GENERALIZED KÄHLER STRUCTURES IN DIMENSION 4

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. These notes are intended to summarize and complete the lectures given by me on Dec 1, 2011, and Jan. 12, 2010 as part of the lectures on generalized complex structures by Prof. Dr. V. Cortés. the main results are contained in [3], [4] and [14], but they rely on [1] and on the important notion of *standard* (or *Gauduchon*) metric [10]. The more recent notion of *conformal product* may help understanding the construction of generalized Kähler structures associated to a pair of commuting complex structures.

Feel free to send me by e-mail your remarks and comments.

1. Generalized Kähler structures

Definition 1.1. [12] *A generalized Kähler structure on a manifold M is a pair $\mathcal{J}_1, \mathcal{J}_2$ of commuting generalized complex structures, such that $G := -\mathcal{J}_1\mathcal{J}_2$ induces (via the tautological symplectic form) a positive definite metric on $TM \oplus T^*M$*

The restrictions of $\mathcal{J}_1, \mathcal{J}_2$ to the eigenspaces $C_\pm$ of G (these eigenspaces are both $\mathcal{J}_1$- and $\mathcal{J}_2$-invariant, because these endomorphisms commute) induce, by projection on TM, almost complex structures $J_\pm$ on M. Integrability of $\mathcal{J}_1, \mathcal{J}_2$ imply the one of $J_\pm$. More precisely, we have:

Proposition 1.2. [12] *A generalized Kähler structure on a manifold M is equivalent to a pair J_+, J_- of integrable complex structures on M, which are both orthogonal with respect to the same Riemannian metric g, such that, if we denote by $\omega_\pm := g(J_\pm\cdot, \cdot)$ the corresponding Kähler forms, the following conditions hold:*

$$d^c_+\omega_+ = -d^c_-\omega_- = h \tag{1}$$

$$h \text{ is exact.} \tag{2}$$

Here the operators $d^c_\pm := i(\bar\partial_\pm - \partial_\pm)$ are obtained from the ∂ and $\bar\partial$ operators corresponding to the complex structures $J_\pm$. In fact, $d^c_+\omega_+$ is a real 3-form equal to $d\omega(J_+\cdot, J_+\cdot, J_+\cdot)$ and an analogue formula holds for $d^c_-\omega_-$.

By analogy (or by considering a *twisted* Courant bracket on $TM \otimes T^*M$), we define a *twisted* generalized Kähler structure to be given by a pair of orthogonal complex structures $J_\pm$ such that the 3-form h above is *closed*, but not exact. Following [4], we will call in these notes a generalized Kähler structure *untwisted* if it satisfied the conditions in the proposition 1.2, and the notion of generalized Kähler structure will be consider to cover both the twisted and untwisted one.

The goal of these lectures is to show that, in dimension 4, the generalized Kähler structures are equivalent to some simpler structures, which are easier to construct. As a consequence, for a compact 4-manifold M admitting a generalized complex structure, it is possible to classify (under some mild assumptions) the compact complex surfaces $(M, J_\pm)$, [3], [14], [4].

Before we can state the main results, we recall some basic facts about 4-dimensional Riemannian geometry. We review then some important properties of the Riemannian (more generally, Weyl) curvature of a hermitian surface; here we need to compare the curvature of two conformal, torsion-free connections (Weyl structures) on a conformal-Riemannian manifold (see the Appendix).

2. Curvature of hermitian surfaces

2.1. **Hermitian surfaces. Main statements.** A *Hermitian surface* is a complex 2-dimensional (real 4-dimensional) manifold (M^4, J) with a hermitian metric g, i.e. g is a Riemannian metric and $J : TM \to TM$ is a skew-symmetric endomorphism of TM with respect to g which is integrable, i.e. its *Nijenhuis tensor*

$$4N^J(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \tag{3}$$

vanishes identically.

We denote by $\omega^J := g(J\cdot, \cdot) \in \Lambda^2 M$ the *Kähler form* of J with respect to g. By the canonical isomorphism $TM \simeq T^*M$ induced by g, $J \in End(TM) = T^*M \otimes TM \simeq T^*M \otimes T^*M$ can be identified with ω^J. This kind of identifications will be often used in the text.

Our first remark is that J induces a canonical volume form on M, $\omega \wedge \omega$, which is twice the Riemannian volume form induced by g and the orientation of TM, for which a basis e_1, Je_1, e_2, Je_2 is positive ($\forall e_1, e_2 \in TM$ linearly independent over $\mathbb{C}$).

Therefore, the case where two different complex structures are simultaneously g-orthogonal is split in two sub-cases:

(1) The case where $J_\pm$ induce the same orientation on M, where $(M^4, g, J_\pm)$ will be called a *bihermitian* surface;
(2) The case where $J_\pm$ induce opposite orientations on M, where $(M^4, g, J_\pm)$ will be called an *ambihermitian* surface.

These two situations will turn out to be geometrically distinct. The main results we will prove are the following:

Theorem 2.1. [3] *Let $(M^4, g, J_\pm)$ be a compact bihermitian surface with even first Betti number. Then there exists a conformal change in the metric such that $(M, g' := e^f g, J_\pm)$ is an untwisted generalized Kähler manifold.*

In fact, the condition on the first Betti number turns out to distinguish the twisted and the untwisted case:

Proposition 2.2. [4] *Let $(M^4, g, J_\pm)$ be a compact generalized Kähler surface. Then this generalized Kähler structure is untwisted iff the first Betti number is even.*

As a consequence, generalized Kähler 4-manifolds (corresponding to bihermitian surfaces) with b_1 even are classified [3].

For the sake of completeness we also recall in Section 5 other results by Pontecorvo [14] about bihermitian surfaces (that also leads to a complete classification [14]), concerning the case when the conformal structure is *anti-self-dual*, which means that there are no local obstruction to existence of compatible complex structures.

The main result for generalized Kähler 4-manifolds associated to *ambihermitian* surfaces is

Theorem 2.3. [4] *Let $(M^4, g, J_\pm)$ be a compact generalized Kähler manifold on an ambihermitian surface. Then the eigenspaces $V_\pm$ of $Q := -J_+J_-$ corresponding to the eigenvalues ± 1 are both holomorphic (complex 1-dimensional) distributions on M, therefore $(M, J_\pm)$ has two transversal holomorphic foliations.*

Conversely, any compact complex surface (M, J) admitting a pair of transversal holomorphic distributions admits a generalized Kähler structure (J being J_+).

This result also leads to a complete classification of the underlying compact complex surfaces [4].

Because the list of surfaces in the above cited classification relies on extensive knowledge of the special field of compact complex surfaces (including the Kodaira classification), we leave to the motivated reader the choice to consult the original papers and we focus in these notes on the geometrical aspects that are the essential ingredients that lead to these lists.

2.2. **Curvature of a 4-dimensional manifold.** On every oriented Riemannian manifold (M^n, g), the line bundle of n-forms is canonically identified with $\mathbb{R}$, by the fact that any n-form is a multiple of the canonical Riemannian volume form v_g.

The Hodge *star* operator of a p-form α is the image of v_g through the adjoint map of the wedge product with α, restricted to Λ^{n-p}:

$$\alpha \wedge \cdot : \Lambda^{n-p} M \to \Lambda^n M,$$

therefore $*\alpha$ is characterized by the following property

$$g(*\alpha, \beta) v_g = \alpha \wedge \beta, \ \forall \beta \in \Lambda^{n-p} M. \tag{4}$$

Equivalently, for a given positive orthonormal basis $e_1, \dots, e_n$ of TM,

$$*_g(e_1 \wedge \ldots \wedge e_p) := e_{p+1} \wedge \ldots \wedge e_n.$$

Recall that the norm on forms is such that the p-forms $e_{i_1} \wedge \ldots \wedge e_{i_p}$, for $1 \leq i_1 < \cdots < i_p \leq n$ form an orthonormal basis of $\Lambda^p M$.

If $n = 4$ and $p = 2$, we obtain that $* : \Lambda^2 M \to \Lambda^2 M$ is an involution, therefore $\Lambda^2 M$ splits in the orthogonal direct sum of the space of *self-dual* forms $\Lambda^+ M$, where $*$ acts as the identity, and the space of *anti-self-dual* forms $\Lambda^- M$, where $*$ acts as -1.

Remark 2.4. *Recalling that $\Lambda^2 M$ is canonically identified to the space of skew-symmetric endomorphisms of TM, i.e. of the* adjoint bundle *of the $O(4)$-principal bundle of orthonormal frames on M, the splitting $\Lambda^2 M = L^+ M \oplus \Lambda^- M$ corresponds to the splitting of $\mathfrak{so}(4)$ as a sum of two simple ideals (both isomorphic to $\mathfrak{so}(3)$)* [11, 1, 5].

