetilde{\mathcal{F}}.\hat{\psi}_{\Re(W)} \left(\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \right)$$
$$= \widetilde{\mathcal{F}}.\left[(\Re(W)^2 + 2\rho\Re(W) + \rho^2)\psi - (\partial_1^2\psi + \partial_2^2\psi + \partial_3^2\psi) \right] \left(\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} \right)$$

Therefore we get with (3) for every $W \in \mathring{H}_0^4$:

$$\lim_{R\to\infty} \int_{B_R^3 \setminus \{0\}} \left[(\Re(W)^2 + 2\rho\Re(W) + \rho^2) \cdot \hat{\psi}_{\Re(W)}(\xi) + \Delta\hat{\psi}_{\Re(W)}(\xi) \right] \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}\xi$$
$$= -2(k+1)\rho \lim_{R\to\infty} \int_{B_R^3 \setminus \{0\}} \hat{\psi}_0(\xi) \cdot \frac{e^{-|\xi|\cdot\Re(W)}}{|\xi|} \cdot e^{\mathrm{i}\langle \xi, \Im(W)\rangle} \, \mathrm{d}\xi.$$

And invoking the injectivity of the Fourier cotransformation on $\mathcal{L}_2(\mathbb{R}, \mathbb{C})$ we get for all $W \in \mathring{H}_0^4$ and all $\xi \in \mathbb{R}^3 \setminus \{0\}$:

$$-\frac{1}{2}\Delta\hat{\psi}_{\Re(W)}(\xi) + \frac{(\Re(W))^2 + 2\rho\Re(W) + \rho^2}{2}\hat{\psi}_{\Re(W)}(\xi) = -\frac{(k+1)\rho}{|\xi|} \cdot e^{-|\xi|\cdot\Re(W)} \cdot \hat{\psi}_0(\xi)$$

Now we consider the limit $\Re(W) \to 0$ on both sides of the equation. Using Lebesgue's theorem of majorized convergence with the majorizing function for ψ given in the proof of lemma 2.2.8, we get

$$\lim_{\Re(W)\to 0} \hat{\psi}_{\Re(W)} = \hat{\psi}_0.$$

With an analogous argument using that ψ is of class $\mathcal{C}^2$ we can show that

$$\lim_{\Re(W)\to 0} \Delta\hat{\psi}_{\Re(W)} = \Delta\hat{\psi}_0.$$

This implies the equality

$$-\frac{1}{2}\Delta\hat{\psi}_0(\xi) + \frac{\rho^2}{2}\hat{\psi}_0(\xi) = -\frac{(k+1)\rho}{|\xi|} \cdot \hat{\psi}_0(\xi).$$

Setting $\kappa := (k+1)\rho$ we get for every $\xi \in \mathbb{R}^3 \setminus \{0\}$:

$$-\frac{1}{2}\Delta\hat{\psi}_0(\xi) + \frac{\kappa}{|\xi|} \cdot \hat{\psi}_0(\xi) = -\frac{\kappa^2}{2(k+1)^2} \cdot \hat{\psi}_0(\xi)$$

and thus $\frac{\kappa^2}{2(k+1)^2} \in \sigma_d\left(\mathcal{H}_\kappa\right)$. This finishes the argument. $\square$

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Appendices

Appendix A

Maple 13 Code

On the following two pages of this appendix we give a copy of the commented source code and output of the maple script that was referred to in the proof of lemma 2.2.15. It is an import of a pdf-file generated directly by Maple 13.

