# GAUGE THEORY

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## 1. Fiber bundles

**Definition 1.1.** *Let $G$ be a Lie group, $\rho : G \times F \to F$ a smooth left action of $G$ on a manifold $F$, and $M$ a manifold. A* fiber bundle $E \overset{\pi}{\to} M$ with structure (gauge) group $G$ and fiber $F$ on the manifold $M$ *is a submersion $\pi : E \to M$ such that there exists an atlas $\{(U, \psi_U) \mid U \in \mathcal{U}\}$ of* local trivializations *of $E$, where:*

(1) *$\mathcal{U}$ is a covering of open sets $U \subset M$;*
(2) *$\psi_U : U \times F \to \pi^{-1}(U)$ are diffeomorphisms such that $\pi|_{\pi^{-1}}(U) \circ \psi_U = \mathrm{pr}_1$, where $\mathrm{pr}_1 : U \times F \to U$ is the projection on the first factor. $\psi_U$ is called a* local trivialization *of $E$ over $U \subset M$;*
(3) *For any pair of intersecting open sets $U, V \in \mathcal{U}$, there is a smooth map $g_{UV} : U \cap V \to G$ (called a* transitition function*) such that*
$$\psi_V^{-1} \circ \psi_U(x, f) = (x, \rho(g_{UV}(x), f)),\ \forall (x, f) \in (U \cap V) \times F.$$

**Remark 1.2.** The atlas of local trivializations can be completed with any pair $(U, \psi_U)$ as above, such that, for any $V \in \mathcal{U}$ that intersects $U$, there exist transition functions $g_{UV}$ and $g_{VU}$ satisfying the relations above. It is possible to construct a maximal such an atlas. Note, however, that the fiber bundle structure does not contain *a priori* the information given by a *particular* atlas of trivializations; it is only required that such an atlas exists, and we can suppose, WLOG, that it is maximal. The notion of fiber bundle over $M$ generalizes the product $M \times F$, which is called the *trivial bundle* with fiber $F$ and basis $M$ (the group $G$ is here irrelevant). The main question is whether a given fiber bundle is *trivial* or not, i.e. whether there exists a *global trivialization* $\psi_M : M \times F \to E$ in the maximal atlas.

**Remark 1.3.** The transition functions $g_{UV} : U \cap V \to G$ depend actually on the corresponding local trivializations $\psi_U, \psi_V$ as well; in a maximal atlas it is common to have different pairs $(U, \psi_U), (U, \psi'_U)$ corresponding to the same open set $U$. In order to keep the (convenient) notation from the definition above, we need to use *different* letters for (possibly identical) open sets in $\mathcal{U}$, if the corresponding local trivializations are not the same.

With this convention, the transition functions $\{g_{UV} \mid U, V \in \mathcal{U}\}$ satisfy the following properties:

(1) $g_{UU} : U \to G$ is the constant map $g_{UU} = e \in G$ ($e$ is the neutral element in $G$)
(2) $g_{UV}(x)$ is the inverse element in $G$ of $g_{VU}(x)$
(3) $g_{UV} \cdot g_{VW} = g_{UW}$ on $U \cap V \cap W$, where the dot denotes the multiplication in $G$ of the values of the transition functions.

A collection of $G$-valued functions as above is called a 1-*cocycle* with values in $G$ (the terminology comes from sheaf theory).

**Example 1.4.** For $G := \mathrm{GL}(k, \mathbb{R})$ and $\rho : \mathrm{GL}(k, \mathbb{R}) \times \mathbb{R}^k \to \mathbb{R}^k$ the standard (linear) action of $\mathrm{GL}(k, \mathbb{R})$ on $\mathbb{R}^k$, the resulting bundle $E$ is a *vector bundle* of *rank* $k$ over $M$. In this case the *fibers* $E_x := \pi^{-1}(x)$ (which, in general, are submanifolds in $E$ of codimension equal to $\dim M$) are vector spaces isomorphic to $\mathbb{R}^k$. Each local trivialization $\psi_U$, for $x \in U$, yields such an isomorphism $\psi_U|_{E_X} : E_x \xrightarrow{\sim} \{x\} \times \mathbb{R}^k$, but as new local trivializations can be constructed by multiplying the second factor of $\psi_U$ with some constant element of $\mathrm{GL}(k, \mathbb{R})$, it turns out that the vector bundle structure on $E$ does not identify *canonically* the fibers $E_x$ with $\mathbb{R}^k$. In fact, *all* such isomorphisms are considered.

If we consider the general setting of a fiber bundle with group $G$ and fiber $F$, we conclude that the relevant objects / properties of a fiber in such a fiber bundle correspond to objects / properties of $F$ that are invariant under the action $\rho$ of $G$. We will come back to this principle.

**Definition 1.5.** *A (local)* section *in a fiber bundle* $E \xrightarrow{\pi} M$ *is a map* $\sigma : M \to E$ *(*$\sigma : U \to E$*, for* $U \subset M$ *open in the local case) such that* $\pi \circ \sigma$ *is the identity of* $M$*, resp. of* $U$*.*

**Remark 1.6.** For every local trivialization $(U, \psi_U)$, the inclusions $\sigma_f : U \to U \times F$ ($f \in F$), defined by $\sigma_f(x) := \psi^{-1}(x, f)$, $\forall x \in U$, induce local sections in $E \xrightarrow{\pi} M$.

**Proposition 1.7.** *If the fiber* $F$ *is contractible, then any fiber bundle* $E \xrightarrow{\pi} M$ *with fiber* $F$ *admits global sections.*

**Proposition 1.8.** *If the base manifold* $M$ *is contractible, then there exists a global trivialization* $\psi : E \to M$*.*

The proofs of these propositions can be found in the classical book by N. Steenrod *The topology of fiber bundles*, where, in order to decide if a certain bundle admits or not global sections, the *obstruction theory* is developped.

## 2. Isomorphic bundles, associated bundles, pull-back

Given a fiber bundles $E \xrightarrow{\pi} M$ with structure group $G$ and fiber $F$, we obtain a 1-cocycle with values in $G$, consisting in the collection of transition functions

$$g_{UV} : U \cap V \to G,\ U \cap V \neq \emptyset.$$

Conversely, using such a collection of smooth maps $g_{UV}$ as above, we can "glue together" the sets $U \times F$ to construct a fiber bundle with group $G$ and fiber $F$, corresponding to the action $\rho$ of $G$ on $F$:

Let $E$ be the following set ($\sqcup$ means *disjoint union*)

$$E := \bigsqcup_{U \in \mathcal{U}} U \times F / \sim, \tag{1}$$

where $(x_1, f_1) \in U \times F$ and $(x_2, f_2) \in V \times F$ are *equivalent* w.r.t. $\sim$ iff $x_1 = x_2$ and $f_2 = \rho(g_{UV}(x_1), f_1)$. The conditions for the collection $\{g_{UV}\}_{U,V \in \mathcal{U}}$ imply then that $\sim$ is an equivalence relation and it is easy to check that $E$ is a manifold and the projections from $U \times F$ to $U$ induce a submersion $\pi : E \to M$. Moreover, we also obtain local trivializations $\psi_U$ for all $U \in \mathcal{U}$. We would like to say that, if we take our data $\{g_{UV}\}_{u \in \mathcal{U}}$

from a fiber bundle $E' \xrightarrow{\pi} M$, that the constructed fiber bundle $E$ is *isomorphic* to $E'$. For this, we define first when two bundles are isomorphic:

**Definition 2.1.** *Let $E_i \xrightarrow{\pi_i} M$, $i = 1, 2$ be two fiber bundles on $M$ with structure group $G$, and fiber $F$, (corresponding to the action $\rho : G \times F \to F$). An* isomorphism *of fiber bundles between $E_1$ and $E_2$ is a diffeomorphism $\Phi : E_1 \to E_2$ such that*

(1) $\pi_2 \circ \Phi = \pi_1$ *($\Phi$ maps the fiber of $E_1$ at $x \in M$ to the fiber of $E_2$ at $x$)*
(2) *for a covering $\mathcal{U}$ of $M$, fine enough such that $\forall U \in \mathcal{U}$, there exist local trivializations $\psi^1_U$ of $E_1$, resp. $\psi^2_U$ of $E_2$ over $U$, the map*
$$\psi^2_U \circ \Phi \circ (\psi^1_U)^{-1} : U \times F \to U \times F_2$$
*can be expressed, in terms of a smooth map $g_U : U \to G$ as*
$$(x, f) \mapsto (x, \rho(g_U(x), f))$$

**Proposition 2.2.** *Two fiber bundles $E_1, E_2$ with group $G$ and fibre $F$ are isomorphic iff, for a fine enough covering $\mathcal{U}$ of $M$, the 1-cocycles $\{g^i_{UV}\}_{U,V \in \mathcal{U}}$, $i = 1, 2$ are* cobordant, *i.e. there exists a family $g_U : U \to G$ of smooth maps such that*
$$g^2_{UV} = g_U \cdot g^1_{UV} \cdot g_V^{-1} \text{ on } U \cap V, \ \forall U, V \in \mathcal{U}.$$

The proof follows directly from the definition.

**Remark 2.3.** The *isomorphism class* of a fiber bundle with structure group $G$ and fiber $F$ turns out to depend only on the transition functions. In terms of *sheaf theory*, the previous proposition establishes an equivalence between an isomorphism class of fiber bundles with group $G$ (and any fiber $F$) and a *Čech cohomology class* in $H^1(M, G)$ (see, e.g., R. Godement, *Topologie algébrique et théorie des faisceaux*).

**Definition 2.4.** *Two bundles $E \to M$ and $E' \to M$ with the same gauge group $G$ and the same transition functions $\{g_{UV} \mid U, V \in \mathcal{U}\}$ (and different fibers $F$, $F'$) are called* associated.

**Remark 2.5.** By (1) we can construct associated bundles for any given $G$-manifold $F$ (a $G$-manifold is a manifold together with a left action of $G$).

We introduce now some operations with fiber bundles; the first one is the *pull-back*, that constructs, starting from a fiber bundle $E \to M$ as above, a fiber bundle with the same structure group and the same fiber over another manifold:

**Definition 2.6.** *If $E \xrightarrow{\pi} M$ is a fiber bundle with group $G$ and fiber $F$ ($G$ acting by $\rho$ on $F$), and $f : N \to M$ is a smooth map, one defines the* pull-back *$f^*E$ of $E$ to $N$ the following fiber bundle with group $G$ and fiber $F$ on $N$:*

(1) $E' := f^*E := \{(x, y) \in N \times E \mid f(x) = \pi(y)\}$;
(2) $\pi' : E' \to N$, $\pi'(x, y) := \pi(y) (= f(x))$, $\forall (x, y) \in E'$; *also denote by $\bar{f} : E' \to E$, $\bar{f}(x, y) := y$, $\forall (x, y) \in E'$, the* induced bundle map;
(3) *the local trivializations are defined on the covering $f^{-1}(\mathcal{U}) := \{f^{-1}(U) \mid U \in \mathcal{U}\}$ as*
$$\psi'_{f^{-1}(U)}(x, y) := (x, \mathrm{pr}_2(\psi_U(y))), \ \forall (x, y) \in {\pi'}^{-1}(f^{-1}(U)) = \bar{f}^{-1}(\pi^{-1}(U)).$$
(4) *the transition functions $g'_{f^{-1}(U) f^{-1}(V)} := g_{UV} \circ f$, $\forall U, V \in \mathcal{U}$.*

The pull-back $f^*E$ can be characterized, up to bundle isomorphism, simply as the manifold $E'$ such that the maps $\pi'$ and $F_f$ below are a submersion, resp. a smooth map such that the fibers of $\pi'$ are mapped diffeomorphically to fibers of $\pi$, and the following diagram commutes:

$$\begin{tikzcd} E' \arrow[r, "\bar f"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

**Exercise 2.7.** It is clear that over a point (seen as a manifold $\bullet$ of dimension 0), there exists only one (trivial) bundle with group $G$ and fiber $F$, namely $F \to \bullet$. Show that a bundle $E \to M$ as above is trivial iff it is isomorphic with the pull-back of any $G, F$-bundle through a constant map.

In the special case when $f$ itself is the projection of a fibre bundle $E_2 \overset{\pi_2}{\to} M$ (and where we denote our initial fiber bundle on $M$ as $E_1 \overset{\pi_1}{\to} M$), we get that $\pi_2^* E_1 = \pi_1^* E_2$ and $\pi_1' = \bar\pi_2$ and $\pi_2' = \bar\pi_1$ are both fiber bundle projections. Moreover, if we denote by $E_1 \times_M E_2 := \pi_2^* E_1 = \pi_1^* E_2$ the *fibered product of* $E_1$ and $E_2$, then the diagonal map $\pi := \pi_1 \circ \pi_2' = \pi_2 \circ \pi_1'$ is a fiber bundle projection, too.

$$\begin{tikzcd} E_1 \times_M E_2 \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] \arrow[dr, "\pi"] & E_1 \arrow[d, "1"] \\ E_2 \arrow[r, "\pi_2"] & M \end{tikzcd}$$

The gauge group is then $G := G_1 \times G_2$ and the fiber is $F := F_1 \times F_2$ (corresponding to the representation $\rho : G \times F \to F$, $\rho((g_1, g_2), (f_1, f_2)) := (\rho_1(g_1, f_1), \rho_2(g_2, f_2))$, $\forall (g_1, g_2) \in G$, $(f_1, f_2) \in F$.

In the particular case where both fiber bundles $E_i \overset{\pi_i}{\to} M$ are vector bundles, the fibered product $E_1 \times_M E_2$ has as fiber at $x \in M$ the direct sum of the fibres $(E_1)_x$ and $(E_2)_x$. For that reason, it is usually denoted by $E_1 \oplus E_2$ and is usually referred to as the *Whitney sum* of the vector bundles $E_1$ and $E_2$. Note that, by construction, the gauge group of $E_1 \oplus E_2 = E_1 \times_M E_2$ is naturally a subgroup of $\mathrm{GL}(n_1, \mathbb{R}) \times \mathrm{GL}(n_2, \mathbb{R})$ (in case the ranks of $E_i$ are $n_i$ and $G_i$ act effectively of $\mathbb{R}^{n_i}$, $i = 1, 2$). To determine whether a given vector bundle $E$ can be decomposed as a Whitney sum is a non-trivial topological problem, and it amounts to *reduce* the structure (gauge) group of $E$ to a subgroup of such a product. More about gauge reduction in the next section.

**Exercise 2.8.** (1) Show that every (local) section of $E \overset{\pi}{\to} M$ induces a (local) section of $f^*E \overset{\pi'}{\to} M$;

(2) Show that a (local) section in $E_1 \times_M E_2 \to M$ is equivalent to a pair of (local) sections in $E_1 \to M$, resp. $E_2 \to M$;

(3) Show that if $f : U \hookrightarrow M$ is the inclusion of an open subset of $M$, then $f^*E \overset{\pi'}{\to} M$ is the restriction to $U$, resp. $\pi^{-1}(U)$ of the bundle projection $\pi : E \to M$. $f^*E$ is sometimes denoted by $E|_U$.

## 3. Principal bundles. Associated bundles

**Definition 3.1.** *A $G$-principal bundle over a manifold $M$ is a fiber bundle $P \stackrel{\pi}{\to} M$ with group $G$ and fiber $F := G$, where the action of $G$ on the fiber $F$ is by left multiplication of $G$ on itself.*

**Remark 3.2.** As we have seen before, the relevant structures that are canonical on $P$ are induced by the structures of the fiber that are invariant w.r.t. the action of the group. E.g., the neutral element is not invariant w.r.t. the left multiplication, so the fibers $P_x := \pi^{-1}(x)$ do not have a particular point, corresponding to the neutral element of $G$, nor is the group multiplication on $G$ invariant w.r.t. the left multiplication. Therefore, the fibers of $P$ are, in general, *not Lie groups*.

They do inherit, however, a special structure: the right multiplication $R_a$ on $G$ with an element $a \in G$ is invariant w.r.t. the left multiplication $L_b$ by any element $b \in G$, i.e.

$$L_b \circ R_a = R_a \circ L_b, \ \forall a, b \in G.$$

(This is the associativity in the group $G$.) Therefore, there is a canonical *right* action of $G$ on the total space $P$. This action is free ($R_a : P \to P$ has no fixed points if $a \neq e \in G$) and *proper* (see below).

The importance of principal bundles is that, among all bundles with the same gauge group and (up to co-boundaries) same 1-cocycle of transition maps (these bundles are called *associated* to each other), it is the only one on which $G$ acts (on the right) and such that all other associated bundles can be retrieved from it by the following procedure:

**Proposition 3.3.** *Let $P \stackrel{\pi}{\to} M$ be a $G$-principal bundle over $M$ and $\rho : G \times F \to F$ a left action of $G$ on a manifold $F$. Then the following manifold is the total space of a fiber bundle over $M$ with group $G$ and fiber $F$ (for the action $\rho$), associated to $P$:*

$$E := P \times_\rho F := P \times F / \sim, \ (p, f) \sim (p \cdot g, \rho(g^{-1}, f)), \ \forall (p, f) \in P \times F, \ \forall g \in G.$$

*The dot denotes the right multiplication of $g \in G$ with the element $p \in P$. The projection $\pi_E : E \to M$ is defined, for an equivalence class $[p, f] \in E$, by $\pi_E([p, f]) := \pi(p)$.*

Because the equivalence relation above is induced by the right free action of $G$ on $P \times F$:

$$(P \times F) \times G \ni ((p, f), g) \mapsto (p \cdot g, \rho(g^{-1}) \in P \times F,$$

the proposition above is a direct consequence of the following Theorem.

**Definition 3.4.** *A continuous map $f : A \to B$ between topological spaces is called* proper *iff $f^{-1}(K)$ is compact in $A$, for any $K \subset B$ compact.*

*A Lie group (right) action $\eta : P \times G \to P$ on a smooth manifold $P$ is said to be* proper *iff the continuous map $f_\eta : P \times P \to P \times P$, defined by*

$$f_\eta(p, g) := (p, \eta(p, g)), \ \forall (p, g) \in P \times G,$$

*is proper.*

**Theorem 3.5.** *Let $P$ be a smooth manifold and let $G$ be a Lie group acting smoothly, freely and properly on $P$. Then the orbit space of $G$ can be given the structure of a smooth manifold $M$, such that the canonical projection $\pi : P \to M$ is a submersion and, moreover, a $G$-principal bundle.*

**Remark 3.6.** We have seen that in each trivialization domain $U$, a fiber bundle $E \stackrel{\pi}{\to} M$ admits, for each $y \in \pi^{-1}(U)$, a section $\sigma_y : U \to \pi^{-1}(U)$ passing through $y$.