As a consequence, the Riemannian curvature of the Levi-Civita connection $\nabla := \nabla^g$, defined by

$$R^{\nabla}_{X,Y} Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z,$$

$X, Y, Z \in TM$, induces a symmetric endomorphism of $\Lambda^2 M$, and, because of the splitting of the latter, we obtain the *Singer-Thorpe* curvature decomposition (see [1] for details):

$$\begin{array}{cc|c|c|c} & & \Lambda^+ M & \Lambda^- M & \\ \cline{3-4} R & = & W^+ + \frac{\mathrm{Scal}}{12} & B_0 & \Lambda^+ M \\ \cline{3-4} & & B_0^t & W^- + \frac{\mathrm{Scal}}{12} & \Lambda^- M \\ \cline{3-4} \end{array} \tag{5}$$

Here, Scal is the scalar curvature of ∇, B_0 is identified with the traceless Ricci tensor (it is actually the *suspension* of half the traceless Ricci tensor, see [2]), and $W_\pm$ are the *self-dual*, resp. *anti-self-dual* Weyl tensors of g. They are both symmetric trace-free endomorphisms of Λ^+M, resp. Λ^-M and depend on the conformal class $[g]$ only (see the Appendix). The factor $1/12$ in front of the Scalar curvature ensures that the trace of R on Λ^2M is half the scalar curvature (the consecutive traces on TM that define Scal yield twice such a result).

Note that the decomposition (5) holds for the Riemannian part of the curvature of an arbitrary Weyl structure as well (see the Appendix).

2.3. **The Lee form of a hermitian surface.** Let (M^4, g) be a Riemannian (oriented) 4-manifold and let J be an *orthogonal* almost complex structure on (M, g), i.e. $J : TM \to TM$, with $J^2 = -Id$ is skew-symmetric w.r.t. g, and $\omega^J := g(J\cdot, \cdot)$ its corresponding Kähler form. We say that (M^4, g, J) is a hermitian surface iff J is integrable (i.e., its Nijenhuis tensor (3) vanishes).

As we have seen at the beginning of this section, J intrinsically defines an orientation on M by the volume form $\frac{1}{2}\omega^J \wedge \omega^J$ which coincides up to sign with the Riemannian volume form v_g. ω^J is thus a self-dual 2-form if J is positively oriented (and anti-self-dual otherwise). Note that the square norm of ω^J is 2.

We get thus a further decomposition of the space of 2-forms on an almost hermitian surface:

Proposition 2.5. *Let (M, g) be a Riemannian 4-manifold and J a g-orthogonal almost complex structure on M. Fix on M the orientation induced by J. Then*

(1) $\omega^J \in \Lambda^+M$
(2) $\Lambda^{1,1}M = \mathbb{R}\omega^J \oplus \Lambda_0^{1,1}M = \mathbb{R}\omega^J \oplus \Lambda^-M$
(3) $\Lambda^{2,0+0,2}M$ *is the orthogonal complement of* $\mathbb{R}\omega^J$ *in* Λ^+M.

The proof is done by checking these claims on an arbitrary hermitian basis.

Remark 2.6. *Note that we consider the* real *parts of the complex vector spaces defined by the type decompositions of forms. More precisely, $\Lambda^{1,1}M$ denotes here the real vector bundle of the J-invariant 2-forms, i.e. for which $\alpha(J\cdot, J\cdot) = \alpha$, and $\Lambda^{2,0+0,2}M$ consists of the real J-anti-invariant 2-forms, i.e. for which $\alpha(J\cdot, J\cdot) = -\alpha$. As skew-symmetric endomorphisms, $\Lambda^{1,1}M$ are those that commute with J, and $\Lambda^{2,0+0,2}M$ those that anti-commute with J.*

Inserting just one J into $\alpha \in \Lambda^{1,1}M$ turns it into a symmetric endomorphism (still commuting with J), and if $\alpha \in \Lambda^{2,0+0,2}M$, then $\alpha(J\cdot, \cdot) = \alpha(\cdot, J\cdot) \in \Lambda^{2,0+0,2}M$. We denote by

$$\mathcal{J} : \Lambda^{2,0+0,2}M \to \Lambda^{2,0+0,2}M \;\; \mathcal{J}(\alpha) := \alpha(J\cdot, \cdot) \tag{6}$$

the complex structure on $\Lambda^{2,0+0,2}M$ given by inserting one J into a J-anti-invariant 2-form.

Before discussing the curvature decomposition of a hermitian surface, let us first introduce a basic notion:

Definition 2.7. *The* Lee form *of a hermitian surface (M, g, J) is the uniquely defined 1-form θ such that*

$$d\omega^J = -2\theta \wedge \omega^J.$$

Remark 2.8. *Of course, we need to show that such a form exists and is unique: Indeed,the wedge product with a non-degenerate 2-form is an injective linear map from Λ^1M to Λ^3M, and therefore is an isomorphism if $\dim M = 4$. So θ in the definition exists and is unique.*

In conformal geometry, there is another notion of *Lee form* (see the Appendix). That these two notions coincide is justified by the following Proposition (that also explains why we have put the factor -2 in the definition of θ):

Proposition 2.9. *Let (M^4, g, J) be a hermitian surface. Then the Weyl structure*

$$\nabla^J := \nabla^g + \tilde{\theta}$$

defined by the Lee form θ of J is the unique Weyl structure ∇ on M such that $\nabla J = 0$.

Proof. Consider an arbitrary Weyl structure $\nabla := \nabla^g + \tilde{\eta} = \nabla^g + (\eta \wedge \cdot) + \eta \otimes Id$ (28-29). From (31), see also Remark 6.5 in the Appendix, we have

$$\nabla_X J = \nabla^g_X J + [\eta \wedge X, J]. \tag{7}$$

The next Lemma shows that the first term in (7) is the image of a $\mathbb{C}$-linear map from TM to $\Lambda^{2,0+0,2}M$:

Lemma 2.10. *Let $(M, [g])$ a conformal Riemannian manifold. An orthogonal almost complex structure J on M is integrable if and only if, for some (and hence any) Weyl structure ∇, the following relation holds:*

$$\nabla_{JX} J = J \nabla_X J, \ \forall X \in TM. \tag{8}$$

Proof. Consider the following tensor:

$$A(X, Y, Z) := g((\nabla_{JX} J - J\nabla_X J)(Y), Z)$$

Because J is a skew-symmetric endomorphism of TM, so is $\nabla_X J$, and because $J^2 = -Id$, this endomorphism anti-commutes with J, therefore A is skew-symmetric in Y, Z.

On the other hand,

$$A(X, Y, Z) - A(Y, X, Z) = 4g(N(X, Y), Z), \tag{9}$$

where $N(X, Y)$is the Nijenhuis tensor of J (3).

If $A = 0$, then $N = 0$, thus J is integrable. On the other hand, using the skew-symmetry of A and the relation ((9) above, it is easy to check that

$$A(X, Y, Z) = 2g(N(X, Y), Z) - 2g(N(Y, Z), X) + 2g(N(Z, X), Y),$$

thus if J is integrable then $N = 0$, therefore $A = 0$. □

$\nabla^g J$ and ∇J are both skew-symmetric endomorphisms that anti-commute with J, hence are elements in $\Lambda^{2,0+0,2}M$. On the other hand, if we consider the complex structure J on TM and the complex structure $\mathcal{J}$ on $\Lambda^{2,0+0,2}M$, the previous Lemma shows that the maps $X \mapsto \nabla^g_X J$ and $X \mapsto \nabla_X J$ are $\mathbb{C}$-linear. Therefore, their difference (see (7))

$$X \mapsto [\eta \wedge X, J]$$

is also a $\mathbb{C}$-linear map from TM to $\Lambda^{2,0+0,2}M$.

Now, the first term of the left hand side of (7) is an unknown element of the vector space

$$\mathrm{Hom}_{\mathbb{C}}(TM, \Lambda^{2,0+0,2}M)$$

which has complex dimension $2 \cdot 1 = 2$. On the other hand, the map

$$\Lambda^1 M \ni \eta \longmapsto [\eta \wedge X, J] \in \mathrm{Hom}_{\mathbb{C}}(TM, \Lambda^{2,0+0,2}M)$$

is a $\mathbb{C}$-linear map from TM (of complex dimension 2) to $\mathrm{Hom}_{\mathbb{C}}(TM, \Lambda^{2,0+0,2}M)$, also of complex dimension 2. This map is surjective iff it is injective, i.e. iff

$$[\eta \wedge X, J] = 0, \ \forall X \in TM \Longrightarrow \eta = 0,$$

which is easily checked (we just need here $\dim TM > 2$).

That implies, therefore, that a unique Weyl structure ∇ (defined by some uniquely determined η_0) always exists, such that $\nabla J = 0$.

