Controlled invariance for DAEs

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We study the concept of locally controlled invariant submanifolds for nonlinear descriptor systems. In contrast to classical approaches, we define controlled invariance as the property of solution trajectories to evolve in a given submanifold whenever they start in it. It is then shown that this concept is equivalent to the existence of a feedback which renders the closed-loop vector field invariant in the descriptor sense. This result is motivated by a preliminary consideration of the linear case.

Local controlled invariance leads to the concept of output zeroing submanifolds. We show that the outcome of the differential-algebraic version of the zero dynamics algorithm yields a locally maximal output zeroing submanifold.

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**1 Motivation - linear systems**

We study controlled invariance for linear descriptor systems governed by differential-algebraic equations (DAEs),

\[ \frac{d}{dt}X(t) = AX(t) + Bu(t), \]  

(1.1)

where \( E, A \in \mathbb{R}^{l \times n} \) and \( B \in \mathbb{R}^{l \times m} \). The set of these systems is denoted by \( \Sigma_{l,n,m} \) and we write \( [E, A, B] \in \Sigma_{l,n,m} \).

Note that we do not assume regularity of the pencil \( sE - A \). The functions \( u : \mathbb{R} \to \mathbb{R}^m \) and \( x : \mathbb{R} \to \mathbb{R}^n \) are called input and state of the system, resp. The behavior of (1.1) is the set

\[ \mathcal{B}_{(1.1)} := \{ (x, u) \in C([\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m]) \mid Ex \in C^1([\mathbb{R}; \mathbb{R}^l]) \} \]  

(1.2)

and \( g \in C(X; \mathbb{R}^{l \times m}) \). The set of these systems is denoted by \( \Sigma_{l,n,m,p} \), and we write \( [E, f, g, h] \in \Sigma_{l,n,m,p} \).

A trajectory \((x, u, y) \in C([0, \infty); \mathbb{R}^m \times \mathbb{R}^p)\) is called a solution of (2.1), if \( I = \text{dom} x \subseteq \mathbb{R} \) is an open interval, \( E \circ x \in C^1([0, \infty); \mathbb{R}^l) \) and \((x, u, y)\) solves (2.1) for all \( t \in I \).

A solution \((x, u, y)\) of (2.1) is called maximal, if any other solution \((\hat{x}, \hat{u}, \hat{y})\) of (2.1) satisfies

\[ J := \text{dom} \hat{x} \cap \text{dom} x \neq \emptyset \land \hat{x}|_J = x|_J \Rightarrow \text{dom} \hat{x} \subseteq \text{dom} x. \]

The behavior of (2.1) is the set of maximal solutions

\[ \mathcal{B}_{(2.1)} := \{ (x, u, y) \in C([I]; \mathbb{R}^m \times \mathbb{R}^p) \mid I \subseteq \mathbb{R} \text{ open interval}, (x, u, y) \text{ is maximal solution of (2.1)} \}. \]

The concept of (locally) controlled invariant submanifolds has been introduced by Isidori and Moog [4], see also the textbooks [5, 6]. Loosely speaking, a connected submanifold \( M \) is locally controlled invariant, if it is invariant under the flow of the closed-loop vector field \( f(x) + g(x)u(x) \) for some feedback \( u(x) \). We show that this “classical” definition in terms of feedback is equivalent to the “natural” definition, that (locally) for any initial value in \( M \) there exists an input such that the corresponding state trajectory remains in the submanifold \( M \) for all times or reaches its boundary in finite time.

Definition 2.1 Let \([E, f, g, h] \in \Sigma_{l,n,m,p} \) and \( M \) be a connected submanifold of \( X \) such that \( 0 \in M \). Then \( M \) is called locally controlled invariant, if there exists an open neighborhood \( U \subseteq X \) of the origin in \( \mathbb{R}^n \) such that

\[ \forall x^0 \in M \cap U \exists (x, u, y) \in \mathcal{B}_{(2.1)}, x \in C^1([0, \infty); \mathbb{R}^n) \]  

\[ \exists t_0 \in \text{dom} x, x(t_0) = x^0 : \]  

\[ (\forall t \in [t_0, \infty) \land x(t) \in M \cap U) \lor (\exists \hat{t} \in \text{dom} x, \hat{t} > t_0 \land x(t) \in M \cap \partial(U \cap \{ \hat{t} \}). \]  

Theorem 2.2 Let \([E, f, g, h] \in \Sigma_{l,n,m,p} \) be such that \( E \in C^2(X; \mathbb{R}^l), f \in C^1(X; \mathbb{R}^l) \) and \( g \in C^1(X; \mathbb{R}^{l \times m}) \) and let \( M \) be a smooth connected submanifold of \( X \) such that \( 0 \in M \). Suppose that there exists an open neighborhood \( V \) of \( 0 \in X \) such that both \( \dim E^*(x)T_xM \) and \( \dim (E^*(x)T_xM + \text{im} g(x)) \) are constant for \( x \in M \cap V \). Then the following statements are equivalent:
(i) $M$ is locally controlled invariant.