Conversely, every local section $\sigma : U \to \pi^{-1}(U)$ in a principal bundle $P \stackrel{\pi}{\to} M$ defines a local trivialization of the bundle over $U$: indeed, the map $F_\sigma : U \times G \to \pi^{-1}(U)$, $F_\sigma(x, g) := \sigma(x) \cdot g$ is a diffeomorphism, and for each local trivialization $\psi_V : \pi^{-1}(V) \to V \times G$, we have on $(U \cap V) \times G$:

$$\psi_V(F_\sigma(x, g)) = \psi_V(\sigma(x) \cdot g) = (x, g_{V\sigma}(x) \cdot g),$$

where $g_{V\sigma}(x)$ is the component in $G$ of $\psi_V(\sigma(x))$.

For principal bundles, a (local) trivialization and a (local) section are thus equivalent notions.

Sections in an associated bundle can also be described using the corresponding principal bundle:

**Proposition 3.7.** *Let $P \stackrel{\pi}{\to} M$ be a principal $G$-bundle, $\rho : G \times F \to F$ a smooth action of $G$ on $F$ and $E := P \times_\rho F$ the associated bundle to $\rho$. There is a 1–1 correspondence between (local) sections $\sigma : U \to \pi_E^{-1}U$ in $E$ and $G$-equivariant maps $f : \pi^{-1}(U) \to F$ (if $U = M$, then $\sigma : M \to E$ and $f : P \to F$), i.e. smooth maps that satisfy the* equivariance *property:*

$$f(p \cdot g) = \rho(g^{-1}, f(p)), \ \forall p \in P, \ g \in G.$$

In case $F$ is a vector space $V$ and $\rho : G \times V \to V$ is a linear representation (one can see $\rho$ as a group homomorphism from $G$ to $\mathrm{GL}(V)$), one considers $E$-valued differential forms on $M$ as sections in the vector bundle $\Lambda^* M \otimes E$ (as a vector bundle, it is constructed by taking fiberwise tensor products of $\Lambda^* M$ and $E$: it is equally the associated bundle to the $\mathrm{GL}(M) \times_M P$ with structure group $\mathrm{GL}(n, \mathbb{R}) \times G$, where $\mathrm{GL}(M)$ is the bundle of frames on $M$ ($\dim M = n$) and $P$ is the associated principal bundle to $E$, by the tensor product representation $\lambda \otimes \rho : \mathrm{GL}(\mathbb{R}^n) \times G \to \mathrm{GL}(\Lambda^*(\mathbb{R}^n)^* \otimes V)$). Such $E$-valued differential forms can be characterized in terms of the associated principal bundle in an analogous way as the sections are equivalent to $G$-equivariant maps on $P$:

**Proposition 3.8.** *Let $P \stackrel{\pi}{\to} M$ be a principal $G$-bundle and $\rho : G \to \mathrm{GL}(V)$ be a linear representation of $G$. Denote by $E := P \times_\rho V$ the associated vector bundle. A $k$-form on $M$ with values in $E$ is equivalent to a $V$-valued $k$-form $\alpha$ on $P$ such that*

1. $R_g^* \alpha = \rho_{g^{-1}}(\alpha)$, $\forall g \in G$
2. *The vertical distribution (the vertical tangent space, or the tangent space to the fibers) $T^\vee P := \ker \pi_*$ lies in the kernel of $\alpha$, i.e. $\alpha(X, \dots) = 0$, $\forall X \in T^\vee P$.*

*Here $R_g : P \to P$ is the right action of $g \in G$ on $P$ and $\rho_{g^{-1}} : V \to V$ is the (left) linear action of $g^{-1}$ on $V$.*

Given a fiber bundle with group $G$, one is usually interested in *reducing* the gauge group to a subgroup $H$ of $G$, in that one attemps to find a sub-atlas of local trivializations for which the transition functions take value in $H$. In case $H = \{e\}$, this amounts to finding a global trivialization of $E$. The concept of *gauge reduction* can be defined in full generality and characterized as follows:

**Definition 3.9.** *Let $E \xrightarrow{\pi} M$ be a fiber bundle with gauge group $G$ and fiber $F$ for $\rho : G \times F \to F$ the corresponding action. Let $\varphi : H \to G$ be a Lie group homomorphism. We say that the gauge group of $E$ reduces to $H$ iff there exists a sub-atlas of local trivializations of $E$ such that the corresponding transition functions $g_{UV} : U \cap V \to G$ factorize through $\varphi$, i.e. they can be expressed as $g_{UV} = \varphi \circ h_{UV}$, where $h_{UV}$ is a 1-cocycle on $M$ with values in $H$.*

For $E = P$ a principal $G$-bundle, this means that $P$ is isomorphic to $P^{red} \times_{\bar{\varphi}} G$, where $P^{red}$ is a principal $H$-bundle (defined by the 1-cocycle $\{h_{UV}\}$ above) and $\bar{\varphi} : H \times G \to G$ is the action of $H$ by left multiplication (via $\varphi$) on $G$: $\bar{\varphi}(h, g) := \varphi(h) \cdot g, \forall h \in H,\ g \in G$.

**Proposition 3.10.** *A reduction to $H$ (for the group homomorphism $\varphi : H \to G$) of the gauge group of the $G$-bundle $P$ is equivalent to a $G$-equivariant ($\varphi$-equivariant) map $F : P^{red} \to P$ from a principal $H$-bundle $P^{red}$ to $P$, i.e. $F(p \cdot h) = F(p) \cdot \varphi(h)$, $\forall p \in P^{red}$, $h \in H$. If $\varphi$ is injective, such a reduction is also equivalent to a section in the bundle $P/\varphi(H) \simeq P \times_L G/\varphi(H)$, where $G$ acts on the right coset space $G/\varphi(H)$ by multiplication on the left.*

A standard case of gauge reduction from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ of the tangent bundle of an $n$-dimensional manifold $M$ is given by the choice of a Riemannian metric, resp. a metric plus an orientation. In view of the previous proposition, the reduction from $\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{O}(n)$ is equivalent to a section in the bundle of symmetric, positive definite bilinear forms on $TM$, whose fiber is the space of positive definite symmetric matrices. This, in turn, is shown to be a contractible space by the standard Gramm-Schmidt orthogonalization procedure, and by the obstruction theory, a fiber bundle with contractible fiber always admits global sections. Thus, the existence of a Riemannian metric on any manifold is a special case of a more general phenomenon: reducing the gauge group of a $G$-bundle to its maximal compact subgroup:

**Theorem 3.11** (Cartan-Iwasawa-Malcev)**.** *Let $G$ be a Lie group with finitely many connected components. Then every compact subgroup of $G$ is contained in a maximal compact subgroup of $G$. Moreover, all such maximal compact subgroups $H$ are conjugated inside $G$, have as many connected components as $G$ has, and the coset space $G/H$ is contractible.*

We conclude that, from a topological viewpoint, the study of principal bundles can be restricted to the case when the structure (gauge) group is compact.

**Exercise 3.12.** Define $or(M) := \mathrm{GL}(M)/\mathrm{GL}_+(n, \mathbb{R})$ to be the bundle of orientations on $M$. Show that $or(M) \to M$ is a two-fold covering, and also a $\mathbb{Z}_2$-principal bundle. Show that a connected manifold $M$ is orientable iff $or(M)$ is not connected.

**Exercise 3.13.** Generalize the previous exercise as follows: Let $P \xrightarrow{\pi} M$ be a $G$-principal bundle over a connected manifold $M$ with $G$ a discrete group (a Lie group of dimension 0). Show that $P$ admits a reduction to a subgroup of $G$ iff the total space $P$ is non connected.

We also mention here the *virtual reduction* of a $G$-bundle $E \to M$: here it is not the gauge group of $E$ itself that is reduced, but the gauge group of $f^*E$, for a suitably chosen smooth map $f : N \to M$.

**Example 3.14.** Let $P \xrightarrow{\pi} M$ be a principal $G$-bundle. Then $\pi^* P \xrightarrow{\pi'} P$, as a $G$-principal bundle over $P$, is trivial: Indeed, the diagonal map $D : P \to P \times P$ induces a global section in $\pi^* P = P \times_M P \subset P \times P$.

The virtual reduction will turn out to be a useful tool for computations with *characteristic classes*, see below.

If the reduced gauge group $H$ is not specified, it is difficult to find out whether the gauge group $G$ of a bundle $E$ admits reductions (exept for the reduction to the maximal subgroup, as seen above). If $H$ is expected to be a closed subgroup of $G$, then the reduction to $H$ amounts to fixing some extra structure on $E$, which is therefore a structure on the standard fiber $F$ extended to the whole space $E$. When $E$ is a vector bundle, this structure may be given by a *tensor*, i.e. a section in some tensor power of $E$. To extend such a tensor $T_x \in \otimes^* E$ from a point $x \in M$ to a curve on $M$ through $x$, and then further to a neighborhood of $x$ and even to the whole manifold $M$, it would be useful to have fixed a way to *lift* curves (a *lift* of a map $f : N \to M$ is a map $\tilde{f} : N \to E$ such that $\pi \circ \tilde{f} = f$), resp. to know that the endpoint of a lifted curve starting at $x$ and located in a small neighborhood of $x \in M$ or, more generally, for arbitrary curves, depends only on the endpoint of the curve on $M$.

The idea of curve lifting leads to the concept of a *connection* (it connects the fibers by *horizontal* – i.e., non-vertical – curves) in a fiber bundle; the independence of the endpoint of the chosen curve turns out to be a property of the *curvature* and *holonomy* of the connection (for local lifts) and of its *monodromy* (for global lifts)

## 4. Connections in a principal bundle. Induced connections

Recall that a Lie group is parallelizable. More precisely, the *left invariant vector fields on $G$* canonically identify the tangent space $T_g G$ in a point $g \in G$ to $\mathfrak{g} = T_e G$. This identification is in fact a 1-form on $G$ with values in its Lie algebra $\mathfrak{g}$:

**Definition 4.1.** *The* Maurer-Cartan form $\omega^{MC} \in \Lambda^1 G \otimes \mathfrak{g}$ *on the Lie group $G$ associates to every vector $X \in T_g G$ the element $(L_{g^{-1}})_{*,g}(X) \in T_e G = \mathfrak{g}$. (recall that $L_h : G \to G$ is the left multiplication with $h$)*

$\omega^{MC}(X)$ is constant, for $X$ a vector field on $G$, iff $X$ is left-invariant. Recall that the flow of a left-invariant vector field $X$ is $R_{\exp(tX_e)}$, the right multiplication with the 1-parameter subgroup $\{\exp(tX_e) \mid t \in \mathbb{R}\}$ generated by $X_e \in \mathfrak{g}$, the value at $e$ of the vector field.

We have that $\omega^{MC}$ is left-invariant, i.e.:

$$(L_g)^* \omega^{MC} = \omega^{MC}, \ \forall g \in G.$$

It is important to know what impact has the *right multiplication* $R_g$ with some element $g \in G$ on the Maurer-Cartan form:

**Proposition 4.2.** *Let $G$ be a Lie group, $g \in G$ and $\omega^{MC}$ the Maurer-Cartan form. Then*

$$(R_g)^* \omega^{MC} = \mathrm{Ad}_{g^{-1}} \circ \omega^{MC}.$$

Here, the left hand side is the pull-back through the diffeomorphism $R_g$ of $\omega^{MC}$ and the right hand side is the composition of the linear maps $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ with $\omega^{MC} : TG \to \mathfrak{g}$.

Because $\omega^{MC}$ is left-invariant, it can be defined on the fibers of a $G$-principal bundle $P \stackrel{\pi}{\to} M$, using the trivializations (that identify the fibers with $G$), and the induced forms on $P_x$ do not depend on the chosen trivialization, since the transition maps act on $G$ by left multiplication, which leaves $\omega^{MC}$ invariant.

**Remark 4.3.** The same argument can be used to show that the left-invariant vector fields on $G$ induce vertical tangent fields (i.e., vector fields that are sections of the vertical tangent space $T^{\vee}P = \ker(\pi_*)$), associated to every element of the Lie algebra $\mathfrak{g}$. These vector fields will be called *fundamental*, and we denote by $\bar{X}$ the fundamental vector field on $P$ corresponding to the element $X \in \mathfrak{g}$. The flow of $\bar{X}$, at time $t$, is the right multiplication on $P$ with $\exp(tX) \in G$.

Equivalently, $\bar{X}$ can be defined by the formula

$$\bar{X}_p := \frac{d}{dt}\Big|_{t=0} \left(R_{\exp(tX)}p\right) = \frac{d}{dt}\,|_{t=0}\,(p \cdot \exp(tX))\,,\ \forall p \in P.$$

Because $\omega^{MC}$ is invariant to the left action of $G$ on itself, the 1-forms $\omega^{MC}_{\phi} : TP_x \to \mathfrak{g}$ defined by the local trivializations $\phi : P_x \to G$ are, in fact, independent of the choice of the trivialization $\phi$, and thus there exists a well-defined map $\omega^{MC,P} : T^{\vee}P \to \mathfrak{g}$, such that

$$\omega^{MC,P}(\bar{X}) = X,\ \forall X \in \mathfrak{g}.$$

$\omega^{MC,P}$ is not a 1-form on $P$ (with values in $\mathfrak{g}$), but, since the right multiplications $R_g : P \to P$, $g \in G$, preserve the vertical distribution, one can consider the pull-back of $\omega^{MC,P}$ through $R_g$. We have then

**Lemma 4.4.** $(R_g)^*\omega^{MC,P} = \mathrm{Ad}_{g^{-1}}(\omega^{MC,P})$, $\forall g \in G$.

*Proof.* It is enough to prove the identity above for the case when $P$ is trivial; in fact it is enough to show

$$(R_g)^*\omega^{MC} = \mathrm{Ad}_{g^{-1}}(\omega^{MC}),$$

for the Maurer-Cartan form of $G$. For this, apply both sides of the identity to a left-invariant vector field $X$ on $G$. On the right hand side we obtain

$$\mathrm{Ad}_{g^{-1}}X_e,$$

and on the left hand side we obtain

$$\omega^{MC}((R_g)_*X).$$

We need thus to show that the vector field $Y := (R_g)_*X$ is also left-invariant and that $Y_e = \mathrm{Ad}_{g^{-1}}X_e$. The flow of $Y$ is

$$\phi^Y_t = R_g \circ \phi^X_t \circ (R_g)^{-1} = R_g \circ R_{\exp(tX_e)} \circ R_{g^{-1}} = R_{g^{-1}\exp(tX_e)g},$$

which is exactly the flow of $\mathrm{Ad}_{g^{-1}}X_e$. □

**Definition 4.5.** *A* connection *in the $G$-principal bundle $P \to M$ is a $G$-invariant horizontal distribution on $P$, i.e. a subbundle $H$ of the tangent bundle of $P$, such that*

$$(R_g)_*H = H,\ \forall g \in G.$$

Equivalently, a connection is given by a *connection 1-form* $\omega \in C^\infty(\Lambda^1 P \otimes \mathfrak{g})$ such that

(1) $\omega|_{T^\vee P} = \omega^{MC,P}$
(2) $R_g^*\omega = \mathrm{Ad}_{g^{-1}}(\omega)$, $\forall g \in G$.

Indeed, the kernel of $\omega$ is then a $G$-invariant horizontal distribution $H^\omega$. Conversely, for a $G$-invariant horizontal distribution $H$ there exists a unique connection 1-form $\omega^H$ such that $\ker \omega^H = H$ (just define $\omega^H$ to be equal to $\omega^{MC,P}$ on $T^\vee P$ and equal to zero on $H$).

4.1. **Induced connections on associated bundles.** Let $P \to M$ be a $G$-principal bundle and $\rho : G \times F \to F$ a $G$ action. For $H$ a connection on $P$, we can construct a horizontal distribution $H^E$ on the associated bundle $E := P \times_\rho F$ as follows: $H^E := pr_*(H \times 0_F)$, where $0_F \subset TF$ is the trivial distribution on $F$. $H \times 0_F \subset TP \times TF$ is then a $n$-dimensional subspace of $T(P \times F)$ which is transversal to the kernel of the projection $pr : P \times F \to P \times_\rho F$, and thus $H^E$ is a $N$-dimensional distribution on $P \times_\rho F$. This distribution is also *horizontal* (i.e., transversal to the vertical tangent space of $E$), because $H \times 0_F$ is transversal to $T^\vee P \times TF$ and $T^\vee E = pr_*(T^\vee P \times TF)$.

Note, however, that not every horizontal distribution on $E$ is induced by a connection on $P$. The condition that such a horizontal distribution must satisfy can be only seen in a local gauge (trivialization):

**Lemma 4.6.** *Let $(U, \psi_U)$ be an atlas of trivializations of a $G$-principal bundle $P$. A connection $H$ on $P$ (or on any associated bundle $E = P \times_\rho F$) is given by a family of 1-forms with values in $\mathfrak{g}$, $\omega^U \in \mathbb{C}^\infty(\Lambda^1 U \otimes \mathfrak{g})$ such that*

$$\omega^V = \mathrm{Ad}_{g_{UV}^{-1}}(\omega^U) + g_{UV}^*\omega^{MC}. \tag{2}$$

*Proof.* A trivialization is equivalent to a local section $\sigma_U : U \to P$. We define $\omega^U := \sigma_U^*(\omega^H)$. We use $\sigma_V = \sigma_U \cdot g_{UV}$ on $U \cap V$ and obtain, by differentiation,

$$(\sigma_V)_{*,x}(X) = (R_{g_{UV}})_{*,\sigma_U(x)}\left((\sigma_U)_{*,x}(X)\right) + \overline{\left((L_{g_{UV}(x)^{-1}})_*\left((g_{UV})_{*,x}(X)\right)\right)}_{\sigma_U(x)}, \tag{3}$$

where the second term is the fundamental vector field that corresponds to the element in $\mathfrak{g}$ (identified with $T_{g_{UV}(x)}G$ by the corresonding left multiplication) defined by $(g_{UV})_{*,x}(X)$.

The equation (2) follows by applying $\omega^H$ to the equation above. $\square$

**Remark 4.7.** We can retrieve the connection form $\omega$ out of a family $(U, \omega_U)$ as above: Indeed, it is enough to specify the kernel $H$ of $\omega$ along the values of a local section $\sigma_U$; then, we define $H$ on $\pi^{-1}(U)$ by using the $G$-invariance. To define $H$, it is necessary and sufficient to define *the horizontal lift* of every vector $X$ on the basis $M$:

**Proposition 4.8.** *Let $x \in U \subset M$ and $\sigma_U$ the corresponding local section in $P$ over $U$. We define*

$$\tilde{X} := (R_g)_*\left((\sigma_U)_{*,x}(X) - \overline{\omega_U(X)}_{\sigma(x)}\right) \in T_{\sigma_U(x)\cdot g}P, \ \forall X \in T_xM, \ \forall g \in G \tag{4}$$

*the* horizontal lift *of the vector $X$ in the gauge (trivialization) $\sigma_U$.*

*Then $\tilde{X}$ is independent of the choice of a trivialization. Therefore, the horizontal distribution*

$$H_{\sigma(x)} := \{\tilde{X} \mid X \in T_x M\}$$

*is a well-defined connection on $P$.*

*Proof.* It suffices to check that $\tilde{X}_{\sigma_V(x)}$, computed in the gauge $\sigma_V$, coincides with the value at $\sigma_V(x) = \sigma_U(x) \cdot g_{UV}(x)$ of the horizontal lift computet in the gauge $\sigma_U$. From (3) it follows that the difference of the first (the non-vertical) terms in the expressions of $\tilde{X}$ in these two gauges, cf. (4), is vertical and coincides with the value at $\sigma_V(x)$ of the fundamental vector field

$$\overline{\omega_V(X)} - \overline{\omega_U(X)},$$

which is the difference of the vertical (correction) terms in the expressions of $\tilde{X}$, cf. (4), in the gauges $\sigma_V$, resp. $\sigma_U$. $\tilde{X}$ is thus well-defined independently of the choice of trivialization.