To check that $\eta_0 = \theta$, the Lee form defined above, we just note that

$$\nabla_X \omega^J = \nabla_X \left(g(J\cdot, \cdot) \right) = (\nabla_X g)(J\cdot, \cdot)$$

since $\nabla_X J = 0$. But $\nabla_X g = -2\eta_0(X) g$, thus

$$\nabla \omega^J = -2\eta_0 \otimes \omega^J.$$

By skew-symmetrizing we get that $d\omega^J = -2\eta_0 \wedge \omega^J$ thus $\eta_0 = \theta$. □

Corollary 2.11. *Let (M, g, J) be a hermitian surface and let $\theta = \theta^J$ be the Lee form of the canonical Weyl structure $\nabla := \nabla^g + \tilde{\theta}$. The following relation holds:*

$$\nabla^g_X J = J\theta \wedge X + \theta \wedge JX \ \forall X \in TM.$$

Proof. We compute

$$[\theta \wedge X, J] = -J\theta \otimes X + JX \otimes \theta - \theta \otimes JX + X \otimes J\theta = -J\theta \wedge X - \theta \wedge JX,$$

and use that $\nabla_X J = 0$. □

Recall that the *codifferential* on k-forms on an oriented Riemannian manifold (M, g) is defined by

$$\delta^g := (-1)^k *^{-1} d*,$$

where d is the exterior differential. As $*^2 = (-1)^{k(n-k)}$ on k-forms, it follows that on an hermitian surface (M, g, J)

$$\delta^g \omega^J = - * d * \omega^J = 2 * (\theta \wedge \omega^J).$$

A simple computation shows

$$*(\eta \wedge \omega^J) = J\eta \text{ and } * \eta = J\eta \wedge \omega^J, \tag{10}$$

for any 1-form η. Here we considered $J\eta$ to be the 1-form dual to the vector $J(\eta^\sharp)$, thus $J\eta = -\eta \circ J$. We get thus

$$\delta^g \omega^J = -2J\theta^J, \tag{11}$$

where we note that J doesn't need to induce the same orientation on M as $*$.

2.4. **Complex structures on a Riemannian 4-manifold.** From the decomposition of $\Lambda^2 M$ on an almost hermitian surface (Prop. 2.5) we obtain the following decomposition of the curvature of a 4-dimensional Riemannian manifold on which an almost complex structure J has been fixed:

$$R \;=\; \begin{array}{ccc|c} \mathbb{R}\omega^J & \Lambda^{2,0+0,2}M & \Lambda^- M & \\ \hline \frac{\mathrm{Scal}^\nabla}{12} + 2a & \Phi & B_1 & \mathbb{R}\omega^J \\ \Phi^t & A_0 + \left(\frac{\mathrm{Scal}^\nabla}{12} - a\right) Id & B_2 & \Lambda^{2,0+0,2}M \\ B_1^t & B_2^t & W^- + \frac{\mathrm{Scal}}{12} & \Lambda^- M \end{array} \tag{12}$$

Here, Scal^∇ is the scalar curvature of the metric g, but it can also be the scalar curvature of a particular Weyl structure (see the Appendix). We will actually see that, if J is integrable, there is a canonical Weyl structure associated to it, and the computations will be made using this Weyl structure.

Note that all components are irreducible with the exception of the 2×2 symmetric matrix $A_0 + \left(\frac{\mathrm{Scal}^\nabla}{12} - a\right) Id$, which decomposes into its traceless part (and also its $\mathcal{J}$-anti-invariant part) A_0 and its diagonal part (also its $\mathcal{J}$-invariant part) which is just a multiple of the identity.

A priori there is no special relation between the entries in the table above, but if J is integrable, much can be said about them:

Theorem 2.12. [1] *Let (M, g) be a Riemannian manifold and let J be a g-orthogonal integrable complex structure on M. Let ∇^J be the unique Weyl structure on M for which $\nabla J = 0$, let θ be the corresponding Lee form, and κ the scalar curvature of this Weyl structure. Then, in the decomposition* (12) *of the Riemannian curvature R^g, the following relations hold:*

(1) $A_0 = 0$,
(2) $\Phi = \mathcal{J}(d\theta)^+$, *the self-dual part of the exterior derivative of the Lee form θ,*
(3) $a = -\kappa/12 = -\mathrm{Scal}^{\mathrm{g}}/12 - (\delta^{\mathrm{g}}\theta - \|\theta\|^2_{\mathrm{g}})/2$

First note that since $d\omega^J = -2\theta \wedge \omega^J$, we have

$$0 = dd\omega^J = -2d\theta \wedge \omega^J$$

thus $d\theta$, and in particular $(d\theta)_+$ is orthogonal to ω^J and is thus equal to $(d\theta)^{2,0+0,2}$ and hence is a skew-symmetric endomorphisms of TM that anti-commutes with J. Also, $(d\theta)_- = (d\theta)^{1,1}$ is the part of $d\theta$ that commutes with J.

Proof. All the claims are about the structure of W^+, which is a conformally invariant tensor (Proposition 6.7 in the Appendix). We can thus consider any Weyl structure, noting that the decomposition (12) refers to the *Riemannian* part of the Weyl curvature, i.e. without the terms in the Faraday form (33).

Consider thus the curvature decomposition for R^∇, for ∇ the canonical Weyl structure associated to J. We recall (see the Appendix) that the curvature of a Weyl structure is

$$R^\nabla = R^{\nabla,Riem} + F^\nabla \otimes Id - \frac{1}{2} F^\nabla \wedge Id, \tag{13}$$

where $F^\nabla = d\theta$ is the Faraday form, and $= R^{\nabla,Riem}$ is the Riemannian part of R^∇. This Riemannian part further decomposes

$$R^{\nabla,Riem} = \rho^\nabla \wedge Id + W$$

into the suspension of the symmetric Schouten tensor and the Weyl tensor (which, for $n = 4$, further decomposes in $W^+ + Wm$). From Proposition 6.7 we know that

$$\rho^\nabla = \rho^g - (\nabla^g \theta)^{sym} + \theta \otimes \theta - \frac{1}{2} g(\theta,\theta) g,$$

and that the Weyl tensor is conformally invariant. By taking the trace of the above relation or from Corollary 6.9 we also get:

$$\kappa = \mathrm{Scal}^\nabla = \mathrm{Scal}^{\mathrm{g}} + 6(\delta^{\mathrm{g}}\theta - \mathrm{g}(\theta,\theta)). \tag{14}$$

We want to compute the image of ω^J by the self-dual Weyl tensor W^+, as relations (2) and (3) in the Theorem will follow from that.

Consider $R^\nabla J = 0$ and use the curvature decomposition (13):

$$R^\nabla(\omega^J) = W^+(\omega^J) + (\rho^\nabla \wedge Id)(\omega^J) - \frac{1}{2}(F^\nabla \wedge Id)(\omega^J) + tr_J F^\nabla Id. \tag{15}$$

Remark 2.13. R^∇ *denotes the curvature operator* $R^\nabla : \Lambda^2 M \to \mathrm{End}(TM)$ *(actually it's just the skew-symmetric endomorphisms plus the multiples of the identity), but when we write* $R^\nabla J$*, we mean that we* apply the curvature to J*, i.e. we consider the* commutator *of the image of* R^∇ *(as an endomorphism of* TM*) with* J*. But even though* J *and* ω^J *are equivalent objects, when we speak of* the image of ω^J through R^∇*, which is an endomorphism of* TM *as well, we denote this by* $R^\nabla(\omega^J)$*.*

Note that, while $R^\nabla J = 0$, this just means that the endomorphism $R^\nabla(\alpha)$ commutes with J for any 2-form α (in particular for ω^J).

In (15), we are interested only in the components in $\Lambda^+ M$: we can therefore ignore the term in ρ_0^∇ that is known to exchange self-dual and anti-self-dual 2-forms. Also note that the term in the identity vanishes (see Corollary 2.11, point 4.).

Note that the left hand side of (15) is not zero, it just commutes with J (see the remark above), but since we only look for the self-dual components, it means that the only possibly non-zero term of $R^\nabla(\omega^J)_+$ is some multiple of J. To compute it, let us fix $e_1, \ldots, e_4$ an orthonormal basis of TM, such that $Je_1 = e_2$ and $Je_3 = e_4$ and take a trace in the Bianchi identity (32):

$$2R^\nabla(\omega^J)X = \sum_{i=1}^4 R^\nabla_{e_i,Je_i} X = -\sum_{i=1}^4 R^\nabla_{X,e_i} Je_i - \sum_{i=1}^4 R^\nabla_{Je_i,X} e_i = -2J\widetilde{Ric^\nabla}(X),$$

where $\widetilde{Ric^\nabla} = (Ric^\nabla)^{sym}$ is the other contraction than the one that yields the Ricci tensor, and which, for $n = 4$, is symmetric (37). We have thus

$$2g(R^\nabla(\omega^J), \omega^J) = \sum_{i=1}^4 g(R^\nabla(\omega^J)e_i, Je_i) = \sum_{i=1}^4 \widetilde{Ric^\nabla}(e_i, J(Je_i)) = -\mathrm{Scal}^\nabla.$$

We conclude that the coefficient of J in $R^\nabla(\omega^J)$ is $\kappa/4$. Thus (15) implies:

$$\frac{\kappa}{4}J = W^+(\omega^J) + \frac{\kappa}{12}J - \frac{1}{2}(F^\nabla \wedge Id)(\omega^J)^+,$$

where we still need to compute the term in the Faraday curvature. Using the definition of the suspension (34), we compute

$$g((F^\nabla \wedge Id)(\omega^J)(X), Y) = g\left(\frac{1}{2}\sum_{i=1}^4 F^\nabla(e_i, \cdot) \wedge Je_i(X), Y\right) =$$

$$= \sum_{i=1}^4 -\frac{1}{2}\left(F^\nabla(e_i, X)g(Je_i, Y) - F^\nabla(e_i, Y)g(Je_i, X)\right) = -\frac{1}{2}\left(F^\nabla(X, JY) - F^\nabla(JX, Y)\right).$$

We see that we obtain automatically a self-dual form (more precisely, a form in $\Lambda^{2,0+0,2}M$). This form is nothing but

$$-\mathcal{J}(F^\nabla)_+ = 2\mathcal{J}(d\theta)_+.$$

Therefore,

$$W^+(\omega^J) = \frac{1}{6}\kappa J + \mathcal{J}(d\theta)_+ \tag{16}$$

Claims (2) and (3) from the Theorem follow immediately.