(ii) There exists an open neighborhood $U$ of $0 \in X$ such that 
$$f(x) \in E'(x) T_x M + \text{im} \ g(x) \text{ for all } x \in M \cap U.$$ 

(iii) There exists an open neighborhood $U$ of $0 \in X$ and $u \in C^1(M \cup U; \mathbb{R}^m)$ such that $f(x) + g(x)u(x) \in E'(x) T_x M$ for all $x \in M \cup U$.

In the remainder of this paper we consider the zero dynamics of (2.1), which is the set of trajectories $ZD_{(2.1)} := \{ (x, u, y) \in \mathfrak{B}_{(2.1)} | y = 0 \}$. The concept of zero dynamics goes back to Byrnes and Isidori [7] and is studied extensively since then, see e.g. [5, 6]. For linear DAEs, the zero dynamics have been investigated in detail recently [8–11]. Zero dynamics are related to the concept of output zeroing submanifolds.

**Definition 2.3** Let $[E, f, g, h] \in \Sigma_{i,n,m,p}^X$ and $M$ be a connected submanifold of $X$ such that $0 \in M$. Then $M$ is called output zeroing, if $M$ is locally controlled invariant and $h(x) = 0$ for all $x \in M$. An output zeroing submanifold $M$ that is called locally maximal, if there exists an open neighborhood $U$ of $0 \in X$ such that any output zeroing submanifold $M$ satisfies $M \cap U \subseteq M \cap U$.

We extend the zero dynamics algorithm developed in [4, 12] to nonlinear DAE systems (2.1).

**Theorem 2.4** Let $[E, f, g, h] \in \Sigma_{i,n,m,p}^X$ be such that $E, f, g$ and $h$ are smooth. Define $M_0 := h^{-1}(0)$ and for any $k \in \mathbb{N}$ the set $M_k$ recursively as follows: Suppose that for some open neighborhood $U_{k-1}$ of $0 \in X$, $M_{k-1} \cap U_{k-1}$ is a submanifold, define $M_{k-1} := \bigcup \{ M_{k-1} \cap U | U \subseteq X \text{ open}, M_{k-1} \cap U \text{ is a submanifold} \}$, let $M_{k-1}^c$ be the connected component of $M_{k-1}$ which contains $0 \in X$ and define $M_k := \{ x \in M_{k-1}^c | f(x) \in E'(x) T_x M_{k-1}^c + \text{im} \ g(x) \}$. Then we have the following:

(i) The sequence $(M_k)$ is nested, terminates and satisfies 
$$\exists k^* \in \mathbb{N}_0 \forall j \in \mathbb{N} : M_0 \supseteq M_1 \supseteq \ldots \supseteq M_{k^*} \supseteq M_{k^*+j} = M_{k^*+j}.$$ 

(ii) If $Z^* := M_{k^*}$ satisfies, for some open neighborhood $U$ of $0 \in \mathbb{R}$, that $\dim E'(x) T_x Z^*$ and $\dim (E'(x) T_x Z^* + \text{im} \ g(x))$ are both constant for $x \in Z^* \cap U$, then $Z^*$ is a locally maximal output zeroing submanifold.

(iii) There exists an open neighborhood $U$ of $0 \in X$ such that for all open $O \subseteq U$ and all $(x, u, y) \in \mathfrak{B}_{(2.1)}$ with $x \in C^1(\text{dom} \ x; X)$ and $x(t) \in O$ for all $t \in \text{dom} \ x$ 
$$x(u, y) \in ZD_{(2.1)} \iff x(t) \in Z^* \cap O \forall t \in \text{dom} \ x.$$ 

If the system (2.1) is linear, then the sequence $(M_k)$ becomes an augmented Wong sequence, see [3, 13] and the references therein, which is based on the Wong sequences [14–16] and which have their origin in [17].

Output zeroing submanifolds can be exploited to study locally autonomous zero dynamics; the latter have been successively used for the analysis of linear time-varying ODEs in [18] and of linear time-invariant DAEs in [9]. Under the assumption of locally autonomous zero dynamics we aim to derive a local zero dynamics form for nonlinear DAE systems (2.1) which would provide the basis for the application of adaptive control techniques. In particular, we aim to use the results of [19] and show feasibility of funnel control for nonlinear descriptor systems which encompass nonlinear electrical circuits, extending the results for the linear case [20].

**References**