$H$ is thus well-defined as well. By definition (4), $H$ is $G$-invariant, so it is a connection on $P$. □

Recall that a gauge transformation $\phi$ is an automorphism of the $G$-principal bundle $P$, i.e., a $G$-equivariant automorphism of $P$. Equivalently, it is a section in $\mathcal{G} = P \times_{AD} G$, where $AD : G \times G \to G$ is the action of $G$ on itself by conjugation: $AD(g, a) := g \cdot a \cdot g^{-1}$. Indeed, the map

$$M \ni x \mapsto [p, g_{\phi(p),p}] \in (P \times_{AD} G)_x,$$

where $p \in P_x$ and $g_{p',p} \in G$, for $p, p' \in P_x$, is the unique element of $G$ such that $p' = p \cdot g_{p',p}$ (one can say, $g_{p',p}$ is the "quotient" of $p'$ by $p$), is a well-defined section in $P \times_{AD} G$ as claimed.

In fact, we have constructed above an $AD$-equivariant map $\bar{\phi} : P \to G$ by the formula $\bar{\phi}(p) = g_{\phi(p),p}$. The equivariance condition:

$$\bar{\phi}(p \cdot g) = AD_{g^{-1}}(\bar{\phi}(p)), \ \forall g \in G, \ \forall p \in P$$

is precisely the one that makes that the class $[p, \bar{\phi}(p)] \in P \times_{AD} G$ is independent of the choice of $p \in P_x$ and thus defines a section in this bundle.

A gauge transformation can be thus expressed by one of the following equivalent objects:

(1) A $G$-equivariant map $\phi : P \to P$ that induces the identity on the basis $M$
(2) A map $\phi : P \to P$ that looks in local trivializations as the left multiplication on the fibers with a $G$-valued function on the base
(3) An $AD$-equivariant map $\bar{\phi} : P \to G$
(4) A section in $\mathcal{G} = P \times_{AD} G$.
(5) if $G$ is connected, a map $\phi : P \to P$ that maps each fiber $P_x$ into itself and preserves the fundamental vector fields $\bar{X}$, $X \in \mathfrak{g}$.

Note that a gauge transformation also induces a transformation on every associated bundle: the second point above realizes directly such a transformation, but also the point 4 (or 3): The fibers $\mathcal{G}_x$, $x \in M$, of $\mathcal{G}$ are Lie groups, identified up to conjugation with $G$. As such, they naturally act on the corresponding fibers $E_x = (P \times_\rho F)_x$ of associated bundles $E = P \times_\rho F$:

$$\mathcal{G}_x \times E_x \ni ([p, a], [p, f]) \mapsto [p, \rho(a, f)] \in E_x.$$

That the above map is well-defined action of $\mathcal{G}_x$ on $E_x$ is an easy exercise.

**Example 4.9.** If $E \to M$ is a real vector bundle, the associated principal bundle $P$ is the bundle $\mathrm{GL}(E)$ of isomorphisms (called *frames*) $f : \mathbb{R}^n \to E_x$, $x \in M$. Here $n := \dim E_x$ is the *rank* of $E$. The bundle $\mathcal{G}$ is then the bundle $\mathrm{Aut}(E)$ of automorphisms of $E$, and the adjoint bundle $\mathrm{Ad}(P)$ to $\mathrm{GL}(E)$ is the bundle $\mathrm{End}(E)$ of endomorphisms of $E$. As a set, $\mathrm{Aut}(E) \subset \mathrm{End}(E)$.

If $E$ is a complex vector bundle of complex rank $n$, then the bundles $P := \mathrm{GL}_{\mathbb{C}}(\mathbb{C}^n, E)$, $\mathcal{G} := \mathrm{Aut}_{\mathbb{C}}(E)$ and $\mathrm{Ad}(P) := \mathrm{End}_{\mathbb{C}}(E)$ are defined analogously.

Two objects are considered to be *gauge equivalent* iff there is a gauge transformation that sends one object into another. A condition is called *gauge invariant* if it is satisfied by a whole class of gauge equivalent objects.

**Example 4.10.** The zero set of a section in a vector bundle is a gauge invariant object/condition: if a section $s$ in $E$ vanishes on the set $A \subset M$, then every section of $E$ which is gauge-equivalent to $s$ vanishes on $A$.

Ther basic idea of gauge theory is to consider either gauge-invariant conditions or to consider the classes of objects up to gauge equivalence. In the following, we investigate the transformation law of connections w.r.t. gauge transformations:

**Proposition 4.11.** *Let $\phi : P \to P$ be a gauge transformation and let $\omega$ be a connection form on the $G$-bundle $P \to M$. The induced connection by $\phi$ on $P$ is the connection form*

$$\omega' = \mathrm{Ad}_{\bar{\phi}^{-1}}(\omega) + (\bar{\phi})^* \omega^{MC}. \tag{5}$$

*Proof.* Let $X \in T_x M$ be a vector on $M$ and let $\sigma$ be a local section in $P$ around $x$. Then $Y := \sigma_{*,x}(X) \in T_{\sigma(x)}P$ is a horizontal vector (i.e., it is not vertical). Then $\sigma' := \phi \circ \sigma = \sigma \cdot \bar{\phi}$ is another local section in $P$ around $x$ and

$$\omega'(Y) = (\sigma')^* \omega'(X) = \mathrm{Ad}_{\bar{\phi}(\sigma(x))^{-1}}(\omega(Y)) + (\bar{\phi}^* \omega^{MC})(Y),$$

where the last equality follows from (2), where we put $\sigma_U := \sigma$, $\sigma_V := \sigma'$ and $g_{UV} := \bar{\phi} \circ \sigma$. But this implies the claimed equality (5) for the argument $Y$.

Considering now all local sections around $x$, such that $\sigma(x) = p$, we obtain that the equality (5) holds for all horizontal (i.e., non-vertical) arguments $Y \in T_p P \smallsetminus T_p^\vee P$, hence for all vectors in $T_p P$. $\square$

Note that a 1-form $\alpha \in \mathbb{C}^\infty(P) \otimes \mathfrak{g}$ that satisfies

$$R_g^* \alpha = \mathrm{Ad}_{g^{-1}} \alpha \text{ and } \alpha|_{T^\vee P} = 0$$

corresponds to a section in $\Lambda^1 M \otimes \mathrm{Ad} P$.

4.2. **Pull-back of a connection.** Let $f : N \to M$ be a smooth map and $P \to M$ a $G$-principal bundle with a connection $\omega \in \Lambda^1 P \otimes \mathfrak{g}$. Let $F : f^* P \to P$ be the ($G$-equivariant) map between the pull-back of $P$ and $P$. Then $F^* \omega \in \Lambda^1(f^* P) \otimes \mathfrak{g}$ is a connection on $f^* P$.

4.3. **Connections on vector bundles. Covariant derivatives.**

**Definition 4.12.** *Let $E \to M$ be a vector bundle (real or complex). A* covariant derivative *on $E$ is a linear map $D : C^\infty(E) \to C^\infty(\Lambda^1 M \otimes_{\mathbb{R}} E)$ such that*

$$D(fs) = fD(s) + df \otimes s, \ \forall S \in C^\infty(E), \ \forall f : M \to \mathbb{R}(\mathbb{C}). \tag{6}$$

**Remark 4.13.** Note that a 1-form $\alpha \in \mathbb{C}^\infty(P) \otimes \mathfrak{g}$ that satisfies

$$R_g^* \alpha = \mathrm{Ad}_{g^{-1}} \alpha \text{ and } \alpha|_{T^\vee P} = 0$$

corresponds to a section in $\Lambda^1 M \otimes \mathrm{Ad}P$. as above, by replacing $\Lambda^1 M$ with its complexification, and taking the tensor product between $\Lambda^1 M \otimes \mathbb{C}$ and $E$ over $\mathbb{C}$; then, the function $f$ can be complex-valued.

The property (6) is called the *Leibniz rule*, and is equivalent to the fact that the symbol of $D$, seen as a first order linear differential operator, is the identity (see below).

**Proposition 4.14.** *Let $P \to M$ be a $G$-principal bundle and $\rho : G \to \mathrm{GL}(V)$ a linear representation of $G$. Then every connection $H$ on $P$ induces a covariant derivative $D^H$ on the vector bundle $E := P \times_\rho V$.*

*Conversely, let $D$ be a covariant derivative on the (real or complex) vector bundle $E \to M$, and denote by $\mathrm{GL}(E) := \mathrm{Hom}^\times(V, E)$, for $V$ the trivial bundle with fiber isomorphic to the fibers of $E$, the set of (real or complex) isomorphisms (or invertible homomorphisms) between $V$ and $E$, in other words the bundle of* frames *of $E$. Then $D$ induces a connection $H^D$ on this principal $\mathrm{GL}(V)$-bundle.*

Here, $V$ stands for $\mathbb{R}^n$, if $E$ is a rank $n$ real vector bundle, and for $\mathbb{C}^n$, if $E$ is a complex vector bundle of complex rank $n$. The field $\mathbb{R}$ or $\mathbb{C}$ is thus "intrinsic" in the notations $\mathrm{GL}(V)$, $\mathrm{End}(E)$, etc.

*Proof.* Let $s$ be a section of $E = P \times_\rho V$. Take $\sigma : U \to P$ a local section in $P$. Then $s$ determines a map $w : U \to V$ such that $s(x) = [\sigma(x), w(x)]$, $\forall x \in U$.

For the action $\rho : G \to \mathrm{GL}(V)$ we define, by differentiation at $e \in G$, the derived *Lie algebra action* $\bar{\rho} : \mathfrak{g} \to \mathrm{End}(V)$, that satisfies

$$[\bar{\rho}(X), \bar{\rho}(Y)] = \bar{\rho}([X, Y]), \ \forall X, Y \in \mathfrak{g},$$

where the bracket on the left hand side is the commutator in $\mathrm{End}(V)$.

We set then

$$D(s) := [\sigma, Id_{\Lambda^1 M} \otimes \bar{\rho}(\omega^\sigma)(w) + dw]. \tag{7}$$

Here, $\omega^\sigma : -\sigma^* \omega$ is a 1-form on $U$ with values in $\mathfrak{g}$, and $\bar{\rho}$ applies only to this latter part, so we finally get a 1-form on $U$ with values in $V$, as is $dw$; the pairing with $\sigma$ of such a 1-form with values in $V$ yields thus a 1-form with values in $P \times_\rho V = E$.

We check now that the resulting 1-form with values in $E$ is independent on the choice of $\sigma$, using (2): Suppose $\sigma' = \sigma \cdot \phi$ is another local section, with $\phi : U \to G$. Then the corresponding $V$-valued function $w'$ is equal to $\rho(\phi^{-1})(w)$ and thus

$$[\sigma', \bar{\rho}(\omega^{\sigma'})(w') + dw'] =$$
$$= [\sigma \cdot \phi, \bar{\rho}(\mathrm{Ad}_{\phi^{-1}}(\omega^\sigma)(\rho(\phi^{-1})(w) + \bar{\rho}(\phi^* \omega^{MC}(w') - \bar{\rho}(\phi^* \omega^{MC})(w') + \rho(\phi^{-1})(dw)] =$$
$$= [\sigma, \bar{\rho}(\omega^\sigma)(w) + dw],$$

thus $Ds = D[\sigma, w]$ is independent on the choice of section $\sigma$ in $P$.

The Leibniz rule follows immediately from the one for $dw$.

Conversely, suppose $D$ is a covariant derivative on a vector bundle $E$.

**Lemma 4.15.** *Let $E \to M$ be vector bundle with a covariant derivative $D$ and $e \in E_x$, $x \in M$. Then there exists a section $s$ of $E$ such that*

(1) *$s(x) = e$ and*
(2) *$(Ds)_x = 0$.*

*Proof.* Let $s$ be a section of $E$ such that $s(x) = e$. Consider a local chart $\phi : U \to \mathbb{R}^n$ around $x \in U$ such that $\phi(x) = 0$ and let $x_1, ..., x_n$ be the coordinate functions on $U$ associated to this chart. Then $dx_1, ..., dx_n$ is a local frame of $\Lambda^1 M$ around $x$, thus $\exists e_1, ..., e_n \in E_x$ such that

$$(Ds)_x = \sum_{i=1}^{n} dx_i \otimes e_i.$$

Let $s_1, ..., s_n$ be local sections in $E$ around $x$ such that $s_i(x) = e_i$, $\forall i \in \{1, ..., n\}$ (these sections can be extended to global sections of $E$ using a cut-off function on $M$). Then the local section $s' := S - \sum_{i=1}^n x_i s_i$ (also extended to a global section using a cut-off function to extend the functions $x_1, ..., x_n$ to $M$) satisfies:

(1) $s'(x) = s(x)$ and
(2) $(Ds')_x = (Ds)_x - \sum_{i=1}^n (dx_i \otimes s_i)_x + x_i(Ds_i)_x = 0$,

since $x_i$ all vanish at $x$. □

We conclude that, for every point $f \in \mathrm{GL}(E)_x$, there exists a local frame $\sigma : U \to \mathrm{GL}(E)$ such that $\sigma(x) = f$ and, $\forall v \in V$, $\sigma(v)$, as a local section in $E$, satisfies $D(\sigma(v))_x = 0$. Then we define the following horizontal distribution $H^E \subset T\mathrm{GL}(E)$ by the condition:

$$H_f^E := \{\sigma_{*,x}(X) \mid X \in T_x M,\ \sigma \in \mathbb{C}^\infty(E)\ :\ \sigma(x) = f \text{ and } (D(\sigma(q))_x = 0,\ \forall q \in V\}.$$

If $\sigma$ is a local section in $\mathrm{GL}(E)$ such that $D(\sigma(q))_x = 0$, $\forall q \in V$, then clearly $\sigma \circ g$ has the same property, $\forall g \in \mathrm{GL}(V)$. Since $\mathrm{GL}(E) \ni f \to f \circ g \in \mathrm{GL}(E)$ is the right action of $\mathrm{GL}(V)$ on $\mathrm{GL}(E)$, the horizontal distribution $H^E$ is $\mathrm{GL}(V)$-invariant, and thus a connection on $\mathrm{GL}(E)$. □

**Remark 4.16.** The covariant derivative of a section $s : M \to E$ at $x$ does not depend on the values of $s$ outside an open set $U \ni x$ (for every such an open set; this means that $(Ds)_x$ depends only on the *germ* at $x$ of the section $s$): indeed, if $s' - s$ is a section of $E$ that vanishes on a neighborhood $U$ of $x$, then $s' - s = f(s' - s)$, where $f : M \to \mathbb{R}$ is a function that vanishes on a open set $U_0 \subset U$, still containing $x$, and being equal to 1 on $M \smallsetminus U$. Then $(D(s' - s))_x = df_x \otimes (s' - s)(x) + f(x)(D(s' - s))_x = 0$. This argument allows us to consider the covariant derivatives of *local sections* in $E$, because the extension of a local section to a global one does not influence the values of its covariant derivative on the original definition domain of the section.

**Terminology:** A section $\sigma$ of a bundle $E \to M$ with a connection (covariant derivative) is *parallel* at a point $x \in M$ iff $\sigma_{*,x}(TM)$ is the horizontal space of the connection, resp. $(D\sigma)_x = 0$.

Like for the connections in principal bundles, we can pull-back covariant derivatives on $f^*E$, for $f : N \to M$ a smooth map and $E \to M$ a vector bundle with a covariant derivative $D$.

We can also construct covariant derivatives on tensor bundles of $E$, i.e. on canonical subbundles in $\otimes E$, like $\Lambda^* E$, $\mathrm{End}(E) \simeq E^* \otimes E$, starting from a covariant derivative on $E$.

More generally, if $E_1$, $E_2$ are vector bundles over $M$ with covariant derivatives $D_1, D_2$, then we can define the covariant derivative $D_1 + D_2$ on $E_1 \oplus E_2$ (we differentiate the components) and $D := 1_{E_1} \otimes D_2 + D_1 \otimes 1_{E_2}$ on $E_1 \otimes E_2$ by the formula:

$$D(s_1 \otimes s_2) := D_1 s_1 \otimes s_2 + s_1 \otimes D_2 s_2,\ \forall s_i \in \mathbb{C}^\infty(E_i),\ i = 1, 2.$$

Indeed, the decomposable sections of type $s_1 \otimes s_2$ linearly span the space of all sections of $E_1 \otimes E_2$ and $D$ can be extended by linearity to define a covariant derivative on $E_1 \otimes E_2$.

We have seen that, for every element of a fiber bundle with connection, there exists a local section in this bundle, extending this element, that is *parallel* at that point. Asking that the section is parallel on a neighborhood is, on the other hand, much more restrictive. The obstruction to the existence of such a parallel extension is the *curvature*

## 5. Curvature of a connection

**Lemma 5.1.** *Let $\omega^{MC}$ be the Maurer-Cartan form of a Lie group $G$. Then the following* structure equation *holds:*

$$d\omega^{MC}(X, Y) + [\omega^{MC}(X), \omega^{MC}(Y)] = 0,\ \forall X, Y \in \mathfrak{g}.$$

The proof is immediate.

In order to write the equation above without arguments, we introduce the following notation

**Definition 5.2.** *Let $\omega_1, \omega_2$ be $k$-, resp. $l$-forms on $M$ with values in a Lie algebra $\mathfrak{g}$. The wedge product $\omega_1 \wedge \omega_2$ is a $k + l$-form on $M$ with values in $\mathfrak{g}$, defined by*

$$(\omega_1 \wedge \omega_2)(X_1, ..., X_{k+l}) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sign}(\sigma) \left[\omega_1(X_{\sigma(1)}, ..., X_{\sigma(k)}), \omega_2(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)})\right],$$

*for all $X_1, ..., X_{k+l} \in TM$.*

**Remark 5.3.** If $\mathfrak{g}$ is non-abelian, then $\omega \wedge \omega$ may be non-zero even if $\omega$ has odd degree. In fact, $\omega_1 \wedge \omega_2 = (-1)^{kl+1} \omega_2 \wedge \omega_1$ so the wedge product is actually precisely then commutative when $\omega_1$ and $\omega_2$ have odd degree, and anti-commutative if one of the factors has even degree. On the other hand, the wedge product on $\mathfrak{g}$-valued forms is not necessary associative.