To prove claim (1) in the Theorem, we will use Lemma 2.10 to prove that the part of the curvature of g that sends $\Lambda^{2,0+0,2}M$ to itself commutes with $\mathcal{J}$, hence is of diagonal type.

Let $D := \nabla^g$ be the Levi-Civita connection of g. From (8) we have

$$D_{JX}D_{JY}J = D_JX(JD_YJ) = J(D_XJ)(D_YJ) + JD_{JX}D_YJ,$$

and, by replacing X, Y with JY, resp. JX, we also have

$$-D_YD_XJ = -J(JD_YJ)(JD_XJ) - JD_YD_{JX}J = J(D_YJ)(D_XJ) - JD_YD_{JX}J.$$

By adding these two relations and skew-symmetrizing in X, Y, we get

$$R^g_{JX,JY} - D_{[JX,JY]}J - R^g_{X,Y}J + D_{[X,Y]}J = JR^g_{JX,Y}J - JD_{[JX,Y]}J - JR^g_{JY,X}J + JD_{[JY,X]}.$$

Note that the derivation arguments of the terms containing only first order derivatives of J add up to the Nijenhuis tensor (which is zero). If we denote by $\alpha := JX \wedge JY - X \wedge Y \in \Lambda^{2,0+0,2}M$, therefore $\mathcal{J}\alpha = JX \wedge Y + X \wedge JY$, we conclude

$$[R^g(\alpha), J] = -\mathcal{J}[R^g(\mathcal{J}\alpha), J], \ \forall \alpha \in \Lambda^{2,0+0,2}M. \tag{17}$$

Note that, for any skew-symmetric endomorphism B on TM, $[B, J]$ is twice the $2,0+0,2$–part of B. If we denote by A the endomorphism of $\Lambda^{2,0+0,2}M$ induced by R^g, (17) reads like

$$A(\alpha) = -\mathcal{J}A(\mathcal{J}\alpha), \ \forall \alpha \in \Lambda^{2,0+0,2}M,$$

but this means precisely that A is $\mathcal{J}$-invariant, hence a multiple of the identity. Thus $A_0 = 0$. $\square$

We conclude that, in case (M, g) admits an orthogonal integrable complex structure compatible with the orientation, then some components of W^+ vanish ($A_0 = 0$) and the value of the other components can be computed in terms of the complex structure. In particular, $\alpha := (d\theta)_+$ is an eigenvector of W^+ for the eigenvalue $\lambda_0 := -\kappa$. Three possibilities occur:

(1) $W^+ = 0$, i.e. $\alpha = 0$ and $\kappa = 0$;
(2) $W^+ \neq 0$ but $\alpha = 0$;
(3) $\alpha \neq 0$.

Consider first that the third case holds on an open dense set of M. We will see that the 3 eigenvalues of W^+ (seen, via a fixed metric g, as a symmetric trace-free endomorphism of Λ^+M) are distinct and we will say that W^+ is *non-degenerate*. Indeed, in the orthonormal basis

$$\frac{\omega^J}{\sqrt{2}}, \mathcal{J}\alpha_0, \alpha_0$$

of Λ^+M (here $\alpha_0 := \alpha/\|\alpha\|_g$ is the unit form determined by α), W^+ is the following 3×3 matrix:

$$W^+ = \begin{pmatrix} -\frac{\kappa}{6} & \|\alpha\|_g & 0 \\ \|\alpha\|_g & \frac{\kappa}{12} & 0 \\ 0 & 0 & -\kappa \end{pmatrix}. \tag{18}$$

The eigenvalues of the upper left 2×2 block are

$$\lambda_\pm = \frac{\kappa}{24} \pm \frac{\sqrt{\kappa^2 + 16\|\alpha\|_g^2}}{8},$$

and the eigenvalue $\lambda_0 = \kappa/12$ lies between (and is distinct of) λ_+ and λ_- unless $\alpha = 0$.

Remark 2.14. *The eigenvalues of W^+ depend on the chosen metric g but not their order and ratios, since the symmetric operator $W^+ : \Lambda^+M \to \Lambda^+M$ is rescaled by a conformal change of metric.*

Conversely, suppose W^+ is almost everywhere non-degenerate on (M, g). Then we can ask whether there is any g-orthogonal complex structure J compatible with the orientation of M and, if yes, how many such J's exist.

Proposition 2.15. [1] *Let (M^4, g) be a connected, oriented 4-manifold with non-degenerate self-dual Weyl tensor W^+. Then there exist at most two* strictly *distinct orthogonal complex structures $J_\pm$ (i.e., $J_+ \neq \pm J_-$ everywhere) compatible with the orientation. In this case, if we denote by $\theta^\pm$ their Lee forms, the following relations hold*

$$(d\theta^+)_+ + (d\theta^-)_+ = 0 \tag{19}$$

$$\|\theta^+\|_g^2 - \delta^g\theta^+ = \|\theta^-\|_g^2 - \delta^g\theta^-. \tag{20}$$

Note that The signs "+" have different significations: $(d\theta^+)_+$ is the self-dual part of $d\theta^+$, i.e. of the exterior derivative of the Lee form of J^+.

Proof. We fix g the Riemannian metric and we use Theorem 2.12 and its implications. Denote by $\omega^\pm := \omega^{J^\pm}$.

We have seen that if J is integrable, then it determines an orthonormal basis of Λ^+ such that W^+ has the form (18), in particular α is an eigenvector for the middle eigenvalue of W^+. It implies that

$$\alpha_0^+ = \pm\alpha_0^-, \tag{21}$$

where $\alpha^\pm := (d\theta^\pm)_+$, $\alpha_0^\pm$ are the unit forms determined by $\alpha^\pm$ and $\mathcal{J}^\pm$ are the complex structures on the orthogonal complements of $\mathbb{R}J^\pm$ in Λ^+M. Note that these spaces are distinct, and (21) implies that their intersection is spanned by α_0^+.

Therefore, $\{\omega^+/\sqrt{2}, \mathcal{J}^+\alpha_0^+\}$ and $\{\omega^-/\sqrt{2}, \mathcal{J}^-\alpha_0^-\}$ are two orthonormal basis of the same 2-dimensional plane, therefore either

$$\begin{pmatrix} \frac{\omega^+}{\sqrt{2}} \\ \mathcal{J}^+\alpha_0^+ \end{pmatrix} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} \frac{\omega^-}{\sqrt{2}} \\ \mathcal{J}^-\alpha_0^- \end{pmatrix}$$

(the two basis induce the same orientation and are thus rotated by the rotation matrix R_τ above), or

$$\begin{pmatrix} \frac{\omega^+}{\sqrt{2}} \\ \mathcal{J}^+\alpha_0^+ \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ \sin\tau & -\cos\tau \end{pmatrix} \begin{pmatrix} \frac{\omega^-}{\sqrt{2}} \\ \mathcal{J}^-\alpha_0^- \end{pmatrix},$$

i.e., the two basis are obtained by some reflection S_τ (and induce opposite orientations). Before we compute the transformation of the 2×2 matrix

$$H := \begin{pmatrix} 2a & b \\ b & -a \end{pmatrix}$$

under the above changes of basis, note that the trace $-a$ and determinant $-2a^2 - b^2$ are invariants, hence

$$\kappa^+ = \kappa^- \text{ and } \|\alpha^+\| = \|\alpha^-\|. \tag{22}$$

This means that, if the 2×2 block in (18) has the form (22) for two different basis $\{\omega^+/\sqrt{2}, \alpha_0^+\}$ and $\{\omega^-/\sqrt{2}, \alpha_0^-\}$, then it must be the same matrix, i.e. the rotation R_τ, resp. reflection S_τ must commute with H, therefore its eigenspaces (labeled by their *distinct* eigenvalues) must be invariant. For a rotation R_τ , this is only possible for $\tau \equiv \pi \pmod{2\pi}$ (which corresponds to the trivial $J^- = \pm J^+$). Thus the only non-trivial orthogonal transformation that could transform H into itself is a reflection about one of the eigenspaces of H (the other reflection would be equal to minus the first one).

Therefore, if $J^\pm$ are integrable orthogonal complex structures on a 4-manifold with non-degenerate W^+, either they coincide up to sign or they are related by a reflection given by W^+ alone. Therefore, up to sign, there are at most two compatible complex structures on (M, g).