Maple 13 script for calculation of the pull-back in lemma 2.2.15

First we define the squared norm on the quaternions:

$N := (z0, z1, z2, z3) \rightarrow z0^2 + z1^2 + z2^2 + z3^2$:

Now we include the package for differential geometry and setup the space $\mathbb{H} \cong \mathbb{R}^4$ as frame for the following calculation. We will denote this frame by E4:

```
> with(DifferentialGeometry) :
> DGsetup([x0, x1, x2, x3], E4)
```

frame name: E4 (1)

Now we define the surface form dS on the three-dimensional unit sphere in $\mathbb{H}$. Then we define a Cayley transformation depending on the parameter ρ, which we will simply denote by C, and which we have calculated by evaluating the Cayley transformation as defined in definition 2.2.5 on the standard basis of $\mathbb{H}$:

```
E4 > dS := evalDG(x0 dx1 &w dx2 &w dx3 - x1 dx0 &w dx2 &w dx3 + x2 dx0 &w dx1
        &w dx3 - x3 dx0 &w dx1 &w dx2)
```

$$dS := -x3\, dx0 \wedge dx1 \wedge dx2 + x2\, dx0 \wedge dx1 \wedge dx3 - x1\, dx0 \wedge dx2 \wedge dx3 + x0\, dx1 \wedge dx2 \wedge dx3 \quad (2)$$

E4 > $C := Transformation\left(E4, E4, \left[x0 = \frac{N(x0, x1, x2, x3) - \rho^2}{N(x0 + \rho, x1, x2, x3)}, x1 = \frac{2 \cdot \rho \cdot x1}{N(x0 + \rho, x1, x2, x3)}, x2 = \frac{2 \cdot \rho \cdot x2}{N(x0 + \rho, x1, x2, x3)}, x3 = \frac{2 \cdot \rho \cdot x3}{N(x0 + \rho, x1, x2, x3)}\right]\right)$:

E4 >

Last we define the pull-back of dS along C as the differential form ω and let Maple explicitly calculate its evaluation on ∂H_0^4 :

E4 > ω := simplify(Pullback(C, dS))

$$\omega := -\frac{16\, \rho^3\, x3\, x0\, dx0 \wedge dx1 \wedge dx2}{\left(x0^2 + 2\, x0\, \rho + \rho^2 + x1^2 + x2^2 + x3^2\right)^4} + \frac{16\, \rho^3\, x2\, x0\, dx0 \wedge dx1 \wedge dx3}{\left(x0^2 + 2\, x0\, \rho + \rho^2 + x1^2 + x2^2 + x3^2\right)^4} - \frac{16\, \rho^3\, x1\, x0\, dx0 \wedge dx2 \wedge dx3}{\left(x0^2 + 2\, x0\, \rho + \rho^2 + x1^2 + x2^2 + x3^2\right)^4} \quad (3)$$

$$-\frac{8\rho^3\left(\rho^2 - x0^2 + x1^2 + x2^2 + x3^2\right) dx1 \wedge dx2 \wedge dx3}{\left(x0^2 + 2\, x0\, \rho + \rho^2 + x1^2 + x2^2 + x3^2\right)^4}$$

E4 > $eval(\omega, [x0 = 0, x1 = z1, x2 = z2, x3 = z3])$

$$0\, dx0 \wedge dx1 \wedge dx2 + 0\, dx0 \wedge dx1 \wedge dx3 + 0\, dx0 \wedge dx2 \wedge dx3 - \frac{8\rho^3\, dx1 \wedge dx2 \wedge dx3}{\left(\rho^2 + z1^2 + z2^2 + z3^2\right)^3} \tag{4}$$

E4 >

By simplification of this last result, using that ρ is a real number, we thus get a confirmation of lemma 2.2.15.

Appendix B

Eigenständigkeitserklärung

Hiermit bestätige ich, dass die vorliegende Arbeit von mir selbständig verfasst wurde und ich keine anderen als die angegebenen Hilfsmittel – insbesondere keine im Quellenverzeichnis nicht benannten Internet-Quellen – benutzt habe und die Arbeit von mir vorher nicht in einem anderen Prüfungsverfahren eingereicht wurde. Die eingereichte schriftliche Fassung entspricht der auf dem elektronischen Speichermedium. Ich bin damit einverstanden, dass die Bachelorarbeit veröffentlicht wird.

Hamburg, den 22. Dezember 2011