With this notation, $[\omega(X), \omega(Y)] = \frac{1}{2}(\omega \wedge \omega)(X, Y)$, thus the structure equation for the Maurer-Cartan form can be written:

$$d\omega^{MC} + \frac{1}{2}\omega^{MC} \wedge \omega^{MC} = 0. \tag{8}$$

**Definition 5.4.** *Let $\omega \in \mathbb{C}^\infty(\Lambda^1 P \otimes \mathfrak{g})$ be a connection form on a $G$-principal bundle. The* curvature form *of $\omega$ is the two-form $\Omega \in \mathbb{C}^\infty(\Lambda^2 P \otimes \mathfrak{g})$ defined by*

$$\Omega := d\omega + \frac{1}{2}\omega \wedge \omega.$$

Because $\omega|_{T^\vee P}$ coincides with the Maurer-Cartan form on the fibers, then (8) implies:

**Proposition 5.5.** *The curvature form $\Omega$ vanishes on $T^\vee P$, thus $\Omega$ can be seen as a section in $\Lambda^2 M \otimes \mathrm{Ad}(P)$.*

*Proof.* From (8), $\Omega(\bar{X}, \bar{Y}) = 0$ for $\bar{X}, \bar{Y}$ fundamental vector fields. Let us consider now one fundamental vector field $\bar{X}$, $X \in \mathfrak{g}$, and a horizontal vector field $\tilde{Y}$, $Y$ vector field on $M$. Then the quadratic term in $\omega$ vanishes on the pair $(\bar{X}, \tilde{Y})$, thus

$$\Omega(\bar{X}, \tilde{Y}) = d\omega(\bar{X}, \tilde{Y}) = \bar{X}.(\omega(\tilde{Y})) - \tilde{Y}.(\omega(\bar{X})) - \omega([\bar{X}, \tilde{Y}]).$$

the first two terms vanish, and the last term, $\omega$ applied to the Lie bracket of the fundamental field $\bar{X}$ and the horizontal vector field $\tilde{X}$ vanishes as well, because the flow of $\bar{X}$ acts by right multiplication on $P$, and thus sends $\tilde{Y}$ into itself, thus $[\bar{X}, \tilde{Y}] = 0$.

So $\Omega$ vanishes on $T^\vee P$. Moreover, it is $G$-equivariant.

**Lemma 5.6.** *Let $\alpha$ be a $k$-form on $P$ with values in $V$, for $\rho : G \to \mathrm{GL}(V)$ a linear representation, such that*

1. *$\alpha(X, ...) = 0$, $\forall X \in T^\vee P$, and*
2. *$R_g^* \alpha = \rho(g^{-1})(\alpha)$, $\forall g \in G$.*

*Then $\alpha$ can be identified with a section $\underline{\alpha}$ in $\Lambda^k M \otimes P \times_\rho V$, such that*

$$[\sigma, \sigma^* \alpha] = \underline{\alpha}, \ \forall \sigma \textit{ local section in } P,$$

*where $[\sigma, \beta]$, for a $k$-form $\beta$ on $M$ with values in $V$ and $\sigma$ a local section in $P$, is the local section in $\Lambda^k M \otimes P \times_\rho V$ defined by*

$$T_x M \ni X_1, .., X_k \mapsto [\sigma(x), \beta(X_1, ..., X_k)] \in P \times_\rho V.$$

*Proof.* The proof is similar to the case of degree 0 (that a $G$-equivariant function frm $P$ to $V$ is equivalent to a section in the corresponding associated bundle): Let $X_1, ..., X_k \in T_x M$ and $p \in P_x$. Take any lifts $Y_1, ..., Y_k \in T_p P$ and compute $\alpha(Y_1, ..., Y_k)$ (the result does not depend on the chosen lifts because $T^\vee P \subset \ker \alpha$). This defines an element $[p, \alpha(Y_1, ..., Y_k)] \in P \times_\rho V$, and, if we consider $p' := p \cdot g \in P_x$, $g \in G$, then if we consider $Y_i' := (R_g)_* Y_i$, $i \in \{1, ..., k\}$, as lifts of $X_1, ..., X_k$ in $p'$, then we immediately get $[p', \alpha(X_1', ..., X_k')] = [p, \alpha(Y_1, ..., Y_k)]$ using the $G$-equivariance of the $k$-form $\alpha$. ☐

It follows that $\Omega$ can be seen as a 2-form $\underline{\Omega}$ on $M$ with values in $\mathrm{Ad}(P)$. ☐

**Remark 5.7.** In the case of a connection on $\mathrm{GL}(E)$ defined by a covariant derivative on $E$, the adjoint bundle $\mathrm{Ad}(\mathrm{GL}(E))$ is just the bundle $\mathrm{End}(E)$ of endomorphisms of $E$.

We define now the *exterior covariant derivative* on $E$-valued forms, where $E$ is a vector bundle with a linear connection (covariant derivative):

Let $E \to M$ be a vector bundle (real or complex) with a connection (covariant derivative) $\nabla$. The operator $\nabla : \mathbb{C}^\infty(L^0(E)) \to \mathbb{C}^\infty(L^1(E))$ induces, by the following formula (based on the usual formula for the exterior differential), the following operators:

$$d^\nabla : C^\infty(\Lambda^k(E)) \to C^\infty(\Lambda^{k+1}(E)),$$

$$\begin{aligned} d^\nabla(\alpha)(X_0,\dots,X_k) \;&:=\; \sum_{j=0}^{k}(-1)^j\nabla_{X_j}\Big(\alpha(X_0,\dots,\hat{X}_j,\dots,X_k)\Big)+ \\ &+\sum_{0\le j<l\le k}(-1)^{j+l}\alpha([X_j,X_l],X_0,\dots,\hat{X}_j,\dots,\hat{X}_l,\dots,X_k), \end{aligned} \tag{9}$$

where the hat $\hat{}$ indicates a missing term. The proof of the following proposition is straightforward (by induction):

**Proposition 5.8.** *Let $\alpha \in C^\infty(\Lambda^k(E))$ and $\beta \in C^\infty(\Lambda^l M)$. Then*

$$d^\nabla(\beta\wedge\alpha) = d\beta\wedge\alpha + (-1)^l\beta\wedge d^\nabla\alpha.$$

*Moreover, this property together with the fact that $d^\nabla = \nabla$ on $\mathbb{C}^\infty(E)$ uniquely determines the operator $d^\nabla : C^\infty(\Lambda^*(E)) \longrightarrow C^\infty(\Lambda^*(E))$.*

**Remark 5.9.** If $\nabla$ is a trivial connection (for example, given by a (local) frame), then $d^\nabla$ is the usual exterior derivative of forms with values in a (fixed) vector space, therefore $(d^\nabla)^2 = 0$.

In general, $(d^\nabla)^2 \neq 0$.

**Definition 5.10.** *Let $\nabla$ be a connection on a vector bundle $E \to M$. Its* curvature tensor *$R^\nabla$ is a 2–form with values in* End$(E)$*, defined by $K^\nabla s := (d^\nabla)^2 s$, more precisely*

$$K^\nabla_{X,Y}s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s,\;\; X,Y\in TM,\;\; s\in C^\infty(E).$$

**Remark 5.11.** Even if $(d^\nabla)^2$ is not zero, it is a zero-order differential operator (i.e. a tensor) - see below.

Using Proposition 5.8 and the fact that the space of sections (sheaf) $C^\infty(\Lambda^*(E))$ is isomorphic to the tensor product $C^\infty(\Lambda^* M)\otimes C^\infty(E)$ (the tensor product is over the ring $C^\infty(M)$ of real/complex–valued functions on $M$), we get

**Proposition 5.12.** *Let $\alpha \in C^\infty(\Lambda^*(E))$. Then $(d^\nabla)^2\alpha = K^\nabla\wedge\alpha$.*

Here we have used the following notation: for a $k$-form $\alpha$ with values in $E$, and a $l$-form $\eta$ with values in End$(E)$, we define their exterior product by taking the tensor product of the following maps:

$$\begin{aligned} \Lambda^k M\otimes\Lambda^l M &\overset{\wedge}{\to} \Lambda^{k+l}M \\ \eta\otimes\xi &\mapsto \eta\wedge\xi, \end{aligned}$$

$$\begin{aligned} E\otimes \mathrm{End}(E) &\to E \\ s\otimes A &\mapsto A(s). \end{aligned}$$

If we consider two forms with values in End$(E)$, $A \in \mathbb{C}^\infty(\Lambda^k(\mathrm{End}(E)))$ and $B \in \mathbb{C}^\infty(\Lambda^l(\mathrm{End}(E)))$, we consider their wedge product

$$A\wedge B\in\Omega^{k+l}(\mathrm{End}(E))$$

by tensoring the wedge product of forms on $M$ with the commutator (linear) map:

$$\begin{array}{ccc} \mathrm{End}(E)\otimes \mathrm{End}(E) & \to & \mathrm{End}(E) \\ A\otimes B & \mapsto & AB-BA. \end{array}$$

We have the following *differential Bianchi identity* (sometimes called the *second Bianchi identity* in the case of connections on the tangent bundle):

**Proposition 5.13.** *The curvature tensor $K^\nabla \in \mathbb{C}^\infty(\Lambda^2(\mathrm{End}(E)))$ of a connection $\nabla$ on $E \to M$ is $d^\nabla$-closed.*

*Proof.* Let us compute $(d^\nabla)^3 s$, for a section $s$ in $E$:

$$(d^\nabla)^3 s = d^\nabla((d^\nabla)^2 s) = d^\nabla\left(K^\nabla s\right) = d^\nabla K^\nabla s + K^\nabla \wedge \nabla s.$$

On the other hand,

$$(d^\nabla)^3 s = (d^\nabla)^2 \nabla s = K^\nabla \wedge \nabla s,$$

therefore $d^\nabla K^\nabla = 0$. □

The differential Bianchi identity has the following formulation in terms of connection 1-forms:

**Proposition 5.14.** *Let $\omega$ be a connection 1-form on the $G$-principal bundle $P \to M$ and $\Omega = d\omega + \frac{1}{2}\omega\wedge\omega$ its curvature form. Then*

$$d\Omega + \omega \wedge \Omega = 0.$$

The proof follows directly by differentiating the formula defining $\Omega$ and using that $\omega \wedge (\omega \wedge \omega) = 0$ (from the Jacobi identity on $\mathfrak{g}$).

We need to show that the curvature tensor $K$ is canonically determined by the curvature form $\Omega$:

**Proposition 5.15.** *Let $P \to M$ be a $G$ principal bundle with a connection form $\omega$, its curvature form $\Omega$ and let $\nabla^\omega : \mathbb{C}^\infty(E) \to \mathbb{C}^\infty(\Lambda^1 M \otimes E)$ be the covariant derivative associated to $\omega$ for an associated vector bundle $E := P \times_\rho V \to M$, for $\rho : G \to \mathrm{GL}(V)$ a linear representation of $G$ on $V$.*

*Then $\forall X, Y \in T_x M$, the curvature tensor*

$$K_{X,Y}[p,v] = [p, \bar\rho(\Omega(\tilde X, \tilde Y))(v)],\ \forall [p,v] \in E_x.$$

*Proof.* We use the description of $\Omega$ as a 2-form on $M$ with values in $\mathrm{Ad}(P)$, as in Proposition 5.5, and then proceed as in the proof of Proposition 4.14, from which we also use the notations $\sigma$ is a local section in $P$ and $w : M \to V$ such that $x \mapsto [\sigma(x), w(x)] \in E_x$ is equal to a fixed section $s \in C^\infty(E)$:

$$\nabla^\omega_X[\sigma, w] = [\sigma, \bar\rho(\sigma^*\omega(X))(w) + dw(X)].$$

Then we obtain

$$\nabla^\omega_X\nabla^\omega_Y s = [\sigma, \bar\rho(\sigma^*\omega(X))(\bar\rho(\sigma^*\omega(Y)))(w)) + \bar\rho(\sigma^*\omega(X))(dw(Y)) + X.\bar\rho(\sigma^*\omega(Y))(w) + X.dw(Y)].$$

Now we use that $X.\sigma^*\omega(Y) - Y.\sigma^*\omega(X) - \sigma^*([X,Y]) = \sigma^* d\omega(X,Y)$ and that

$$\bar\rho(\sigma^*\omega(X))\circ\bar\rho(\sigma^*\omega(X)) - \bar\rho(\sigma^*\omega(X))\circ\bar\rho(\sigma^*\omega(Y)) = \bar\rho(\sigma^*[\omega(X),\omega(Y)]) = \tfrac{1}{2}\bar\rho(\sigma^*(\omega\wedge\omega)(X,Y)).$$

Also note that $X.dw(Y) = Y.dw(X) - dw([X,Y]) = d(dw)(X,Y) = 0$. Then we get

$$K_{X,Y}s = \nabla^\omega_X\nabla^\omega_Y s - \nabla^\omega_Y\nabla^\omega_X s - \nabla^\omega_{[X,Y]}s = [\sigma, \bar\rho(\sigma^*\Omega(X,Y))(w)].$$

□

**Remark 5.16.** If $P \subset \mathrm{Aut}(E)$ (and $G \subset \mathrm{GL}(V)$), we can simply write $K = \underline{\Omega}$. However, it is not true that every Lie group $G$ can be emedded in a linear group (the universal covering of $\mathrm{SL}(2,\mathbb{R})$ is such an counterexample). For a general Lie group $G$, we need to distinguish between $K = (d^{\nabla^\omega})^2$ and $\underline{\Omega}$. Indeed, the bundle $\mathrm{Ad}(P)$ where $\underline{\Omega}$ takes values, reflects the structure of $E$ to a better extent as $\mathrm{End}(E)$: a $\mathrm{U}(n)$ (or unitary) connection on a rank $n$ complex vector bundle is a complex linear connection (i.e., the covariant derivative is complex-linear) and *metric*, i.e., there exist a symmetric, positive definite, *Hermitian* tensor $h \in C^\infty(E \otimes_{\mathbb{R}} E)$, (Hermitian means that such that $h(iX, iY) = h(X,Y)$ for all $X, Y \in E$, where $iX$ is the multiplication with the complex number $i$ on the fibers of $E$) that is *parallel* w.r.t. the covariant derivative.) In this case, $P = \mathrm{U}(E,h)$, the space of unitarz frames of $E$, $\mathrm{Ad}(P)$ can be identified with the subbundle of $\mathrm{End}(E)$ of antihermitian endomorphisms of $E$.

**Remark 5.17.** Although such a statement is false for Lie groups (see above), it is in general true that every Lie algebra admits a finite-dimensional faithful representation (i.e., $\exists \rho : G \to \mathrm{GL}(V)$ s.t. $\bar\rho : \mathfrak{g} \to \mathrm{End}(V)$ is injective (thus $\rho$ is an immersion)), and thus there is no loss of information on $\Omega$ if we consider the curvature tensor $K$ for the bundle associated to such a representation $\rho$.

From now on, we will consider that $E$ is a vector bundle with a covariant derivative that comes from a $G$-connection. The difference of two such covariant derivatives comes thus from a section $\theta$ of $\Lambda^1 M \otimes \mathrm{Ad}(P)$, and is precisely the image of $\theta$ through $\bar\rho$ in $\Lambda^1 M \otimes \mathrm{End}(E)$.

Let us compute the change of the curvature tensor when the connection changes: Let $\nabla' := \nabla + \theta$ be another connection on $E$, where $\theta \in \mathbb{C}^\infty(\Lambda^1(E))$. Then $d^{\nabla'} = d^\nabla + \theta \wedge \cdot$, therefore

$$(d^{\nabla'})^2 s = d^\nabla(d^\nabla s + \theta s) + \theta \wedge (d^\nabla s + \theta s) = (d^\nabla)^2 s + d^\nabla \theta s - \theta \wedge \nabla s + \theta \wedge \nabla s + \theta \wedge \theta s.$$

the last term is not zero, even if $\theta$ is a 1–form with values in $\mathrm{End}(E)$; this is because $\mathrm{End}(E)$ is not commutative. We get

$$R^{\nabla'} = R^\nabla + d^\nabla \theta + \frac{1}{2}[\theta, \theta]. \tag{10}$$

**Remark 5.18.** The gauge transformation law of the curvature form $\Omega$ is, for a gauge transformation $\bar\phi : M \to AD(P)$

$$\Omega' := \phi^* \Omega = \mathrm{Ad}_{\bar\phi^{-1}} \Omega,$$

which is, of course, the transformation law of a section in $\Lambda^2 \otimes \mathrm{Ad}(P)$.

Note that, if $E = E_1 \oplus E_2$, and the connection $\nabla$ on $E$ is the sum of a connection $\nabla^1$ on $E_1$ and $\nabla^2$ on $E_2$, then the curvature tensor $K^\nabla$ admits a block decomposition, being equal to $K^{\nabla^i}$ on $\mathrm{End}(E_i)$, $i = 1, 2$ and zero on the "mixed blocks" $E_1^* \otimes E_2$ and $E_2^* \otimes E_1$.

Also, if $\nabla = 1_{E_1} \otimes \nabla^2 + \nabla^1 \otimes 1_{E_2}$ on $E_1 \otimes E_2$, then we have $K^\nabla = K^{\nabla^1} \otimes 1_{E_2} + 1_{E_1} \otimes K^{\nabla^2}$.

We will use now the curvature of a connection in $E$ to define cohomology classes in $M$, that turn out to depend only on the vector bundle $E$ on $M$ and not on the chosen connection.

## 6. Yang-Mills theory

Let $(M,g)$ be a compact Riemannian manifold and let $P \to M$ be a $G$-principal bundle over $M$, and suppose there is a fixed Ad-invariant metric $h_0$ on $\mathfrak{g}$, i.e., $h_0(X,Y) = h_0(\mathrm{Ad}_a X, \mathrm{Ad}_a Y)\ \forall a \in G,\ X,Y \in \mathfrak{g}$ (this is possible for all compact Lie groups $G$). Then $h$ defines a metric $h$ on the vector bundle $\mathrm{Ad}P$, i.e. a *positive definite* section $h$ in the vector bundle $S^2(\mathrm{Ad}P^*)$, and for every $G$-connection on $P$, this section is parallel (or covariant constant) for the induced covariant derivative.

The *Yang-Mills theory* looks for connections $\nabla$ on $P$ that are minimize (or, more generally, are *critical points* of the *Yang-Mills functional*, see below) the *total norm* of the curvature

$$\int_M \|K^\nabla\| \mathrm{vol}^g.$$

Here, the pointwise norm $\|K^\nabla_x\| \in \Lambda^2 M_x \otimes \mathrm{Ad}P_x$ is determined by the metric $g$ on $\Lambda^2 M$ and by $h$ on $\mathrm{Ad}P$.