Equation (20) follows from (22) and the last relation from Theorem 2.12. From (21) and (22) we only obtain

$$(d\theta^+)_+ = \pm(d\theta^-)_+.$$

In order to show that the correct sign is -1, we note that the orientation induced on Λ^+M by the basis $\{\omega^J, \alpha_0, \mathcal{J}\alpha_0\}$ is independent on the complex structure $\mathcal{J} \in \Lambda^+M$. As the reflection S_τ relating $\{\omega^-, \mathcal{J}^-\alpha_0^-\}$ to $\{\omega^+, \mathcal{J}^+\alpha_0^+\}$ is orientation-reversing, in order to complete this 2×2 transformation to an orientation-preserving 3×3 orthogonal transformation T of Λ^+M, we need to add a -1 on the diagonal:

$$T = \begin{pmatrix} S_\tau & 0 \\ 0 & -1 \end{pmatrix}.$$

It follows that $\alpha_0^+ = -\alpha_0^-$, thus $(d\theta^+)_+ + (d\theta^-)_+ = 0$, as claimed. □

If we look for compatible complex structures on a conformal 4-manifold, we have seen that, if W^+ is non-degenerate, a bihermitian structure is the most we can expect, and in this case, the compatible complex structures are related by (19-20). These relations trivially hold on the set where W^+ is degenerate, but not zero, because in this case there is only one compatible complex structure (up to sign), namely the eigenvector of norm $\sqrt{2}$ for the simple eigenvalue $2a$ (the other double eigenvalue is $-a$). They also hold on the subset where $W^+ = 0$, because Theorem 2.12 implies that on this subset $\kappa^J = 0$ and $(d\theta^J)_+ = 0$ for any orthogonal complex structure.

Note, however, that if W^+ vanishes on an open set U, there are infinitely many *local* complex structures on U compatible with the metric an the orientation (see Section 5).

A theorem by Pontecorvo [14] says that the open subset U where two orthogonal complex structures $J^\pm$ are linearly independent (as elements in Λ^+M) is dense in M. As a consequence, the conformal structure is determined by a non-trivial bihermitian structure on M: Indeed, if $J^\pm$ are linearly independent on U, the commutator $K := [J^+, J^-]$ is a skew-symmetric endomorphism that doesn't vanish on U, and, together with J^+ and J^-, spans Λ^+M (the identifications are made for any metric g compatible with both $J^\pm$). It is easy to see that the space E^+ spanned by K, J^+, J^- is stable under the commutator of endomorphisms and that each element of norm[1] $1/4$ is an almost complex structure which should be orthogonal w.r.t. to any bihermitian metric g. But this fixes the conformal structure on the open dense set U, hence on M.

We end this section with the following concluding remarks (see [14] for details):

Remark 2.16. (1) *The relations (19-20) hold for any two compatible integrable complex structures on a given oriented Riemannian 4-manifold (M, g).*
(2) *The subset where $J^\pm$ are linearly independent is open and dense.*
(3) *A non-trivial bihermitian structure determines the underlying conformal structure.*
(4) *If (M, g) admits three independent compatible complex structures, then $W^+ \equiv 0$ (the manifold is* anti-self-dual*).*

3. Proof of theorem 2.1

Let (M, J^+) be a compact complex surface with b_1 even and suppose that there exists a J^+-hermitian metric g on M and another g-orthogonal complex structure J^- that induces the same orientation as J^+, i.e. $(M, g, J^\pm)$ is bihermitian.

In the following Lemma, we use the notion of a *standard metric* (also called *Gauduchon metric*):

Definition 3.1. *Let (M, c) be a conformal manifold and ∇ a fixed Weyl structure on M. A metric $g \in c$ is* standard *for ∇ if $\delta^g\theta = 0$, where θ is the Lee form of ∇ w.r.t. g.*

We will use the case where $\nabla = \nabla^J$ is the canonical Weyl structure of a complex structure on a 4-dimensional manifold. In this case, we say that g is standard for J.

An deep result by Gauduchon [10] states that if M is compact, there is always a standard metric for a given Weyl structure and this standard metric is unique up to a constant rescaling. Moreover, in the case of a complex hermitian surface (M, c, J) with b_1 even, another result by Gauduchon states that the Lee form of J w.r.t. the standard metric is co-exact [10], see also [4], Proposition 1.

Lemma 3.2. [3] *Let $(M, c, J^\pm)$ be a bihermitian structure on a compact 4-manifold M with b_1 even. Then the standard metric g of J^+ is also standard for J^- and their Lee forms satisfy*

$$\theta^+ + \theta^- = 0. \tag{23}$$

Proof. From (19) we have $d(\theta^+ + \theta^-)_+ = 0$, which implies that $\theta^+ + \theta^-$ is closed (as M is compact). Let g be the standard metric of J^+, thus $\delta\theta^+ = 0$. Let β be the harmonic part in the Hodge decomposition of $\theta^+ + \theta^-$, i.e.

$$\theta^+ + \theta^- = \beta + da,$$

[1] the norm on the space of endomorphisms, equal to $A \mapsto tr(A^tA)$, is independent on any metric

where a is some function. From [10], we have that $\theta^+ = \delta\alpha$, for some 2-form α. Therefore

$$\theta^- = da + \beta + \delta\alpha$$

is the Hodge decomposition of θ^-, i.e. these three components are L^2-orthogonal:

$$\int_M \|\theta^-\|^2 v_g = \int_M \|\beta + da - d\alpha\|^2 v_g = \int_M \left(\|\beta\|^2 + \|da\|^2 + \|d\alpha\|^2\right) v_g.$$

On the other hand, from (20), we have

$$\int_M \|\theta^-\|^2 v_g = \int_M \|\theta^+\|^2 v_g = \int_M \|\delta\alpha\|^2 v_g,$$

thus $\beta \equiv 0$ and $da \equiv 0$, i.e., $\theta^+ + \theta^- = 0$. □

Recall now that, by definition of the Lee form, $d\omega^\pm = -2\theta^\pm \wedge \omega^\pm$, therefore, from (10), we have

$$d^c_\pm \omega^\pm = 2J^\pm \theta^\pm \wedge \omega^\pm = 2 * \theta^\pm. \tag{24}$$

Let $M, c, J^\pm$ be a bihermitian 4-manifold with b_1 even. Let g be the standard metric for both J^+ and J^-. Then the above equation and (23) imply

$$d^c_+ \omega^+ + d^c_- \omega^- = 0$$

and

$$dd^c_\pm \omega^\pm = 2d * \theta^\pm = -2\delta^g \theta^\pm = 0, \tag{25}$$

i.e., $(M, g, J^\pm)$ is generalized Kähler.

This finishes the proof of Theorem 2.1

As a consequence, in [3], the authors classify all compact complex surfaces with b_1 even that admit bihermitian metrics. The bihermitian structures with b_1 odd have also been intensively studied (see the citations in [4]), but they are usually not generalized Kähler. If, additionally, the bihermitian structure is *anti-self-dual*, then the classification is complete (see section 5).

4. Proof of Theorem 2.3

First note that the first integrability condition (1) for a generalized Kähler manifold is equivalent, in case of a 4-manifold, to

$$J^+\theta^+ \wedge \omega^+ + J^-\theta^- \wedge \omega^- = 0,$$

where we have used the notations from Section 2.1. Using (24) and the fact that J^+ is self-dual, and J^- anti-self-dual, we obtain thus that (1) is equivalent to

$$\theta^+ = \theta^-.$$

That means, if we want to obtain an ambihermitian metric on M such that it is associated to a generalized Kähler structure, we need to look for orthogonal complex structures $J^\pm$ that have the same Lee form, or, equivalently, they have the same canonical Weyl structure ∇ (such that $\nabla J^\pm = 0$).

In that case, $Q := -J^+J^-$ (which is a symmetric involution, as J^+ and J^- commute) is also ∇-parallel, therefore the 2-dimensional distributions

$$H^\pm := \{X \in TM \mid J^+X = \pm J^-X\} = \{X \in TM \mid QX = \pm X\},$$

are ∇-invariant. But if a distribution is invariant w.r.t. a torsion-free connection, it is integrable, i.e. it is tangent to a foliation.

We have thus, that an ambihermitian structure $(M^4, g, J^{\pm})$ undelying a generalized Kähler structure implies the existence of 2 transversal foliations by surfaces which, because $H^{\pm}$ are $J^{\pm}$-invariant, are $J^{\pm}$-holomorphic as well.

That proves one direction of the claimed equivalence.

Conversely, let (M, J) be a complex surface admitting two transversal holomorphic foliations by complex curves. Let $H^{\pm}$ be the distributions tangent to these foliations. We set $Q := Id_{H^+} - Id_{H^-}$ and denote $J^+ := J$ and $J^- := QJ^+$. Then it is easy to check that J^- is an integrable complex structure on M.

Let g be a metric on M such that

(1) H^+ and H^- are orthogonal for g
(2) the restriction of g to $H^{\pm}$ is hermitian w.r.t. J

We claim that this metric is ambihermitian w.r.t. $J^{\pm}$ and that there exists a unique Weyl structure ∇ such that the distributions $H^{\pm}$ are ∇-invariant.