For non-compact groups and $(M,g)$ pseudo-Riemannian, we need to reformulate the above integral: on $\mathfrak{g}$, we replace the positive-definite scalar product $h_0$ with a *non-degenerate* Ad-invariant symmetric form $B$ (not all Lie groups admit such a form, however a large class – including $\mathrm{GL}(n,\mathbb{R}),\ \mathrm{GL}(n,\mathbb{C}), \mathrm{SO}(n,1)$, the latter being the group of oiented automorphisms of the Minkowski space – does), and we replace the metric induced by $g$ on $\Lambda^k M$ with a non-degenerate scalar product. For the latter, it is useful to introduce the *Hodge star* operator

$$* : \Lambda^k M \to \Lambda^{n-k} M$$

on an oriented, pseudo-Riemannian $n$-manifold. As the definition is fiberwise, it is enough to define the $*$ operator for an oriented vector space $V$, on which a scalar product $g$ of signature $(p,q)$ (where $p+q=n$) has been fixed:

Let $e^1, ..., e^p, e^{p+1}, ..., e^n$ be an oriented orthonormal basis of $V^*$, such that

$$g(e^i, e^j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \leq p \\ -1, & i = j > p \end{cases}$$

The metric volume element is then $\mathrm{vol}^n := e^1 \wedge ... \wedge e^n \in \Lambda^n V^*$.

Then, for $I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$, we define $e^I := e^{i_1} \wedge ... \wedge e^{i_k}$ and we set

$$g(e^I, e^J) := \begin{cases} 0, & I \neq J \\ \varepsilon(i_1)...\varepsilon(i_k), & I = J \end{cases},$$

where $\varepsilon(i) := g(e^i, e^i)$. We define then the *Hodge* $*$ operator by the property

$$g(*\alpha, \beta)\mathrm{vol}^g = \alpha \wedge \beta, \quad \forall \alpha \in \Lambda^k V^*, \forall \beta \in \Lambda^{n-k} V^*.$$

It is easy to compute

**Proposition 6.1.** *The Hodge $*$ operator satisfies:*

(1) $*(e^{i_1} \wedge ... \wedge e^{i_k}) := \varepsilon(i_1)...\varepsilon(i_k)\varepsilon(i_1, ..., i_k, j_1, ..., j_{n-k}) e^{j_1} \wedge ... \wedge e^{j_{n-k}}$, *$\forall 1 \leq i_1 < ... < i_k \leq n$, where $j_1 < ... < j_{n-k}$ are the remaining indices, and $\varepsilon(i_1, ..., i_k, j_1, ..., j_{n-k})$ is the signum of this permutation of $\{1, ..., n\}$.*

(2) $*(*\alpha) = (-1)^{k(n-k)+p}\alpha$, *$\forall \alpha \in \Lambda^k V^*$.*

If we have a vector bundle $E \to M$, then we define, on the fiber at $x \in M$, the $*$ operator on $\Lambda^k M_x \otimes E_x$ as the tensor product of the $*$ operator defined above (with $V := T_x M$) with the identity of $E_x$.

Let $B_0 : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$, in case $\mathfrak{g}$ is a complex Lie algebra) be a symmetric, non-degenerate form (for *semisimple* Lie algebras $g$ there exist such bilinear forms $B_0$; for $\mathfrak{g}$ *simple* such a $B_0$ is unique up to a factor). Then $B_0$ defines a section $B$ in $S^2(\mathrm{Ad}g^*)$, the second symmetric power of the dual $\mathrm{Ad}P^*$ of the adjoint bundle. (Note that, as $B$ is determined by an invariant element of the representation $S^2\mathfrak{g}^*$, it is a $\nabla$-parallel section of $S^2(\mathrm{Ad}g^*)$, for any $G$-connection $\nabla$.)

We define then $B(\alpha \wedge \beta)$, for $\alpha \in C^\infty(\Lambda^k M \otimes \mathrm{Ad}P)$, $\beta \in C^\infty(\Lambda^{n-k} M \otimes \mathrm{Ad}P)$, as follows: consider the wedge pairing

$$\wedge : \Lambda^k M \otimes \Lambda^{n-k} M \to \Lambda^n M,$$

and take the tensor product of this pairing with $B$ to obtain a bilinear pairing (symmetric if $k$ is even, skew-symmetric if $k$ is odd)

$$(\Lambda^k M \otimes \mathrm{Ad}P) \otimes (\Lambda^{n-k} M \otimes \mathrm{Ad}P) \to \Lambda^n M.$$

**Example 6.2.** For $G := \mathrm{GL}(k, \mathbb{R})$, the trace of the product is a non-degenerate symmetric bilinear form on the Lie algebra $\mathfrak{gl}(k, \mathbb{R})$. If we restrict it to the Lie subalgebra $\mathfrak{so}(k)$ of skew-symmetric matrices, it is *negative definite*

$$\mathrm{tr}(A^2) = -\mathrm{tr}(A \cdot {}^t\!A) = -\|A\|^2,$$

if we denote by $\|\cdot\|^2$ the Euclidean squared norm on $\mathrm{End}(\mathbb{R}^n)$ (the sum of the squares of all entries of a matrix.)

**Example 6.3.** For $G := \mathrm{GL}(k, \mathbb{C})$, the trace of the product is a non-degenerate symmetric bilinear form on the *complex* Lie algebra $\mathfrak{gl}(k, \mathbb{C})$ with values in $\mathbb{C}$. If we restrict it to the Lie subalgebra $\mathfrak{u}(k)$ of antihermitian matrices, it is real-valued and negative definite

$$\mathrm{tr}(A^2) = -\mathrm{tr}(A \cdot {}^t\!\bar{A}) = -\|A\|^2,$$

if we denote by $\|\cdot\|^2$ the Euclidean squared norm on $\mathrm{End}(\mathbb{C}^n)$ (the sum of the square norms of all entries of a matrix.)

**Definition 6.4.** *A connection $\nabla$ on the $G$-bundle $P \to M$ (where an invariant symmetric non-degenerate form $B \in C^\infty(S^2\mathrm{Ad}P^*)$, and a pseudo-Riemannian metric on the compact, oriented manifold $M$ have been fixed) is said to be* Yang-Mills *iff it is a* critical point *of the* Yang-Mills functional

$$YM(\nabla) := \frac{1}{2}\int_M B(K^\nabla \wedge *K^\nabla),$$

*i.e., for every family $\{\nabla^t\}_{t\in[0,1]}$ of $G$-connections on $P$, depending smoothly on $t$*[1]*, and such that $\nabla^0 = \nabla$, the derivative*

$$\left.\frac{d}{dt}\right|_{t=0} YM(\nabla^t) = 0.$$

**Proposition 6.5.** *A connection $\nabla$ as above is Yang-Mills iff*

$$d^\nabla * K^\nabla = 0.$$

[1]because the space of connections is an affine space modelled on $C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P)$, we can always construct "linear" paths $\nabla^t := \nabla + t\theta$, for $\theta$ some section of $\Lambda^1 M \otimes \mathrm{Ad}P$.

**Example 6.6.** If $n = 4$ and $(M, g)$ is a Riemannian manifold, or a manifold with signature $(2, 2)$, then $*^2 : \Lambda^2 M \to \Lambda^2 M$ is the identity (see Proposition 6.1), so $\Lambda^2 M$ splits as the direct sum of the bundles $\Lambda^2_+ M \oplus \Lambda^2_- M$ of *self-dual* (SD) and *anti-self-dual* (ASD) 2-forms. The SD forms satisfy $*\alpha = \alpha$, and the ASD forms satisfy $*\alpha = -\alpha$, so if the curvature $K^\nabla$ of $\nabla$ is SD, resp. ASD, then the condition in the Porposition above is equivalent to the Bianchi identity. Therefore, SD and ASD connections are automatically Yang-Mills. We well see later that, for Riemannian $(M, g)$ and compact $G$, they are the absolute minimizers of the Yang-Mills functional.

## 7. Chern-Weil theory

**Definition 7.1.** *Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G$. An (ad-)invariant polynomial $Q : \mathfrak{g} \to \mathbb{C}$, resp. an (ad-)invariant symmetric multilinear form $S : \mathfrak{g} \otimes \ldots \otimes \mathfrak{g} \longrightarrow \mathbb{C}$, such that*

$$Q(\mathrm{Ad}_g A) = Q(A), \ \forall A \in \mathfrak{g}, \ g \in G,$$

*resp. that*

$$S(\mathrm{Ad}_g A, \ldots, \mathrm{Ad}_g A_k) = S(A_1, \ldots, A_k), \ \forall A_i \in \mathfrak{g}, \ g \in G.$$

From a symmetric multilinear form $S$ we get a polynomial $Q^S(A) := S(A, \ldots, A)$ and, conversely, from a polynomial $Q$ on $\mathfrak{g}$, homogeneous of degree $k$, we get the symmetric multilinear form

$$S^Q(A_1, \ldots, A_k) := \frac{1}{k!} \frac{d^k}{dt_1 \ldots dt_k} |_{t_1 = \cdots = t_k = 0} Q(t_1 A_1 + \cdots + t_k A_k).$$

It is clear that $Q^{S^Q} = Q$ and that $Q$ is Ad-invariant iff $S^Q$ is.

**Example.** The trace, determinant and, more generally, the homogeneous components of $A \mapsto \det(Id + A)$ are Ad-invariant polynomials on the Lie algebra $\mathrm{End}(V)$, for $V$ a real or complex vector space.

**Proposition 7.2.** *Let $G$ be a connected Lie group. A symmetric multilinear form $S : \mathfrak{g}^{\otimes k} \to \mathbb{C}$ is invariant iff*

$$\sum_{j=1}^{k} S(A_1, \ldots, [B, A_j], \ldots, A_k) = 0, \ \forall B, A_i \in \mathfrak{g}. \tag{11}$$

*Proof.* Let $g_t := \exp(tB) \in G$, for $t \in \mathbb{R}$, $B \in \mathfrak{g}$. All elements in the neighborhood of the identity of $G$ can be written like this. Fix $A_1, \ldots, A_k \in \mathfrak{g}$ and consider $F(t) := S(g_t A_1 g_t^{-1}, \ldots, g_t A_k g_t^{-1})$. Then note that

$$\frac{d}{dt} \exp(tB) = B \exp(tB), \text{ thus } \frac{d}{dt} g_t = B g_t \text{ and } \frac{d}{dt} g_t^{-1} = -g_t^{-1} B.$$

If $S$ is invariant, then $F$ is constant, therefore $F'(0) = 0$, which implies (11). Conversely, if (11) holds for every $A_j$ and $B \in \mathfrak{g}$, in particular for $g_t A_j g_t^{-1}$ and $B$, it follows that $F'(t) = 0 \ \forall t \in \mathbb{R}$, therefore

$$F(t) = F(0) = S(A_1, \ldots, A_k).$$

This implies that the map $g \mapsto S(gA_1 g^{-1}, \ldots, gA_k g^{-1})$ is constant on a neighborhood of $Id \in G$ (for fixed $A_1, \ldots, A_k \in \mathfrak{g}$). But $G$ is connected and the considered map is analytic (even polynomial), thus it has to be constant on whole $G$. ☐

Let $E \to M$ be a vector bundle with fiber $V$ and (this holds for the other examples of Lie groups above) let $S$ be an invariant symmetric multilinear form on $\mathrm{End}(V)$. Then $S$ induces a multilinear bundle map

$$S^E : \mathrm{End}(E) \otimes \ldots \otimes \mathrm{End}(E) \to \mathbb{C},$$

given, in a frame $f \in \mathrm{GL}(E)$, by

$$S^f(A_1, \ldots, A_k) := S(A_1^f, \ldots, A_k^f),$$

where $A_j^f$ is the element in $\mathrm{End}(E)$ defined by $A_j \in \mathfrak{g}$ and the frame $f$. The invariance of $S$ ensures that $S^f$ is independent of the frame and hence $S^E$ is well-defined.

By using again the sheaf isomorphism

$$\mathbb{C}^\infty(\Lambda^*(End(E))) \simeq \mathbb{C}^\infty(\Lambda^* M) \otimes_{\mathbb{C}^\infty(M)} \mathrm{End}(\mathcal{E})$$

we extend $S^E$ to a multilinear map

$$S^E : \Lambda^*(\mathrm{End}(E))^{\otimes k} \to \Lambda^* M,$$

$$S^E(\alpha_1 \otimes A_1, \ldots, \alpha_k \otimes A_k) := \alpha_1 \wedge \ldots \wedge \alpha_k \cdot S^E(A_1, \ldots . A_k),$$

forall $A_j \in End(E),\ \alpha_j \in \Lambda^* M$.

We can also define $P^E(\alpha) := S^E(\alpha, \ldots, \alpha) \in \Lambda^{kp} M$, $\alpha \in \Lambda^p(End(E))$, where $P$ is the polynomial associated to the symmetric multilinear map $S$.

**Theorem 7.3. (Chern-Weil)** *Let $E \to M$ be a vector bundle with fiber $V$, associated to a $G$-principal bundle $P$. Let $Q$ be an invariant polynomial on $\mathfrak{g}$ of degree $k$ and $\nabla$ a $G$-connection on $E$, with curvature $K^\nabla$. Then the $2k$–form $Q^E(K^\nabla) \in \mathbb{C}^\infty(\Lambda^{2k} M) \otimes \mathbb{C}$ is closed, and its class in the de Rham cohomology group $H^{2k}(M, \mathbb{C})$ is independent of $\nabla$.*

*Proof.* First we show that $Q^E(R^\nabla)$ is $d^\nabla$–closed. This is an immediate consequence of the following

**Lemma 7.4.** *Let $\alpha_j \in \mathbb{C}^\infty(\Lambda^{p_j}(\mathrm{Ad}(P)))$ and $S$ an invariant symmetric multilinear form of degree $k$. Denote by $\epsilon_j := p_1 + \cdots + p_{j-1}$. We have then*

$$d\left(S(\alpha_1, \ldots \alpha_k)\right) = \sum_{j=1}^{k} S(\alpha_1, \ldots, (-1)^{\epsilon_j} d^\nabla \alpha_j, \ldots, \alpha_k).$$

*Proof.* We proceed by induction over the sum $\epsilon$ of all degrees of the forms $\alpha_j$, that we suppose to be arranged such that $p_1 \leq \cdots \leq p_k$. For all $p_j = 0$ the claimed formula is just the covariant derivative of

$$S^E(A_1, \ldots, A_k),$$

for $A_j \in \mathrm{Ad}(P)$, where we note that $S^E$ is induced by a *constant* element of $\mathrm{Hom}(\mathfrak{g}^{\otimes k}, \mathbb{C})$, therefore its derivative vanishes.

Suppose now the claim is true for any $\alpha_j \in C^\infty(\Lambda^{p_j}(\mathrm{Ad}(P)))$ such that the sum of the degrees is $\sum p_j = \epsilon \geq 0$ and let now $\alpha'_j := \alpha_j$ for $1 \leq j < k$ and $\alpha'_k = \alpha_k \wedge \beta \in C^\infty(\Lambda^{p_k+1}(\mathrm{Ad}(P)))$, where $\beta$ is a 1–form on $M$. If we show that the claim holds for $S$ and for all $\alpha_1, \ldots \alpha_{k-1}, \alpha'_k$ as above and every 1–form $\beta$, then (again using that $C^\infty(\Lambda^{p+1}(\mathrm{Ad}(P))) = C^\infty(\Lambda^1 M) \otimes C^\infty(\Lambda^p(\mathrm{Ad}(P)))$ as a sheaf), the claim will be proven for $\epsilon + 1$.

Note that

$$S^E(\alpha_1, \ldots, \alpha'_k) = S^E(\alpha_1, \ldots, \alpha_k) \wedge \beta,$$

thus

$$\begin{aligned} dS^E(\alpha_1, \ldots, \alpha'_k) = & \\ = & \sum_{j=1}^{k} (-1)^{\epsilon_j} S^E(\alpha_1, \ldots, d^\nabla \alpha_j, \ldots, \alpha_k) \wedge \beta + (-1)^{\epsilon - 1} S^E(\alpha_1, \ldots, \alpha_k) \wedge d\beta = \\ = & \sum_{j=1}^{k-1} (-1)^{\epsilon_j} S^E(\alpha_1, \ldots, d^\nabla \alpha_j, \ldots, \alpha'_k) + (-1)^{\epsilon_k} S^E(\alpha_1, \ldots, d^\nabla \alpha'_k). \end{aligned}$$

□

To show that the cohomology class defined by $P^E(R^\nabla)$ is independent of $\nabla$, we need to compute the difference between $Q^E(K^{\nabla'})$ and $Q^E(K^\nabla)$, for a new connection $\nabla' = \nabla + \theta$ and show that it is an exact form.

Actually, we will consider a path of connections $\nabla^t$ between $\nabla$ and $\nabla'$, for example $\nabla^t := \nabla + t\theta$. Then $\nabla^0 = \nabla$ and $\nabla^1 = \nabla'$. Denote by $K^t := K^{\nabla^t}$. In order to show that

$$Q^E(K^1) = Q^E(K^0) + d\beta, \tag{12}$$

we will show that

$$\frac{d}{dt} Q^E(K^t) = d\beta^t,$$

and conclude by integration that the class of cohomology of $Q(K^t)$ is constant. More precisely, (12) holds with

$$\beta = \int_0^1 \beta^t dt$$

Let us compute the derivative in $t$ at $t = t_0$ of $Q^E(K^t)$: first note that

$$\frac{d}{dt}|_{t=t_0} K^t = \lim_{t \to t_0} K^t - K^{t_0} = \lim_{t \to t_0} \left( d^{\nabla^{t_0}} (t - t_0)\theta + \frac{1}{2}[(t - t_0)\theta, (t - t_0)\theta] \right) = d^{\nabla^{t_0}} \theta.$$

Therefore

$$\frac{d}{dt}|_{t=t_0} Q^E(K^t) = \sum_{j=1}^{k} S^E(K^{t_0}, \ldots, \overset{(j)}{d^{\nabla^{t_0}} \theta}, \ldots, K^{t_0}),$$

but this is, according to the previous lemma (using that $d^{\nabla^{t_0}} K^{t_0} = 0$), equal to $d\beta^{t_0}$, where

$$\beta^{t_o} := \sum_{j=1}^{k} S^E(K^{t_0}, \ldots, \overset{(j)}{\theta}, \ldots, K^{t_0}).$$

We get thus $Q^E(K^1) - Q^E(K^0) = d\beta$, where

$$\beta := \int_0^1 \left( \sum_{j=1}^{k} S^E(K^t, \ldots, \overset{(j)}{\theta}, \ldots, K^t) \right) dt$$

is the integral in $t$ of the forms $\beta^t$ found above. $ChS(\nabla^0, \nabla^1) := \beta$ is called the *relative Chern-Simons form* of the connections $\nabla^0$ and $\nabla^1$, corresponding to the invariant polynomial $Q$ of order $k$. □

**Corollary 7.5. (Chern classes)** *Let $E \to M$ be a complex vector bundle and let $C_k$ be the homogeneous polynomial of order $k$ in the invariant (non-homogeneous) polynomial expression $A \mapsto \det(Id + \frac{A}{2\pi i})$. Let $\nabla$ be a connection on $E$. Then the cohomology classes*

$$c_k^{\mathbb{R}}(E) := \left[C_k^E(K^\nabla)\right]$$

*are real, i.e. the above forms represent cohmology classes $c_k^{\mathbb{R}}(E) \in H^{2k}(M, \mathbb{R})$, called the (real)* Chern classes *of $E$. Denote by $c(E) := 1 + c_1(E) + c_2(E) + ... \in H^*(M, \mathbb{R})$ the* total Chern class *of $E \to M$.*

**Remark 7.6.** The classes $c_k^{\mathbb{R}}(E)$ defined above are actually induced by the *integer Chern classes $c_k(E) \in H^{2k}(M, \mathbb{Z})$*, defined in algebraic topology [2].