Indeed, we only need to prove the second claim. Locally, (M, J) looks like a product of two complex curves $C^{\pm}$, i.e. there exists a covering of M with open sets $U \simeq C^+ \times C^-$ (product of complex manifolds). The projections $\pi^{\pm} : U \to C^{\pm}$ are holomorphic maps, therefore the tangent maps $\pi^{\pm}_* : TU \to TC^{\pm}$ induce conformal equivalences between $(\ker \pi^{\pm}_*) \perp$ and $TC^{\pm}$. This means that $[g]$ is a *conformal product* of some conformal structures $[g^{\pm}]$ on $C^{\pm}$ (see [6]), more precisely

$$g = e^{2f^+} g^+ + e^{2g^-} g^-, \tag{26}$$

where $g^{\pm}$ are randomly chosen[2] hermitian metrics on $C^{\pm}$ and $f^{\pm} : U \to \mathbb{R}$. Let

$$\theta := -d^+ f^- - d^- f^+$$

be the sum of $d^- f^+$, the derivative of f^+ in the H^--directions (tangent to the leaves of U biholomorphic to C^-), and the H^+-derivative of f^-, $d^+ f^-$. We claim:

Lemma 4.1. [6] *The Weyl structure $\nabla := \nabla^g + \tilde{\theta}$ is the unique Weyl structure on U such that the distributions $H^{\pm}$ are ∇-parallel.*

Proof. Let $X, Y \in H^+$ be vectors in $T_{(x^+,x^-)}U \simeq T_{x^+}C^+ \times T_{x^-}C^-$. Because $X, Y \in H^+$ we have that the second components of X, Y are zero. Note that the component of $\nabla_X Y$ in H^- does not depend on the extensions of X, Y to vector fields in H^+ (the argument is similar to the definition of the (second) fundamental form of an immersed submanifold).

Extend then X, Y to be vector fields in H^+ that project on vector fields on C^+ (or lift to U some vector fields X^+, Y^+ from C^+) such that $X^+, Y^+ \in TC^+$ are parallel at $x^+ \in C^+$ w.r.t. the Levi-Civita connection of g^+ on C^+. Then

$$C^- \ni y^- \longmapsto g(X, Y)_{(x \cdot y^-)} = e^{2f^+(x^+, y^-)} g^+(X^+, y^+)_{x^+}$$

is the product of the e^{2f^+} factor with a *constant* factor. Moreover, $[X, Y]_{(x^+, y^-)} = 0$.

Let Z be a lift of a vector field on C^-. It follows that X and Y commute with Z: $[X, Z] = [Y, Z] = 0$. Then we compute, using the Koszul formula:

$$2g(\nabla^g_X Y, Z) = -Z.(e^{2f^+}) g^+(X, Y) = 2\theta(Z) g(X, Y). \tag{27}$$

Compute then, using (28-29),

$$\nabla_X Y = \nabla^g_X Y + \theta(X) Y + \theta(Y) X - \theta^{\sharp} g(X, Y),$$

[2] all hermitian metrics on a complex curve are conformally equivalent

therefore $\nabla_X Y \in H^+$. Similarly we get that, for $X, Y \in H^-$, $\nabla_X Y \in H^-$. This is enough to prove that $H^\pm$ are ∇-invariant.

To show that ∇ is the only Weyl structure that leaves $H^\pm$ invariant, consider $\nabla' = \nabla + \tilde{\eta}$ and suppose that ∇' also has this property. By choosing $X, Y \in H^+$, $\nabla'_X Y \in H^+$ implies that η vanishes on H^-, and $\nabla'_X Y \in H^-$ $\forall X, Y \in H^-$ implies $\eta|_{H^+} = 0$. □

Remark 4.2. *We have actually shown that M is locally a* conformal product *of two complex curves, which abstractly means that the (local) projections $\pi^\pm$ are* conformal submersions, *and more concretely that the metric is given by a formula like (26). The previous Lemma holds in general: any conformal product admits a unique Weyl structure that preserves the product structure. See* [6] *for details.*

Now, ∇ is a Weyl structure for which $H^\pm$ are parallel distributions, i.e. $\nabla Q = 0$. Moreover, as ∇ is a conformal connection, it preserves the conformal structure on $H^\pm$ which, together with the orientation of $H^\pm$, is equivalent to the complex structre J on $H^\pm$. Thus $J = J^+$ is ∇-parallel, and $J^- = QJ^+$ is parallel as well.

Therefore, the Lee forms of $J^\pm$ are both equal to θ, which is equivalent to the first generalized Kähler condition (1).

The second generalized Kähler condition,

$$dd^c_+ \omega^+ = 0,$$

see also (2) for the untwisted version, is equivalent (like in (25)) to

$$\delta\theta = 0,$$

resp. that θ is co-exact. But $\delta\theta = 0$ characterizes the standard metric in the conformal class.

Therefore, if we denote by $h \in [g]$ the standard metric on the compact conformal manifold $(M, [g])$ w.r.t. ∇, then $(M, h, J^\pm)$ is a generalized Kähler manifold.

It is untwisted iff θ is co-exact. But a standard metric has co-exact Lee form iff the first Betti number is even ([4], Proposition 3).

This finishes the proof of Theorem 2.3. Note that one of the implications (generalized Kähler $\Rightarrow$ transverse foliations) holds in higher dimensions, too, [4]. The difficulty of the converse implication in higher dimensions lies in the fact that not every hermitian metric that is adapted to a hermitian product is a conformal product, as it is if the factors are complex 1-dimensional.

On the other hand, in dimension 4 the characterization of compact complex surfaces that admit an ambihermitian metric underlying a generalized Kähler structure yields a complete classification of those surfaces [4].

5. Anti-self-dual bihermitian surfaces

5.1. **Twistor theory.** Let (M, g) be an oriented Riemannian 4-manifold. If $W^- \equiv 0$ we say that g is *self-dual* and if $W^+ \equiv 0$ we say that g is *anti-self-dual* (in short ASD). Of course, this condition depends only on the conformal class $[g]$ of g. The main tool of *local* investigation of orthogonal complex structures on (M, g) is the *twistor theory,* [5]:

Let $Z \to M$ be the sphere bundle of self-dual 2-forms (or endomorphisms) of square norm 2, equivalently the bundle of orthogonal, positive-oriented, pointwise complex structures on M. An orthogonal, positive-oriented almost complex structure J is thus a section of $Z \to M$.

Every Weyl structure ∇ induces a connection on $Z \to M$ that decomposes $T_J Z$ in a vertical part (tangent to the fiber Z_x, if $J \in \Lambda^2_x M$) and a horizontal part, isomorphic to $T_x M$. As the fiber is a sphere, it admits a canonical complex structure J_0. The horizontal part, being isomorphic to T_xM, also has a complex structure, namely J itself. We add these to get an almost complex structure $\mathcal{J}$ on Z. The main points of the twistor theory are (see [5],[7] for details):

(1) $\mathcal{J}$ is independent on the Weyl structure and depends on the conformal structure $[g]$ alone.
(2) $\mathcal{J}$ is integrable iff $W^+ \equiv 0$.
(3) An integrable complex structure J on M is equivalent to a section J in Z that is $\mathcal{J}$-holomorphic (i.e., the tangent space of $J(M)$ is $\mathcal{J}$-invariant).

We see thus that, if $W^+ = 0$, then there are infinitely many local complex structures on M compatible with g and the orientation. Global complex structures need not exist (see the conformally flat, hence self-dual, round sphere S^4).

5.2. **ASD bihermitian metrics.** The global compatible complex structures on an ASD manifold turn out to be very special:

Proposition 5.1. [14] *Let M be a compact oriented 4-manifold. An ASD metric g which is bihermitian w.r.t. the complex structures $J^\pm$ is also* locally conformally Kähler *(l.c.K.) w.r.t. both these metrics.*

A metric g is *locally conformally Kähler* (l.c.K.) iff it admits local Kähler metrics h (i.e., for which ω^h is closed) in the conformal class. If, additionally, $b_1(M)$ is even, then g is globally conformally Kähler. Equivalently, a metric g on (M, J) is l.c.K. iff the Lee form of J w.r.t. g is closed; it is globally conformally Kähler iff that Lee form is exact.

Proof. In Theorem 2.12, since $W^+ = 0$, we conclude that $(d\theta^\pm)_+ = 0$ and hence both $\theta^\pm$ are closed. But this means that $\theta = df$ on some open set, and there the local metric $h := e^{2f}g$ will have

$$d\omega^h = d(e^f\omega^g) = 2df\omega^h - 2e^f\theta \wedge \omega^g = 0.$$

We also conclude that $\kappa^J = 0$ for all compatible complex structures J (i.e., there are local Kähler scalar-flat metrics in the conformal class).

Conversely, assume g is l.c.K. w.r.t both $J^\pm$ and assume by the Theorem of Pontecorvo [14] cited at the end of Section 2.4 that $J^+ \neq \pm J^\pm$ everywhere.

On the other hand, for the complex structure J^+, the curvature of a Kähler metric (M, g, J^+) takes values only in $\Lambda^-M \oplus \mathbb{R}J^+$, in particular the Weyl tensor must be degenerate and, if $\kappa^+ \neq 0$, then J^- should be collinear to J^+, contradiction. But $\kappa^+ = 0$ implies $W^+ = 0$.