*Proof.* It suffices to consider a particular connection and show that the resulting form $C_k^E(K^\nabla)$ is a real $2k$–form.

We consider a hermitian metric on $E$ and take $\nabla$ a hermitian connection. Then its curvature will be a 2–form with values in $\mathrm{End}(E)$, actually in the subspace (or rather, Lie subalgebra) of antihermitian endomorphisms of $E$. This space is isomorphic, via an *unitary frame* $f : U \times V \to E|_U$ (a frame that is a unitary isomorphism on the fibers), to the space of antihermitian matrices $\mathfrak{u}(V) \subset End(V)$.

What we need to show is that $C_k(iA) \in \mathbb{R}$, for any antihermitian matrix $A$. But $iA$ is hermitian and $Id + \frac{1}{2\pi i}A$ too, therefore $\det(Id + \frac{1}{2\pi i}A)$ is real, and so must be all its homogeneous components. □

**Proposition 7.7.** *Let $T \to \dot{\gamma}^1$ be the tautologic line bundle: $\forall p \in \dot{\gamma}^1$, $T_p := p \subset \mathbb{C}^2$. . Then the integral over $\dot{\gamma}^1$ of first Chern class of $T$ is equal to $-1$.*

This *normalization* property, together with the following two:

(1) *characteristic:* For every complex vector bundle $E \to N$ and every smooth map $f : M \to N$, it holds $c(f^*E) = f^*c(E)$.
(2) *multiplicativity:* For $E, F \to M$ two complex vector bundles, it holds $c(E \oplus F) = c(E) \wedge c(F)$

characterize completely the total Chern class, [2].

## 8. Characteristic classes

**Definition 8.1.** *A (real)* characteristic class *$q$ associated to the Lie group $G$ is a map*[2] *from the space $\mathcal{P}G$ of $G$-principal bundles on a manifold $M$ with values in the de Rham cohomology $H^*(M, \mathbb{R})$ such that, for every $G$-bundle $P \to M$ and any smooth (continuous) map $f : N \to M$, we have $q(f^*P) = f^*q(P)$.*

**Remark 8.2.** One can define *integer* or $\mathbb{Z}_2$ characteristic classes, by considering appropriate cohomology theories [2]. In fact most characteristic classes that we construct here are *realizations* of integer characteristic classes.

It is clear that the Chern-Weil theory constructs characteristic classes out of invariant polynomials. The theory of *classifying spaces* shows that the converse is true: every real characteristic class is obtained from an invariant polynomial by the Chern-Weil theory.

[2] functor

We focus thus on determining the ring of invariant polynomials on a Lie algebra $\mathfrak{g}$:

**Definition 8.3.** *Let $A \in \mathfrak{u}(n)$ be a antihermitian matrix. Then its eigenvalues are imaginary, thus the* total Chern polynomial

$$C(A) := \det\left(\mathrm{id} + \frac{1}{2\pi i}A\right) = \prod_{j=1}^{n}(1 + x_j),$$

*where $\{2\pi i x_j \mid j = 1, ..., n\}$ is the set of the eigenvalues of $A$, is a real-valued invariant polynomial on $\mathfrak{u}(n)$. Its homogeneous component (as a polynomial in $x_1, ..., x_n$) of degree $k$ is*

$$C_k(A) = \sum_{1 \leq j_1 < ... < j_k \leq n} x_{j_1} ... x_{j_k},$$

*the $k$th Chern polynomial on $\mathfrak{u}(n)$.*

**Exercise 8.4.** Show that $C_1(A) = \mathrm{tr}\left(\frac{A}{2\pi i}\right)$, $C_n(A) = \det\left(\frac{A}{2\pi i}\right)$ and $(C_1^2 - 2C_2)(A) = \mathrm{tr}\left(\frac{A}{2\pi i}\right)^2$.

**Theorem 8.5.** *The ring of invariant real-valued polynomials on $\mathfrak{u}(n)$ is isomorphic to*

$$\mathbb{R}[\mathfrak{u}(n)]^{\mathrm{U}(n)} \simeq \mathbb{R}[x_1, ..., x_n]^{sym} \simeq \mathbb{R}[C_1, ..., C_n],$$

*where $\mathbb{R}[x_1, ..., x_n]^{sym}$ is the ring of* symmetric *polynomials in the indeterminates $x_1, ..., x_n$.*

One of the main invariant polynomial (power series) on $\mathfrak{u}(n)$ is the *Chern character poynomial* (power series)

$$CH_k^n := \left(\sum_{j=1}^{n} e^{x_j}\right)\Bigg|_{\deg \leq k} = \left(\sum_{j=1}^{n}(1 + x_j + \frac{1}{2}{x_j}^2 + ...)\right)\Bigg|_{(\deg \leq k)}.$$

The subscript $(\deg \leq k)$ means that we truncate the corresponding power series at degree $k$ ($CH^n$ or $CH_\infty^n$ is the non truncated power series). For example:

$$CH_2^n = n + C_1 + \frac{1}{2}(C_1^2 - 2C_2).$$

**Exercise 8.6.** Show that, if $A \in \mathfrak{u}(n)$, then

$$CH(A) = \mathrm{tr}(\exp(A)),$$

where the exponential is defined by the series $\exp(A) = \sum_{k \in \mathbb{N}} \frac{A^n}{n!}$ (for $n = 0$ we set $A^0 := \mathrm{Id}$).

When applied to the curvature of a connection on a $G$-bundle over $M$, these invariant polynomials produce, via the Chern-Weil theory, *characteristic classes*, i.e. cohomology classes on $M$. Here, we see that the components of degrees $k > n/2$ of an invariant polynomial are irrelevant, since $H^{2k}(M, \mathbb{R}) = 0$. This means that, for purposes of Chern-Weil theory, truncating power series (as $CH$) at a sufficiently high degree is not an issue, we can as well work with the full power series as if they were polynomials.

The Chern character power series defines, by Chern-Weil theory, the *Chern character* of a complex vector bundle $E$ over a manifold $M$:

$$ch(E) := [CH(K^\nabla)] \in H^*(M, \mathbb{R}),$$

for $\nabla$ some (unitary) connection on $E$. (Note: the Chern character power series can be expressed in terms of the Chern polynomials, which can be defined independently if $\nabla$ is a unitary connection or not, but the resulting cohomology class is independent of the connection, in particular we can choose $\nabla$ to be unitary, which implies that the corresponding cohomology class id *real.* It is, in fact, even *integer*, see [2])

The Chern character polynomial has the following properties:

**Proposition 8.7.** *Let $A_j \in \mathfrak{u}(n_j)$, $j = 1, 2$. Then denote by $A_1 \oplus A_2 \in \mathfrak{u}(n_1 + n_2)$ the "diagonal-block" matrix made of $A_1$ and $A_2$, and by $A_1 \otimes A_2 \in \mathfrak{u}(n_1 n_2)$ the tensor product of the two matrices. Then we have*

(1) $C(A_1 \oplus A_2) = C(A_1) \cdot C(A_2)$
(2) $CH^{n_1+n_2}(A_1 \oplus A_2) = CH^{n_1}(A_1) + CH^{n_2}(A_2)$
(3) $CH^{n_1 n_2}(A_1 \otimes A_2) = CH^{n_1}(A_1) \cdot CH^{n_2}(A_2)$

In other words, the total Chern polynomial is *multiplicative* w.r.t. direct sums, and the Chern character polynomial is *additive* w.r.t. direct sums, and *multiplicative* w.r.t. tensor products.

Thus implies that, for complex vector bundles $E, F$ on $M$, we have

$$c(E) \wedge c(F) = c(E \oplus F);\ ch(E \oplus F) = ch(E) + ch(F);\ ch(E \otimes F) = ch(E) \wedge ch(F).$$

For a complex line bundle $L$ on $M$, we have

$$ch(L) = 1 + c_1(L).$$

For every compact Lie group $G$, it is possible to construct invariant polynomials on its Lie algebra $\mathfrak{g}$ by means of unitary representations:

**Proposition 8.8.** *Let $\rho_j : G \to \mathcal{U}(V_j)$ be unitary (linear) representations of $G$ on the complex vector spaces $V_j$ ($j = 1$ or $2$). Let $Q_j \in \mathbb{R}[\mathfrak{u}(V_j)]^{\mathcal{U}(V_j)}$ be invariant polynomials. Then $a_1 Q_1 \circ \rho'_1 + a_2 Q_2 \circ \rho'_2$ is an invariant polynomial on $\mathfrak{g}$. (Here $\rho'_j : \mathfrak{g} \to \mathfrak{u}(V_j)$ are the Lie algebra actions on $V_j$ associated to $\rho_j$, $j = 1, 2$.)*

In particular, if we take $Q_j := CH^{V_j}$ to be the corresponding Chern character power series, and $a_1 = a_2 = 1$, then we simply obtain

$$CH^{V_1} \circ \rho'_1 + CH^{V_2} + \rho'_2 = CH^{V_1 \oplus V_2} \circ (\rho'_1 \oplus \rho'_2),$$

the Chern character power series of the direct sum representation $\rho_1 \oplus \rho_2$. But, using the Proposition above, we can also define the Chern character power series of any "linear combination of representations", like $CH(a_1 \rho'_1 + ... + a_k \rho'_k)$ , in particular for a "formal difference" $V_1 - V_2$, we can compute $CH(V_1 = V_2) := CH \circ \rho'_1 - CH \circ \rho'_2$.

Every analytic function, symmetric in $x_1, ..., x_n$, determines by truncation an invariant polynomial on $\mathfrak{u}(n)$. In particular:

**Definition 8.9.** *The* Todd power series *is*

$$TD := \prod_{j=1}^{n} \frac{x_j}{1 - e^{-x_j}} \in \mathbb{R}[[\mathfrak{u}(n)]]^{\mathrm{U}(n)}.$$

*The corresponding characteristic class (of a complex vector bundle $E \to M$) is called the* Todd class $Td(E) \in H^*(M, \mathbb{R})$.

**Exercise 8.10.** Show that $Td(E \oplus F) = Td(E) \wedge Td(F)$ for any complex vector bundles on $M$. The Todd class is thus *multiplicative* w.r.t. direct sums.

Let us determine the rings of invariant polynomials for $\mathrm{SU}(n), \mathrm{O}(n)$ and $\mathrm{SO}(n)$:

**Proposition 8.11.** $\mathbb{R}[\mathfrak{su}(n)]^{\mathrm{SU}(n)} \simeq \mathbb{R}[C_2, C_3, ..., C_n]$.

Idea of proof: Clearly, all invariant polynomials on $\mathfrak{u}(n)$ induce ones on $\mathfrak{su}(n)$, and $C_1$ (essentially the trace) vanishes on $\mathfrak{su}(n)$. One also shows that all polynomials on $\mathfrak{su}(n)$ can be extended to $\mathfrak{u}(n)$. □

For the real orthogonal group, note that $\mathrm{O}(n) = \mathrm{GL}(n, \mathbb{R}) \cap \mathrm{U}(n)$, the intersection taking place inside $\mathrm{GL}(n, \mathbb{C})$. As in the case of $\mathrm{SU}(n)$, it can be shown that all $\mathrm{O}(n)$-invariant polynomials on $\mathfrak{so}(n)$ (the Lie algebra of $\mathrm{O}(n)$, but also of $\mathrm{SO}(n)$, the connected component of $\mathrm{id} \in \mathrm{O}(n)$) can be extended to $\mathrm{U}(n)$-invariant polynomials on $\mathfrak{u}(n)$. It is also easy to see that $C_k|_{\mathfrak{so}(n)} = 0$ if $k$ is odd: the eigenvalues of $\frac{1}{2\pi i}A$, for a generic matrix $A \in \mathfrak{so}(n) \subset \mathfrak{u}(n)$, are pairwise opposite real numbers

$$\{x_1, -x_1, ..., x_{[\frac{n}{2}]}, -x_{[\frac{n}{2}]}\} \text{ for } n \text{ even, or } \{x_1, -x_1, ..., x_{[\frac{n}{2}]}, -x_{[\frac{n}{2}]}, 0\} \text{ for } n \text{ odd,}$$

where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$, thus $A$ and $-A$ are conjugate to each other by some matrix in $\mathrm{O}(n)$. Therefore, $Q(A) = Q(-A)$ for every $\mathrm{O}(n)$-invariant polynomial on $\mathfrak{so}(n)$, which implies that all invariant polynomials contain only terms of even degree. These facts can be summarized as follows:

**Proposition 8.12.** $\mathbb{R}[\mathfrak{so}(n)]^{\mathrm{O}(n)} = \mathbb{R}[x_1^2, ..., x_{[\frac{n}{2}]}^2] = \mathbb{R}[P_1, ..., P_{[\frac{n}{2}]}]$, *where*

$$P_j := (-1)^j C_{2j}|_{\mathfrak{so}(n)}$$

*are the* Pontryagin polynomials.

**Exercise 8.13.** Using Exercise 8.4 show that, for $A \in \mathfrak{so}(n)$, $P_1(A) = -\frac{1}{8\pi^2}\mathrm{tr}A^2$.

It can easily be shown that, for odd $n$, the $\mathrm{O}(n)$-invariant polynomials on $\mathfrak{so}(n)$ are also $\mathrm{SO}(n)$-invariant, thus

**Proposition 8.14.** *Let $n$ be odd. Then* $\mathbb{R}[\mathfrak{so}(n)]^{\mathrm{SO}(n)} = \mathbb{R}[x_1^2, ..., x_{[\frac{n}{2}]}^2\} = \mathbb{R}[P_1, ..., P_{[\frac{n}{2}]}\}$.

On the other hand, for $n$ even, there is another $\mathrm{SO}(n)$-invariant polynomial on $\mathfrak{so}(n)$:

**Definition 8.15.** *Let $n := 2k$ and $A \in \mathfrak{so}(n) \simeq \Lambda^2(\mathbb{R}^n)$. Then $A \wedge ... \wedge A$ ($k$ times) is a $2k$-form on $\mathbb{R}^n$, thus a volume form. It is thus a real multiple of the standard,* $\mathrm{SO}(n)$*-invariant volume form on $\mathbb{R}^n$ $vol_n$. We define the* Pfaffian $Pf : \mathfrak{so}(n) \to \mathbb{R}$ *as*

$$Pf(A) := \frac{A \wedge ... \wedge A}{n!(2\pi)^n vol_n}.$$

The caharcteristic class induced on a real, oriented vector bundle $E$ of rank $2k$ over a manifold $M$ by the Pfaffian on $\mathfrak{so}(2k)$ is a cohomology class in $H^{2k}(M, \mathbb{R})$ called the *Euler class* of $E$.

**Exercise 8.16.** Show that $Pf(BAB^{-1}) = \det B \cdot Pf(A)$, for all $B \in \mathrm{O}(n)$, and if $\{x_1, -x_1, ..., x_k, -x_k\}$ are the eigenvalues of $2\pi i A$, corresponding to an oriented basis $\{e_1, ..., e_n\}$ of $\mathbb{R}^n$ (more precisely, an eigenvector for the eigenvalue $2\pi i x_j$ is $e_{2j-1} + ie_{2j}$, and an eigenvector for the eigenvalue $-2\pi i x_j$ is $e_{2j-1} - ie_{2j}$), then $Pf(A) = x_1 \cdot ... \cdot x_k$.

It follows that $Pf^2 = (-1)^k P_k$. One can prove

**Proposition 8.17.** *Let $n := 2k$. Then* $\mathbb{R}[\mathfrak{so}(n)]^{\mathrm{SO}(n)} = \mathbb{R}[x_1^2, ..., x_k^2, x_1 \cdot ... \cdot x_k\} = \mathbb{R}[P_1, ..., P_{k-1}, Pf]$.

**Remark 8.18.** One can show in general (Th. Chevalley) that the ring of Ad-invariant polynomials on a compact Lie algebra is a polynomial ring generated by a finite number of such invariant polynomials.

The characteristic classes induced, by the Chern-Weil theory, by these polynomials are

(1) on $U(n)$-bundles (complex vector bundles $E$ of rank $n$): the Chern classes $c_i(E) \in H^{2i}(M, \mathbb{R})$, induced by the polynomials $C_i$, $i = 1, ..., n$.
(2) on $SU(n)$-bundles (complex vector bundles $E$ of rank $n$, with a nowhere-vanishing section of $\Lambda^n E$): the Chern classes $c_i$, induced by the polynomials $C_i$, $i = 2, ..., n$.
(3) on $SO(2n+1)$, $O(2n+1$ or $O(2n)$ bundles (on real vector bundles of any rank, or oriented real vector bundles $E$ of odd rank): the Pontryagin classes $p_j(E) \in H^{4j}(M, \mathbb{R})$, induced by $P_j$, $j = 1, ..., n$
(4) on $SO(2n)$ bundles (on oriented real vector bundles $E$ of even rank): the Pontryagin classes $p_j(E) \in H^{4j}(M, \mathbb{R})$, induced by $P_j$, $j = 1, ..., n$ and $e(E) \in H^{2n}(M, \mathbb{R})$ the Euler class of $E$, induced by $Pf$.

Every other real characteristic class of a bundle associated to these groups can be written as a polynomial expression in the Chern, resp. Pontryagin classes, possibly also the Euler class.

**Remark 8.19.** If $E$ is a complex vector bundle of rank $k$, then its Euler class (seen as an oriented real vector bundle of rank $2k$) $e(E_{\mathbb{R}})$ coincides with the top Chern class $c_k(E_{\mathbb{C}})$.

We focus now on Yang-Mills theory for 4-dimensional oriented Riemannian manifolds, and for $G$-bundles on them with $G$ compact:

**Proposition 8.20.** *Let $G \subset \mathrm{U}(k)$ be a compact Lie group and $B(X, Y) := -\mathrm{tr}(XY)$, $\forall X, Y \in \mathfrak{g} \subset \mathfrak{u}(k)$. Let $P \to M$ be a $G$-bundle on the oriented Riemannian 4-manifold $(M^4, g)$. Then*

$$YM(\nabla) \geq 4\pi^2 \left| \int_M p_1(\mathrm{Ad}P) \right|,$$

*and, if $\nabla$ satisfies the equality case, then $\nabla$ is SD (and $\int_M p_1(\mathrm{Ad}P) \geq 0$), or ASD (and $\int_M p_1(\mathrm{Ad}P) \leq 0$).*

*Proof.* If we decompose $K^\nabla = K_+^\nabla + K_-^\nabla$ in the SD, resp. ASD parts, we have:

$$B(K^\nabla \wedge *K^\nabla) = -\mathrm{tr}(K_+^\nabla \wedge K_+^\nabla) + \mathrm{tr}(K_-^\nabla \wedge K_-^\nabla) = \|K_+^\nabla\|^2 + \|K_-^\nabla\|^2,$$

because the wedge product of a SD form with an ASD one is always zero. On the other hand, applying the first Pontryagin polynomial to $K^\nabla$ (and using Exercise 8.13) yields:

$$P_1(K^\nabla) = \frac{1}{8\pi^2}\mathrm{tr}\left(K^\nabla \wedge K^\nabla\right) = \frac{1}{8\pi^2}\left(\|K_+^\nabla\|^2 - \|K_-^\nabla\|^2\right).$$

The result follows by integration. (Note that $\nabla$ is SD, resp. ASD, iff $K_-^\nabla$, resp. $K_+^\nabla$ identically vanishes.) $\square$

## 9. Chern-Weil theory on noncompact manifolds

We can define, using the Chern-Weil Theorem, classes in some more special cohomology groups, for example: Let $M$ be noncompact and consider only *$G$-bundles with compact support*, i.e. $G$-bundles $p : P \to M$, such that there exists a compact set $K$ and a trivialization of $P$ on $M \setminus K$. We can then consider the space of $G$-connections $\nabla$ on $P$ *with compact support*, i.e. $\nabla$ on $p^{-1}(M \setminus K)$ is the trivial connection (given by the fixed trivialization of $P$ over $M \setminus K$. A version "with compact support" of the Chern-Weil theorem states then that, for every invariant polynomial $Q$ on $\mathfrak{g}$, the induced differential forms $Q(K^{\nabla^j})$, $j = 1, 2$, for $\nabla^1, \nabla^2$ connections with compact support as above, are

1. with compact support contained in $K$ (they vanish outside $K$)
2. the relative Chern-Simons form $ChS(\nabla^1, \nabla^2)$ also has compact support.

This implies that every invariant polynomial $Q$ on $\mathfrak{g}$ defines, for such $G$-bundles with compact support on $M$, a cohomology class

$$[Q(K^{\nabla})] \in H^*_c(M, \mathbb{R}),$$

the *cohomology with compact support on $M$*.

Another version of the Chern-Weil Theorem will be needed for the Atiyah-Singer Index Theorem, where cohomology classes on the total space of a vector bundle (the tangent bundle) $T \to M$ are needed.

**Definition 9.1.** *Let*

$$(\sigma) \quad ... \to F_i \overset{\sigma_i}{\to} F_{i+1} \to ...$$

*be a finite* complex *of vector bundles over $T$. We say that this complex has* fiberwise compact support *(FCS) iff there exists a* tubular neighborhood $U \subset T$ *of the zero section $M_0 \subset T$ such that the sequence $(\sigma)$ is exact when restricted to $T \setminus U$.*

Recall that a sequence of linear maps between vector bundles $(\sigma)$ is called a *complex* iff $\sigma_{i+1} \circ \sigma_i = 0$, $\forall i \in \mathbb{Z}$. The complex is *finite* iff $F_i \neq 0$ only for finitely many $i \in \mathbb{Z}$. An open set $U \supset M_0$ in $T$ is *tubular* iff $U \cap T_x$ is relatively compact, $\forall x \in M$.

**Example 9.2.** If the complex only contains 2 non-zero terms, $F_0$ and $F_1$, then it is FCS iff $\sigma_0$ is a vector bundle isomorphism outside a tubular neighborhood $U \supset M_0$.

**Exercise 9.3.** Let $h_i$ be Hermitean metrics on the vector bundles $F_i$ (we need in the sequel only the case when $F_i$ are complex vector bundles). Then the linear maps $\sigma_i^* : F_{i+1} \to F_i$ are the *adjoint maps* to $\sigma_i$:

$$h_i(\sigma_i^*(y_{i+1}), y_i)) = h_{i+1}(y_{j+1}, \sigma_i(y_i)), \quad \forall y_i \in F_i, y_{i+1} \in F_{i+1}.$$

Consider a sequence $(\sigma)$ of linear maps between vector bundles as above. Show that:

1. $(\sigma)$ is a complex iff the *adjoint sequence* $(\sigma^*)$ is a complex, and that $(\sigma^{**}) \simeq (\sigma)$.
2. $(\sigma)$ is exact over $U \supset M_0$ iff $(\sigma^*)$ is exact over $U$ iff the map $\sigma_{even} \oplus \sigma^*_{odd} : F_{even} \to F_{odd}$ is a isomorphism over $U$ of the vector bundles

$$F_{even} := \bigoplus_{j \in \mathbb{Z}} F_{2j}, \quad F_{odd} := \bigoplus_{j \in \mathbb{Z}} F_{2j-1},$$

$$\sigma_{even} \oplus \sigma^*_{odd}(..., y_{2j}, ...) := (..., \sigma_{2j-2}(y_{2j-2}) + \sigma^*_{2j-1}(y_{2j}), ...).$$

(3) On an open set where $(\sigma)$ is exact, the subspaces $\ker \sigma_i \subset F_i$ and $\sigma_i(F_i) \subset F_{i+1}$ are vector subbundles (i.e. they have constant rank).

**Definition 9.4.** *Let $(\sigma)$ be a finite complex with FCS on $T$ as above. A set of linear connections $(\nabla) := (\nabla^i)_{i\in\mathbb{Z}}$ is called* admissible *or with* fiberwise compact support *(FCS) iff there exists a tubular neighborhood $U \supset M_0$ such that $(\sigma)$ is exact over $T \setminus U$ and*

(1) *there exists a set $(h)$ of Hermitean metrics on the bundles $(F_i)_{i\in\mathbb{Z}}$ such that $\nabla^i h_i = 0$ and $\sigma_i(F_{i-1})$ are $\nabla^i$-stable, $\forall i \in \mathbb{Z}$*

A subbundle $F$ in a vector bundle $E$ is said to be $\nabla$-stable (for $\nabla$ a linear connection on $E$) iff

$$s \in C^\infty(F) \Rightarrow \nabla s \in C^\infty(\Lambda^1 M \otimes F).$$

We also need to define the cohomology with *fiberwise compact support* (FCS cohomology) on the total space $T$ of a vector bundle over $M$, as the cohomology groups

$$H^*_{fc}(T) := \ker d / \operatorname{Im} d$$

of the following complex

$$\ldots \xrightarrow{d} C^\infty(\Lambda^i_{fc} T) \xrightarrow{d} C^\infty(\Lambda^{i+1}_{fc}) \xrightarrow{d} \ldots \quad ,$$

where $\Lambda^i_{fc} T$ are the $i$-forms on $T$ whose support is contained in a tubular neighborhood of $M_0$. These forms are called differential forms with fiberwise compact support.

**Remark 9.5.** Note that a closed form with FCS which has vanishing FCS cohomology class iff it is the derivative on an FCS form (being exact is also not enough, it needs a primitive with FCS).

**Theorem 9.6** (Chern-Weil theorem for complexes with FCS)**.** *Let $(\sigma)$ a finite complex of vector bundles, with FCS, on the total space $T$ of a vector bundle over $M$*

(1) *For every admissible set of connections $(\nabla)$ on $(\sigma)$, the differential form*

$$ch_0(\nabla) := \sum_{i\in\mathbb{Z}} (-1)^i CH(K^{\nabla^i})$$

*is a closed FCS form*

(2) *For every two admissible sets of connections $(\nabla)$, $(\nabla')$ on $(\sigma)$, the relative Chern-Simons differential form $ChS(\nabla, \nabla')$ defined as in Theorem 7.3 is a FCS form.*

*Therefore, there exists a FCS cohomology class $ch_0(\sigma)$ (represented by one of the $ch_0(\nabla)$ above) on $T$ that depends on the finite FCS complex $(\sigma)$ alone*

## 10. Elliptic linear differential operators on vector bundles

We introduce now the notion of a linear differential operator.

First, define the space $\mathcal{F}_x$ of *germs* of functions (real or complex-valued) on $M$ at $x \in M$ as the space of equivalence classes of functions that coincide on some open set around $x$. This is a local ring, and its maximal ideal is $\mathfrak{m}_x := \{f \in \mathcal{F}_x \mid f(x) = 0\}$, the space of germs vanishing at $x$. The space $\mathfrak{m}_x^k$ is the space of germs that can be written as products of $k$ germs in $\mathfrak{m}_x$ or, equivalently, the space of germs that vanish *up to order* $k-1$ at $x$, i.e.

$$f \in \mathfrak{m}_x \Leftrightarrow f(x) = 0, df(x) = 0, ..., d^{k-1}f(x) = 0,$$

where note that the derivative of order $l$ at $x$, $d^l f(x)$ is well-defined (without using coordinates) as long as the previous derivatives at $x$ vanish.

**Remark 10.1.** We have $\mathcal{F}_x/\mathfrak{m}_x \simeq \mathbb{R}$ (if we take real-valued functions, if we take complex-valued functions we need to tensorize all relations with $\mathbb{C}$), $\mathfrak{m}_x^{k-1}/\mathfrak{m}_x^k \simeq S^{k-1}T_x^*M$, in particular $\Lambda_x^1 M = \mathfrak{m}_x/\mathfrak{m}_x^2$.

Quotienting out by $\mathfrak{m}_x^k$ means "ignoring" the derivatives of higher order than $k-1$.

**Remark 10.2.** One defines the *jet bundle* on $M$ as the vector bundle $J^k M$ on $M$ whose fibers are $\mathcal{F}_x/\mathfrak{m}_x^{k+1}$. The construction of jet bundles can be made precise by glueing local trivial parts (for chart domains) by $k$-order derivatives of the coordinate changes (the formulas are complicated).

We have the following exact sequence:

$$0 \to \mathfrak{m}_x^k/\mathfrak{m}_x^{k+1} \to \mathcal{F}_x/\mathfrak{m}_x^{k+1} \to \mathcal{F}_x/\mathfrak{m}_x^k \to ]\!], \tag{13}$$

where the first term is $S^k T_x^* M$.

If we have a vector bundle $E$ on $M$, we define $\mathcal{E}_x$ as the space of germs at $x$ of sections of $E$ (which is a rank $r = \dim E_x$ free module over $\mathcal{F}_x$ - a basis is given, for example, by the germ of a local frame in $E$ around $x$), and we have the $\mathcal{F}_x$ modules $\mathfrak{m}_x^{k+1} \otimes_{\mathcal{F}_x} \mathcal{E}_x$ (of germs at $x$ of sections of $E$ that vanish up to order $k$ at $x$), such that $(\mathfrak{m}_x^k \otimes_{\mathcal{F}_x} \mathcal{E}_x)/(\mathfrak{m}_x^{k+1} \otimes_{\mathcal{F}_x} \mathcal{E}_x) \simeq S^k T_x^* M \otimes E_x$ (in particular $E_x = \mathcal{E}_x/(\mathfrak{m}_x \otimes_{\mathcal{F}_x} \mathcal{E}_x)$ and $(\Lambda^1 M \otimes E)_x = (\mathfrak{m}_x \otimes_{\mathcal{F}_x} \mathcal{E}_x)/(\mathfrak{m}_x^2 \otimes_{\mathcal{F}_x} \mathcal{E}_x))$, and we have, by tensorizing over $\mathcal{F}_x$ with $\mathcal{E}_x$, the exact sequences

$$0 \to \left(\mathfrak{m}_x^k/\mathfrak{m}_x^{k+1}\right) \otimes_{\mathcal{F}_x} \mathcal{E}_x \to \left(\mathcal{F}_x/\mathfrak{m}_x^{k+1}\right) \otimes_{\mathcal{F}_x} \mathcal{E}_x \to \left(\mathcal{F}_x/\mathfrak{m}_x^k\right) \otimes_{\mathcal{F}_x} \mathcal{E}_x \to 0. \tag{14}$$

**Definition 10.3.** *Let $E$ and $F$ be vector bundles over $M$. A* linear differential operator *$D$ from $E$ to $F$ of order at most $k$ is a linear map $D : C^\infty(E) \to C^\infty(F)$ such that $\forall x \in M$, $D(f)_x = 0$, $\forall f \in \mathfrak{m}_x^{k+1} \otimes_{\mathcal{F}_x} \mathcal{E}_x$ and the induced linear map*

$$\sigma^D : S^k T_x^* M \otimes E_x \to F_x, \ \sigma^D(df_1(x) \otimes ... \otimes df_k(x) \otimes e) := D(f_1...f_k s)_x,$$

$$\forall f_1, ..., f_k \in \mathfrak{m}_X \ \textit{and} \ s \in \mathcal{E}_x, \ s(x) = e$$

*is called the symbol of $D$. We say that $D$ is of order $k$ if it is of order at most $k$ and its symbol ist not identically zero.*

*Such a operator $D$ is called* elliptic *iff $\sigma^D(\lambda, ...\lambda) : E_x \to F_x$ is an isomorphism for all $x \in M$ and all $\lambda \in T_x^* M \smallsetminus \{0\}$.*

**Example 10.4.** A covariant derivative on a vector bundle $E$ is a first order linear differential operator from $E$ to $\Lambda^1 \otimes E$ with symbol equal to the identity of $\Lambda^1 \otimes E$. It is elliptic iff $n = 1$, otrherwise $\dim E_x < \dim(\Lambda^1 \otimes E)_x$.

**Example 10.5.** The exterior differential $d : C^\infty(\Lambda^k M) \to C^\infty(\Lambda^{k+1} M)$ is a first order linear differential operator with symbol the wedge product

$$\sigma^d(\lambda) : \Lambda^k M \to \Lambda^{k+1} M, \ \sigma^d(\lambda)(\alpha) = \lambda \wedge \alpha.$$

In an analogue way, if $(M, g)$ is a Riemannian manifold, then the *codifferential* $\delta^g : C^\infty(\Lambda^k M) \to C^\infty(\Lambda^{k-1} M)$,

$$\delta^g(\alpha) := \sum_{i=1}^{n} e_i \lrcorner \nabla_{e_i} \alpha, \text{ for } e_1, ...e_n \text{ an o.n.b. of } TM,$$

is a first order linear differential operator with symbol the interior product

$$\sigma^\delta(\lambda) : \Lambda^k M \to \Lambda^{k-1} M, \ \sigma^d(\lambda)(\alpha) = \lambda \lrcorner \alpha.$$

None of these operators is elliptic, but

$$\Delta^g := d\delta^g + \delta^g d : C^\infty(\Lambda^k M) \to C^\infty(\Lambda^k M)$$

is the Laplacian on forms, is a second-order linear differential operator and its symbol

$$\sigma^\Delta(\lambda_1, \lambda_2) : \Lambda^k M \to \Lambda^k M, \ \sigma^\Delta(\lambda_1, \lambda_2)(\alpha) = g(\lambda_1, \lambda_2)\alpha.$$

**Exercise 10.6.** Show that the Laplacian is elliptic iff the metric $g$ is definite (e.g. Riemannian).

If we compose two linear differential operators, their orders add up, if we add two linear differential operators $D_1 + D_2$, the order is at most the maximum of the two orders (like the degree of polynomials). In the first case, the symbols are composed $\sigma^{D_1 \circ D_2}(\lambda) = \sigma^{D_1}(\lambda) \circ \sigma^{D_2}(\lambda)$ (here, and from now on, we write the multiple argument $\lambda \in T^*M$ only once), and if the polynomials $D_1$ and $D_2$ have the same order, their symbols add up (possibly resulting zero, i.e. $D_1 + D_2$ has smaller order) or, if the order of $D_1$ is larger than the order of $D_2$, then $\sigma^{D_1+D_2} = \sigma^{D_1}$.

It is useful to consider *complexes* of differential operators

**Definition 10.7.** *A sequence of linear differential operators*

$$... \to C^\infty(E_k) \stackrel{D_k}{\to} C^\infty(E_{k+1}) \stackrel{D_{k+1}}{\to} C^\infty(E_{k+2}) \to ...$$

*is a* complex *iff* $D_{k+1} \circ D_k = 0$, $\forall k \in \mathbb{Z}$. *Such a complex is called* elliptic *iff the corresponding sequence of symbols*

$$... \to \mathcal{E}_k \stackrel{\sigma^{D_k}(\lambda)}{\to} \mathcal{E}_{k+1} \stackrel{\sigma^{D_{k+1}}(\lambda)}{\to} \mathcal{E}_{k+2} \stackrel{\sigma^{D_{k+2}}(\lambda)}{\to} ...$$

*is exact for all* $\lambda \in T^*M$ *non-zero.*

**Remark 10.8.** If the sequence contains only two non-trivial bundles, say $E_0$ and $E_1$, then the corresponding complex is elliptic iff $D_0 : C^\infty(E_0) \to: C^\infty(E_1)$ is an elliptc differential operator in the usual sense.

**Example 10.9.** The *de Rham complex*

$$... \stackrel{d}{\to} C^\infty(\Lambda^k M) \stackrel{d}{\to} C^\infty(\Lambda^{k+1} M) \stackrel{d}{\to} ...$$

is elliptic.

The main property of an elliptic operator is

**Theorem 10.10.** *Let* $M$ *be a compact manifold and* $D : C^\infty(E) \to C^\infty(F)$ *an elliptic operator. Then* $\ker D \subset C^\infty(E)$ *and* $cokerD = C^\infty(F)/D(C^\infty(E))$ *are finite dimensional. Denote by* $\mathrm{ind}(D) := \dim \ker D - \dim cokerD$ *the index of* $D$.

For an elliptic complex, the theorem states:

**Theorem 10.11.** *Let $M$ be a compact manifold and $(D) = (D_i)_{i\in\mathbb{Z}}$ a* finite *elliptic complex of linear differential operators on vector bundles over $M$. Then the quotient spaces* $\ker D_i/\mathrm{Im} D_{i-1}$ *are finite dimensional. Denote by*

$$\mathrm{ind}(D) := \sum_{i\in\mathbb{Z}} (-1)^i \dim\left(\ker D_i/\mathrm{Im} D_{i-1}\right)$$

*the* index *of $(D)$.*

The Atiyah-Singer Index Theorem gives a formula for the index of $D$ uniquely in terms of the topological properties of the symbol.

Let us first describe the relevant topological objects:

(1) Let $(D)$ be an elliptic complex. Then the symbol determines a complex of vector bundles

$$(\sigma(D)) \qquad\qquad \dots \overset{\sigma(D_{i-1})}{\to} \pi^* E_i \overset{\sigma(D_i)}{\to} \pi^* E_{i+1} \overset{\sigma(D_{i+1})}{\to} \dots$$

over the total space of the cotangent bundle $\pi : T^*M \to M$, which has FCS, because of the ellipticity of $(D)$. We will, however, identify $TM$ with $T^*M$, because they are isomorphic bundles.

(2) Using *admissible* sets of connections on the bundles $F_i := \pi^* E_i$, the pull-backs to $T^*M$ of the bundles $E_i$, we can define the *Chern character* $ch_0(\sigma(D))$ of the complex $(\sigma(D))$ (see Theorem 9.6).

(3) Consider $T^*M$ as an oriented manifold as follows: if $e^1, ..., e^n$ is a basis of $T_x M$, and $e_1^*, ..., e_n^*$ is the dual basis of $T_x^* M$, then $e_1, e_1^*, ..., e_n, e_n^*$ defines an *oriented* basis for the tangent space in $\alpha \in T_x^* M$ of $T^*M$.