The last claim follows from Lemma 3.2: Indeed, the Hodge decomposition of the closed form θ^+ contains an exact form and a harmonic form. But g is standard, thus θ^+ is co-exact, therefore it must vanish. □

Therefore, if $(M, [g], J^\pm)$ is bihermitian ASD and $b_1(M)$ is even, then the standard metric to $J^\pm$ is actually Kähler (w.r.t. both $J^\pm$), but this means that $J^\pm$ are both parallel w.r.t. the Levi-Civita connection of g, and since they are linearly independent, the whole Λ^+M is spanned by global parallel sections. i.e. (M, g) is *hyperkähler* (see also [14]).

If, on the other hand, $b_1(M)$ is odd, then it can be shown that $b_1 = 1$ [8], [13]. This implies that the conclusion of Lemma 3.2 also holds in this case:

Proposition 5.2. [14] *Let $(M, c, J^\pm)$ be a compact ASD bihermitian manifold with $b_1(M)$ odd. Then the J^+-standard metric g is also standard for J^- and if the Lee forms w.r.t. g are*

$\theta^\pm$, then

$$\theta^+ = \pm\theta^-.$$

Proof. As in the proof of Lemma 3.2 we show that $\theta^\pm$ have the same L^2-norm w.r.t. any metric in the conformal class. Then, because they are both closed 1-forms, and that $H^1(M)$ is 1-dimensional, it follows that

$$[\theta^+] = k[\theta^-],$$

for some $k \in \mathbb{R}$.

If $k = 1$, let g be J^+-standard and then $\theta^+ = \theta^-$ by Hodge theory, as in Lemma 3.2.

If $k \neq 1$, then $\theta^+ = k\theta^- + df$ and after a conformal change with the factor e^{2h} with $h := f/(k-1)$ we end up with the metric g' and the Lee forms $\eta^\pm = \theta^\pm + dh$ that satisfy

$$d\eta^+ = kd\eta^-,$$

which implies (by taking the L^2-norms), that $k^2 = 1$, thus $k = -1$ and (20) applied for g' implies that

$$\|\eta^+\|^2 - \delta\eta^+ = \|\eta^-\|^2 - \delta\eta^- = \|\eta^+\|^2 + \delta\eta^+,$$

thus $\delta\eta^+ = 0$ and g' is standard for J^+ (and thus for J^-, too). □

These results can be used to classify completely all compact complex surfaces that admit ASD bihermitian metrics [14].

6. Appendix: Riemannian and conformal geometry

6.1. **Weight bundles.** Let M be a n-dimensional manifold with density bundle $|\Lambda|M$. This is an oriented, hence trivial real line bundle, whose positive sections are the *volume elements* of M, allowing the integration of functions on the manifold; it is isomorphic, if M is oriented, with $\Lambda^n M$, the bundle of n-forms on M.

A *conformal structure* on M is a *non-degenerate* symmetric bilinear form on TM with values in the line bundle $L^2 := L \otimes L$, also seen as a section $c \in C^\infty(S^2M \otimes L^2)$. (here we denote by S^2M the bundle of symmetric bilinear forms on TM.)

In this paper, we restrict to the case where c is *positive definite* (note that L is canonically oriented), thus (M, c) is a conformal *Riemannian* manifold.

Remark 6.1. *Each positive section l of L trivializes it, hence*

$$g_l := cl^{-2} : TM \otimes TM \to \mathbb{R}$$

is a Riemannian metric on M. If $l' := e^f l$ is another positive section (for $f : M \to \mathbb{R}$ a smooth function), then the metric $g_{l'} = e^{-2f} g_l$ is conformally equivalent to g_l, and they belong to the same conformal class, defined by c.

6.2. **Weyl structures.** Unlike in (semi-) Riemannian geometry, a conformal manifold does not carry a canonical affine connection. Instead, there is a family of adapted connections, the *Weyl structures*:

Definition 6.2. *A Weyl structure ∇ on a conformal manifold (M, c) is a torsion-free,* conformal *connection on TM,* i.e. $\nabla c = 0$.

Remark 6.3. *The expression ∇c has the following meaning: ∇ induces a connection on all associated bundles to TM, in particular on $|\Lambda|M$ and on L itself, therefore ∇ induces a connection on $S^2M \otimes L^2$, and ∇c is the covariant derivative of c with respect to this connection.*

The fundamental theorem of conformal geometry can now be stated:

Theorem 6.4. *Let (M, c) be a conformal manifold and denote, for a Weyl structure ∇, by ∇^L the connection induced by ∇ on L. The correspondence*

$$\{\textit{Weyl structures on } M\} \longrightarrow \{\textit{connections on } L\},$$

given by $\nabla \mapsto \nabla^L$ is one-to-one.

More precisely, we have the following *conformal Koszul formula*

$$\begin{aligned} 2c(\nabla_X Y, Z) &= \nabla^L_X(c(Y,Z)) + \nabla^L_Y(c(X,Z)) - \nabla^L_Z(c(X,Y)) + \\ &\quad c([X,Y],Z) + c([X,Z],Y) - c([Y,Z],X). \end{aligned}$$

Why do we need Weyl structures: first, because any integrable c–orthogonal complex structure on a 4-manifold is parallel for a (uniquely determined) Weyl structure. Second, the formalism is helpful when dealing with the way curvature is modified after a conformal change of the metric (see Proposition 6.7)

An important consequence of Theorem 6.4 and (28) is the relation between two Weyl structures: as the difference between two linear connections ∇'^L and ∇^L in the line bundle L is a 1–form θ, the difference between the corresponding Weyl structures ∇' and ∇ must be given by a tensor that depends linearly on θ. More precisely, applying (28) we get:

$$\nabla'_X Y - \nabla_X Y = \tilde{\theta}_X Y := (\theta \wedge X)(Y) + \theta(X)Y, \tag{28}$$

where $\theta \wedge X$, the wedge product of a 1–form and a vector, is the skew-symmetric endomorphism of TM defined by

$$(\theta \wedge X)(Y) := \theta(Y)X - c(X,Y)\theta. \tag{29}$$

Here, note that the 1–form θ is a section of $T^*M \simeq TM \otimes L^{-2}$ and thus $c(X,Y)\theta$ is a section of $TM \otimes L^{-2} \otimes L^2 \simeq TM$.

Convention. In this paper, we use the usual identifications of vectors and co-vectors, of 2–forms and skew-symmetric endomorphisms of the tangent space from Riemannian geometry, but we keep in mind that, in the conformal setting, such an identification may involve the addition of a weight factor L^k. As a rule, vectors have weight 1, co-vectors have weight -1 and an (r,s) tensor $A \in \otimes^r T^*M \otimes \otimes^s TM$ has weight $s-r$. The identifications can be made then as in Riemannian geometry, but we need to check that, in the end, the conformal weights are the same.

In the particular case where $\nabla = \nabla^g$ is the Levi-Civita connection of a Riemannian metric g, the form θ above is called the *Lee form* of ∇' w.r.t. g. Note that, if ∇' is the Levi-Civita connection of $g' = e^{-2f}g$, $\theta = df$. It turns out that the Lee form of a fixed Weyl structure ∇ changes with an exact form when the referring metric changes, therefore $d\theta$ depends on ∇ alone (not surprising since $d\theta$ is the curvature of ∇^L and, as we shall see below, the Faraday form of ∇).

Remark 6.5. *The difference tensor $\tilde{\theta}_X$ is an endomorphism of TM which lies in the adjoint bundle $\mathfrak{co}(M)$ of the bundle of conformal frames $CO(M)$; this adjoint bundle is split in the bundle of skew-symmetric endomorphisms of TM and the line bundle of multiples of the identity $Id : TM \to TM$. The corresponding components of $\tilde{\theta}_X$ are $(\theta \wedge X)$ and, respectively $\theta(X)I$, and they act on a tensor by the Lie algebra action of $\mathfrak{co}(n)$ on the corresponding tensor powers of $\mathbb{R}^n$ and of $(\mathbb{R}^n)^*$. In particular, if the action of the skew-symmetric part is the usual $\mathfrak{so}(n)$ action and thus all identifications between vectors and co-vectors are allowed (disregarding the conformal weights), the action of Id on a tensor is the multiplication by its conformal weight.*

For example, for a section l of L^k we have

$$\nabla'_X l - \nabla_X l = k\theta(X)l, \tag{30}$$

for a 1–form α, we have

$$\nabla'_X\alpha - \nabla_X\alpha = \tilde{\theta}_X\alpha = (\theta \wedge X)\alpha - \theta(X)\alpha = -\alpha \circ (\theta \wedge X) - \theta(X)\alpha,$$

and, for an endomorphism $A : TM \to TM$, we have

$$\nabla'_X A - \nabla_X A = [\theta \wedge X, A], \tag{31}$$

where the square bracket is the commutator of endomorphisms. Note that the factor $\theta(X)Id$ acts trivially on the weightless tensor A.

6.3. **Curvature.** The curvature of a Weyl structure ∇ is defined by

$$R^{\nabla}_{X,Y}Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

and can be seen as a 2–form with values in $\mathfrak{co}(M)$. The identity component of this 2–form is related to the *Faraday* form $F^{\nabla} \in \Lambda^2 M$, which is the curvature of the connection ∇^L, more precisely

$$R^{\nabla}_{X,Y} = (R^{\nabla}_{X,Y})^{skew} + F^{\nabla} \otimes Id.$$

R^{∇} satisfies the Bianchi identities (tensorial and differential), the tensorial (or the *first*) Bianchi identity being:

$$R^{\nabla}_{X,Y}Z + R^{\nabla}_{Y,Z}X + R^{\nabla}_{Z,X}Y = 0, \ \forall X, Y, Z \in TM, \tag{32}$$

and this identity is equally satisfied by the tensor

$$\tilde{F}^{\nabla} := -\frac{1}{2}F^{\nabla} \wedge Id + F^{\nabla} \otimes Id. \tag{33}$$

Notation. The *suspension* of a bilinear form $A \in T^*M \otimes T^*M$ by the identity is the following tensor

$$(A \wedge Id)_{X,Y} := A(Y, \cdot) \wedge X - A(X, \cdot) \wedge Y, \tag{34}$$

where we note, as in (29), that the wedge product between a vector and a 1–form (here $A(Y, \cdot)$) is a skew-symmetric endomorphism.

The curvature R^{∇} decomposes thus in two components, both of which satisfy the tensorial Bianchi identity (32):

$$R^{\nabla} = R^{\nabla,Riem} + \tilde{F}^{\nabla}.$$

$R^{\nabla,Riem}$ is a 2–form with values in $\mathfrak{so}(M)$, the bundle of skew-symmetric endomorphisms, and satisfies the Bianchi identity (32), therefore it is a Riemannian curvature tensor and, for $\dim M \geq 3$, it decomposes thus as

$$R^{\nabla,Riem} = W + \rho^{\nabla} \wedge Id,$$

where ρ^{∇} is a symmetric bilinear form on TM and W is the trace-free part of $R^{\nabla,Riem}$ and is called the *Weyl tensor* of (M, c) (that it depends only on c will become clear in the Proposition 6.7 below).

Recall that if $F^{\nabla} = 0$, and thus ∇ is the Levi-Civita connection of a (local) metric g, then $R^{\nabla} = R^0$ and, in this case, ρ^{∇} is the Schouten-Weyl tensor (a renormalization of the Ricci tensor) of g. If ∇ is a general Weyl structure, we have:

Proposition 6.6. *The curvature R^{∇} of a Weyl structure ∇ on a conformal manifold (M, c) decomposes as*

$$R^{\nabla} = R^{\nabla,Riem} + \tilde{F^{\nabla}} = \left(\rho^{\nabla} \wedge Id + W\right) + \left(\frac{1}{2}F^{\nabla} \wedge Id + F^{\nabla} \otimes Id\right),$$

where W is the Weyl tensor, F is the Faraday (exact) 2-form, and ρ^∇ is the symmetric Schouten-Weyl tensor. Equivalently,

$$R^\nabla = h^\nabla \wedge Id + W + F^\nabla \otimes Id,$$

where $h^\nabla = \rho^\nabla - \frac{1}{2}F^\nabla$ is the full Schouten-Weyl tensor of ∇.

We give now the transformation rule for the curvature tensors corresponding to two Weyl structures.

Proposition 6.7. *For two Weyl structures $\nabla' = \nabla + \tilde{\theta}$, the corresponding Schouten-Weyl tensors are related by:*

$$h^{\nabla'} - h^\nabla = -\nabla\theta + \theta \otimes \theta - \frac{1}{2}c(\theta,\theta)c. \tag{35}$$

Moreover, the Weyl tensor W is independent on the Weyl structure and depends on the conformal structure only. The Faraday curvature changes as follows:

$$F^{\nabla'} = F^\nabla + d\theta.$$

Proof. Let us compute the curvature of ∇' by deriving (28), and using (31) at a point where all ∇–derivatives of the involved vector fields vanish:

$$\nabla'_X\nabla'_Y Z = (\nabla_X\theta \wedge Y)(Z) + [\theta \wedge X, \theta \wedge Y](Z) + (\nabla_X\theta)(Y)Z. \tag{36}$$

Thus

$$R^{\nabla'}_{X,Y}Z = R^\nabla_{X,Y}Z - (\nabla\theta \wedge Id)_{X,Y}Z + 2[\theta \wedge X, \theta \wedge Y](Z) + d\theta(X,Y)Z.$$

We compute directly

$$\begin{aligned} [\theta \wedge X, \theta \wedge Y] &= (\theta \otimes \theta)(Y) \wedge X - (\theta \otimes \theta)(X) \wedge Y - c(\theta,\theta)X \wedge Y \\ &= \frac{1}{2}\left(((\theta \otimes \theta) \wedge Id)_{X,Y} - \frac{1}{2}c(\theta,\theta)(c \wedge Id)_{X,Y}\right), \end{aligned}$$

that implies

$$F^{\nabla'} = F^\nabla + d\theta, \quad W^{\nabla'} = W^\nabla$$

and the claimed result. □

Remark 6.8. *While the Schouten-Weyl tensor is only defined by the curvature decomposition for $\dim M \geq 3$, the curvature itself, and the Ricci tensor in particular, are well-defined in all dimensions, and we have*

$$h^\nabla = \frac{1}{n-2}Sym_0(Ric^\nabla) + \frac{1}{2n(n-1)}Scal^\nabla \cdot c - \frac{1}{2}F^\nabla,$$

where the Ricci tensor is defined by the usual trace of the curvature:

$$\mathrm{Ric}^\nabla(X,Y) := \mathrm{tr}(R^\nabla_{\cdot,X}Y).$$

Equivalently,

$$\mathrm{Ric}^\nabla = (n-2)\rho^\nabla + \mathrm{tr}_c\rho^\nabla \cdot c - \frac{n}{2}F^\nabla,$$

which also implies

$$\mathrm{Scal}^\nabla = 2(\mathrm{n}-1)\mathrm{tr}_c\rho^\nabla = 2(\mathrm{n}-1)\mathrm{tr}_c\mathrm{h}^\nabla.$$

Note that, for ∇ a general Weyl structure, it is important which trace of the curvature we consider: the other trace (in the arguments corresponding to the vectors X and Y in the above expression) is a related Ricci-like tensor:

$$\widetilde{\mathrm{Ric}^\nabla}(X,Y) := \mathrm{tr}_c\left(c(R^\nabla_{X,\cdot}\cdot, Y)\right),$$

which has the same symmetric part, but the skew-symmetric part changes:

$$\widetilde{\mathrm{Ric}^\nabla} = \mathrm{Ric}^\nabla - 2F^\nabla. \tag{37}$$

Since the skew part of Ric^∇ is $-(n/2)F^\nabla$, it means that $\widetilde{\mathrm{Ric}^\nabla}$ is *symmetric* if $n = 4$.

Corollary 6.9. *The scalar curvature of a Weyl structure ∇, with Lee form θ with respect to a Riemannian metric g (i.e., $\nabla = \nabla^g + \tilde{\theta}$) is given by*

$$\mathrm{Scal}^\nabla = \mathrm{Scal^g} + (\mathrm{n}-1)\left(2\delta^{\mathrm{g}}\theta + (2-\mathrm{n})\|\theta\|^2_{\mathrm{g}}\right).$$

References

[1] V. Apostolov, P. Gauduchon, *The Riemannian Goldberg-Sachs Theorem*, Internat. J. Math. **8** (1997), 421-439.

[2] V. Apostolov, P. Gauduchon, *Self-dual Einstein Hermitian 4-manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **1** (2002), 203-243.

[3] V. Apostolov, P. Gauduchon, G. Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414-428; *Corrigendum* **92** (2006), 200-202.

[4] V. Apostolov, M. Gualtieri, *Generalized Kähler Manifolds, Commuting Complex Structures, and Split Tangent Bundles*, Commun. Math. Phys.**271** (2007), 561-575.

[5] M.F. Atiyah, N.J. Hitchin, I.M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. Lond. A **362** (1978), 425-461.

[6] F. Belgun, A. Moroianu, *Weyl-parallel forms, conformal products and Einstein-Weyl manifolds*, Asian. J. Math. **15** (2011), 499-520.

[7] A.L. Besse, *Einstein manifolds*, Springer 1987.

[8] C.P. Boyer, *Conformal duality and compact complex surfaces* , Math. Ann. **274** (1986), 517-526.

[9] P. Gauduchon, *Fibrés Hermitiens à endomorphisme de Ricci non negatif*, Bull. Soc. Math. France, **105** (1970), 113-140.

[10] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte* , Math. Ann. **267** (1984), 495-518.

[11] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. reine angew. Math. **469** (1995), 1-50.

[12] M. Gualtieri, *Generalized complex geometry*, D. Phil. Thesis, University of Oxford, 2003, http://arxiv.org/list/math.DG/0401221 (2003).

[13] M. Pontecorvo, *Hermitian surfaces and a twistor space of algebraic dimension 2*, Proc. Am. Math. Soc. **113** (1991), 177-186.

[14] M. Pontecorvo, *Complex structures on Riemannian four-manifolds*, Math. Ann. **309** (1997), 159-177.

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