**Theorem 10.12** (Atiyah-Singer Index Theorem)**.** *Let $M$ be an even-dimensional*[3] *manifold and $(D)$ be an elliptic complex with symbol $(\sigma^D)$. Then it holds*

$$\mathrm{ind}(D) = \int_{TM} ch_0(\sigma(D)) \wedge \pi^* td(TM \otimes \mathbb{C}),$$

*for the above defined orientation of $TM$. The integration convention is that only the homogeneous component of degree $2n$ in the right hand side is integrated over $TM$, all other components are ignored.*

## 11. Applications of the Index Theorem

The Index Theorem is already a spectacular result, because it states that two numbers are equal, one coming from an analytic problem ("counting" the solutions of some differential equations), the other from a topological one, that only involves the symbol of the operator. Sometimes it is not obvious at all that the number from the right hand side should be an integer, but this obviously follows from the Index Theorem.

However, there are many examples where the right hand side of the Index Theorem can be simplified even further, so that in the end only the bundles $(E_i)$ matter (the reference to the $\sigma(D)$ totally vanishes)!

This happens if the symbol complex $(\sigma(D))$ is an *universal complex*:

[3] one can show that the index of a differential operator on an odd-dimensional manifold is always zero

**Definition 11.1.** *Let $\pi : T \to M$ an oriented real vector bundle (the corresponding group being thus* $\mathrm{SO}(k)$*, where $k$ is the rank of $T$. Denote by $P$ the* $\mathrm{SO}(k)$*-principal bundle on $M$ defining $T$. Let $F_i := \pi^* E \otimes \pi^* L_i$, where $E$ is some complex vector bundle and $L_i$ are vector bundles associated to $P$ via representations $\rho_i : \mathrm{SO}(k) \to \mathrm{GL}(V_i)$.*

*Suppose the bundle maps from $F_i$ to $F_{i+1}$ are the tensor products of* $\mathrm{Id}_E$ *with some maps $\sigma_i^0 : \pi^* L_i \to \pi^* L_{i+1}$, such that:*

$$(L) \qquad \ldots \stackrel{\sigma_{i-1}^0}{\to} \pi^* L_i \stackrel{\sigma_i^0}{\to} \pi^* L_{i+1} \stackrel{\sigma_{i+1}^0}{\to} \ldots$$

*is a complex of vector bundles, and $\sigma_i^0$ is defined by a* $\mathrm{SO}(k)$*-equivariant map*

$$[\sigma_i^0] : S^{k_i}(\mathbb{R}^k)^* \otimes V_i,$$

*where $k_i \in \mathbb{N}$. (thus the dependance of the map $\sigma_i^0$ on the fiber elements of $T$ is polynomial, and can be expressed by a formula in terms of representations of* $\mathrm{SO}(k)$*). Then the complex*

$$(\sigma) \qquad \stackrel{\mathrm{Id}_E \otimes \sigma_{i-1}^0}{\to} \pi^*(E \otimes L_i) \stackrel{\mathrm{Id}_E \otimes s_i^0}{\to} \pi^*(E \otimes L_{i+1}) \stackrel{\mathrm{Id}_E \otimes s_{i+1}^0}{\to} \ldots$$

*is called* universal. *A complex of linear differential operator is called universal iff its symbol complex is.*

**Remark 11.2.** One can define universal complexes and operators by stating that they are obtained by pull-backs from the *classifying space* of $\mathrm{SO}(k)$.

The classifying space $BG$ of a compact Lie group is a (in general infinite-dimensional) manifold, that admits a *universal* $G$-principal bundle $PG \to BG$ such that $PG$ is a contractible space. As a consequence, every $G$-principal bundle $P \to M$ can be obtained as the pull-back of $PG$, for a suitable map $f : M \to BG$, in fact there is a bijective correspondence between isomorphism classes of principal $G$-bundles on $M$ and homotopy classes of maps from $M$ to $BG$.

In fact, a real *characteristic class* for a certain Lie group $G$ corresponds to a cohomology class in $H^*(BG, \mathbb{R})$, which turns out to be isomorphic (as a ring), to the ring of invariant polynomials over $\mathfrak{g}$, thus it has no zero divisors, unlike the cohomology of a finite-dimensional manifold.

The theory of the classifying spaces is an important and classical part of the Lie group and gauge theory and characteristic classes, but in these lectures we will not develop it any further and "go around it".

The *universality* of the symbol is satisfied by most differential operators that appear in geometry, when they are determined by the structure of the the manifold $M$.

The idea of reducing an integration of a form with FCS on $TM$ to an integral on $M$ is the *Thom isomorphism*:

**Theorem 11.3** (Thom isomorphism)**.** *Let $\pi : T \to M$ be an oriented real vector bundle of rank $k$ over a manifold $M$. There exists a unique cohomology class $U$ in $H^k_{fc}(T, \mathbb{R})$ (cohomology with fiberwise compact support) such that*

$$\int_{E_x} U = 1, \ \forall x \in M.$$

*Moreover, the following hold:*

(1) *the wedge product with $U$ of the pull-back $\pi^*$ is an isomorphism of $H^*(M,\mathbb{R})$-modules*

$$\psi : H^*(M,\mathbb{R}) \to H^{*+k}_{fc}(T,\mathbb{R}), \quad \psi(\alpha) := U \wedge \pi^*\alpha,$$

*called the* Thom isomorphism.

(2) *The restriction of $U$ to the zero section $M_0 \subset T$, or its pull-back through the inclusion $i : M \to T$ of the zero section, is the Euler class $e(T)$ of the real vector bundle $T \to M$.*

The above Theorem claims that the equation

$$\psi(\alpha) = \beta,$$

where the right hand side is a cohomology class in $H^{k+p}_{fc}(T,\mathbb{R})$ admits a unique solution in $H^p(M,\mathbb{R})$. On the other hand, this solution satisfies

$$u \wedge \pi^*\alpha = \beta \ \Rightarrow \ e(T) \wedge \alpha = i^*\beta,$$

of which the right hand side equation does not, unfortunately, admit unique solutions in the cohomology ring $H^*(M,\mathbb{R})$.

The idea is to solve this equation (and hence find an expression for $\psi^{-1}(\beta)$) is that, if $\beta$ is the Chern character of an *universal* bundle (or an universal complex of bundles $(\sigma)$), then $\beta = \pi^* ch(E) \wedge \beta_0$, where $\beta_0$ is the Chern character of the complex $(L)$ of Definition 11.1, and it can be expressed as $ch_0(\nabla)$, for some admissible set of connections $(\nabla)$, see Theorem 9.6.

As $e(T)$ is expressed by $Pf(K^{\nabla^{(0)}})$, where $\nabla^{(0)}$ is an SO$(k)$-connection on $T$, we can construct a representative $U^{(0)}$ of the Thom class $U$ by the formula

$$U^{(0)} := Pf(K^{\nabla^T}),$$

where $\nabla^T$ is a connection on $\pi^*T$, which coincides on a tubular neighborhood $V_0$ of $M_0$ with $\pi^*\nabla^{(0)}$, and satisfies

$$\nabla^T_v (|v|w) = 0, \ \nabla^T_{\tilde{X}} w = 0, \ \forall v \in T_x, \ X \in T_X M, \ w \in C^\infty(T)$$

outside a suitable tubular neighborhood $V_1 \supset \bar{V}_0$ of $M_0$. Here $v$ is seen as a vertical vector field on $T$, and $\tilde{X}$ is the $\nabla^{(0)}$-horizontal lift of $X$ to a horizontal vector on $T$, and $w$ is a section in $\pi^*T$ that is constant on the fibers (it is the pull-back to $T$ of a section $w$ on $M$). Note that the pulled-back connection $\pi^*\nabla^{(0)}$ would satisfy

$$\nabla^T_v w = 0, \ \nabla^T_{\tilde{X}} w = 0, \ \forall v \in T_x, \ X \in T_X M, \ w \in C^\infty(T),$$

without the factor $|v|$. The presence of the factor $|v|$ in the definition of $\nabla^T$ ouside $V_1$ implies that the *tautological section* of $\pi^*T \to T$ which has in $v \in T$ the value $v \in \pi^*T_v$ is $\nabla^t$-parallel outside this tubular neighborhood. This, in turn, implies that the associated connections $\nabla^i$ on $\pi^*L_i$ form an admissible set.

We skip the technical details concerning the choice of the tubular neighborhoods $V_0, V_1$ such that

$$\int_{T_x} Pf(K^{\nabla^T}) = 1, \ \forall x \in M,$$

but it is easy to see that on $V_0$ and on $T \smallsetminus \bar{V}_1$ the curvature of $\nabla^T$ (and thus of the associated $\nabla^i$ as well) has only horizontal components, in particular

$$i^* Pf(K^{\nabla^T}) = Pf(K^{\nabla^{(0)}}),$$

and thus it represents the Euler class.

At this point we see that, if the polynomial

$$\sum_{i\in\mathbb{Z}}(-1)^i CH\circ\rho_i'$$

is divisible by $Pf$, and the quotient is $Q^\sigma \in \mathbb{R}[\mathfrak{so}(k)]^{\mathrm{SO}(k)}$, then we can write

$$ch_0(\nabla) = Pf(K^{\nabla^T}) \wedge Q^\sigma(\nabla),$$

thus

$$\psi^{-1}ch_0(\nabla) = Q^\sigma(K^{\nabla^{(0)}}). \tag{15}$$

In most cases, we can show that the divisibility by the Pfaffian is automatic, hence we can reduce the computation of the index of an elliptic complex to a computation of characteristic classes on $M$:

$$\mathrm{ind}(D) = (-1)^{n/2}\int_M Q^\sigma(K^{\nabla^{(0)}}) \wedge Td(TM\otimes\mathbb{C}).$$

(the sign $(-1)^{n/2}$ is due to the fact that the orientation of $TM$ needed for the Index Theorem is different from the orientation coming from the Thom isomorphism).

The details of the following examples can be found in [4]:

11.1. **The de Rham complex.** Consider the *de Rham complex*

$$...\xrightarrow{d} C^\infty(\Lambda^k M)\xrightarrow{d} C^\infty(\Lambda^{k+1}M)\xrightarrow{d} ...$$

on a $2n$-dimensional oriented manifold $M$, and its symbol

$$...\xrightarrow{v\wedge\cdot}\Lambda^k M\xrightarrow{v\wedge\cdot}\Lambda^{k+1}M\xrightarrow{v\wedge\cdot}...,$$

for $v\in T^*M$. This symbol is a universal complex, thus we compute the polynomial

$$\sum_{k\in\mathbb{N}} CH(\Lambda^k\cdot) = \prod_{k=1}^n (1-e^{x_i})(1-e^{-x_i}),$$

which is divisible by $Pf = x_1...x_n$, because $x_i$ are clearly zeros of the above polynomial (power series). Denote by $Q^d$ the corresponding quotient polynomial. The formula for $Q^d\cdot TD$ is very simple: it is

$$\frac{\prod_{k=1}^n (1-e^{x_i})(1-e^{-x_i})}{x_1...x_n}\cdot\frac{\prod_{k=1}^n x_k(-x_k)}{\prod_{k=1}^n (1-e^{x_i})(1-e^{-x_i})} = (-1)^n Pf.$$

Thus

$$\chi(M) = \mathrm{ind}(d) = (-1)^n\int_M (-1)^n Pf(K^\nabla) = \int_M e(TM),$$

which is a well-known fact: the *Euler characteristic* $\chi(M)$, which is the alternated sum of the Betti numbers $\dim H^i(M,\mathbb{R})$, is equal to the integral of the Euler class.

11.2. **The signature operator.** . The de Rham complex is equivalent, via Hodge theory, to the following operator

$$d+\delta : C^\infty(\Lambda^{even}M) \to C^\infty(L^{odd}M),$$

where $\delta$ is the adjoint of $d$ for a fixed Riemannian metric. If we denote by $D := d+\delta : C^\infty(\Lambda^*M) \to C^\infty(L^*M)$, then $D$ is clearly self-adjoint, so its index is zero. But if we restrict $D$ to the bundles $\Lambda^{even}$ and $L^{odd}$ as above, then its index is not trivial and is equal to the Euler characteristic.

But we can restrict $D$ to some other subbundles of $\Lambda^*M$, as follows: recall that, on an oriented, even-dimensional manifold $M^{2n}$, the Hodge star operator has the property

$$*^2|_{\Lambda^k M} = (-1)^k \mathrm{Id},$$

so it is almost an involution. The following linear operator, defined on the complexified space $\Lambda^*M \otimes \mathbb{C}$ is an involution (exercise!, use Proposition 6.1):

$$\tau(\alpha) := i^{k(k+1)+n} * \alpha,\ \forall \alpha \in \Lambda^k M \otimes \mathbb{C}.$$

**Exercise 11.4.** Check that $D \circ \tau = -\tau \circ D$.

We denote by $\Lambda^+M$ and $\Lambda^-M$ the eigenspaces of $\tau$ for the eigenvalues 1, resp. $-1$, and we can decompose $D$ as $D = D^+ + D^-$, where

$$D^+ : C^\infty(\Lambda^+M) \to: C^\infty(\Lambda^-M), \quad D^- : C^\infty(\Lambda^-M) \to: C^\infty(\Lambda^+M).$$

These operators are adjoint to each other, so $\mathrm{coker} D_+ = \ker D_-$.

The bundle $\Lambda^+M$ is isomorphic to the direct sum of all $\Lambda^k M \otimes C$, for $k < n$, and of $(\Lambda^n M \otimes \mathbb{C})^+$, and $\Lambda^-M \simeq \Lambda^1 M \oplus ... \oplus \Lambda^{n-1}M \oplus (\Lambda^n M \otimes \mathbb{C})^-$. It can also be shown that

$$\ker D_+ \simeq \left( \bigoplus_{k=0}^{n-1} \mathcal{H}^k(M) \right) \otimes \mathbb{C} \oplus \left( (\mathcal{H}^n(M) \otimes \mathbb{C}) \cap (\Lambda^n M \otimes \mathbb{C})^+ \right),$$

where $\mathcal{H}^k(M)$ is the space of harmonic $k$-forms (thus $\ker D \cap \Lambda^k M$). Also

$$\ker D_- \simeq \left( \bigoplus_{k=0}^{n-1} \mathcal{H}^k(M) \right) \otimes \mathbb{C} \oplus \left( (\mathcal{H}^n(M) \otimes \mathbb{C}) \cap (\Lambda^n M \otimes \mathbb{C})^- \right),$$

thus

$$\mathrm{ind}(D_+) = \dim_{\mathbb{C}} \left( (\mathcal{H}^n(M) \otimes \mathbb{C}) \cap (\Lambda^n M \otimes \mathbb{C})^+ \right) - \dim_{\mathbb{C}} \left( (\mathcal{H}^n(M) \otimes \mathbb{C}) \cap (\Lambda^n M \otimes \mathbb{C})^- \right).$$

If $n$ (half of the dimension of $M$) is odd, then $\tau : \Lambda^n \to \Lambda^n$ has its square equal to –Id, thus the eigenspaces of $\tau$ are conjugated to each other. Moreover, every real harmonic $n$-form $\alpha$ defines harmonic forms $\alpha \pm \tau(\alpha) \in ((\mathcal{H}^n(M) \otimes \mathbb{C}) \cap (\Lambda^n M \otimes \mathbb{C})^{\pm})$, and thus the index of $D_+$ is zero.

If $n$ is even (thus $\dim M$ is divisible by 4), then $\tau$ coincides with $*$ on $\Lambda^n M$ and the eigenspaces of $\tau$ are real, thus

$$\mathrm{ind}(D_+) = \dim \mathcal{H}^n_+(M) - \dim \mathcal{H}^n_-(M),$$

(here $\mathcal{H}^n_\pm(M) := \mathcal{H}^n(M) \cap L^\pm(M)$) which is the signature of the *intersection form*

$$q : H^n(M) \times H^n(M) \to \mathbb{R},\ q(\alpha, \beta) := *(\alpha \wedge \beta).$$

Noet that, for even $n$, $q$ is bilinear and symmetric, and it induces on $\mathcal{H}^n(M)$ a bilinear symmetric form which restricted to $\mathcal{H}^n_+(M)$ is positive-definite, and restricted to $\mathcal{H}^n_-(M)$ is negative-definite.

The signature of an oriented manifold of dimension $4k$ (equal, by definition, to the signature of its intersection form) is thus equal to the index of $D_+$.

To compute this index, we need to compute

$$CH(\Lambda^+ - \Lambda^-),$$

where $\Lambda^\pm$ are the corresponding complex representations of $\mathrm{SO}(2n)$. It is useful, at this point, not to restrict to even $n$.

**Lemma 11.5.** *If $V_1$, $V_2$ are evendimensional oriented real vector spaces, then*

$$\Lambda^+(V_1 \oplus V_2) = \left(\Lambda^+(V_1) \otimes \Lambda^+(V_2)\right) \oplus \left(\Lambda^-(V_1) \otimes \Lambda^-(V_2)\right),$$
$$\Lambda^-(V_1 \oplus V_2) = \left(\Lambda^+(V_1) \otimes \Lambda^-(V_2)\right) \oplus \left(\Lambda^-(V_1) \otimes \Lambda^+(V_2)\right),$$

*thus*

$$CH(\Lambda^+ - \Lambda^-)(V_1 \oplus V_2) = CH(\Lambda^+ - \Lambda^-)(V_1) \cdot CH(\Lambda^+ - \Lambda^-)(V_2).$$

*Proof.* The first part (the relations between $\Lambda^\pm$ of $V_1, V_2$ and $V_1 \oplus V_2$) is left as an exercise. For the second part note that block matrices $A_1 \oplus A_2 \in \mathfrak{so}(V_1 \oplus V_2)$, made of the two blocks $A_i \in \mathfrak{so}(V_i)$, cover all conjugacy classes of $\mathfrak{so}(V_1 \oplus V_2)$ under the adjoint action of $\mathrm{SO}(V_1 \oplus V_2)$ (Note: this only happens because $V_i$ are both even-dimensional). Therfore, the Chern character polynomial of the difference $(\Lambda^+ - \Lambda^-)(V_1 \oplus V_2)$ is determined by evaluating this polynomial on such block matrices. $\square$

The last step in computing $CH(\Lambda^+ - \Lambda^-)(R^{2n})$ is thus to compute it for $n = 1$, where it can by done by direct computation: Take $e_1, e_2$ an oriented ONB of $\mathbb{R}^2$, such that $e_1 \pm ie_2 \in \mathbb{R}^2 \otimes \mathbb{C}$ is a basis of eigenvectors of $A$ for the eigenvalues $\pm 2\pi i x$. Then

$$1 + \ ie_1 \wedge e_2, e_1 + ie_2$$

forms a basis of eigenvectors for the action of $A$ on $\Lambda^+(\mathbb{R}^2)$ (the eigenvalues are zero, resp $2\pi i x$), and

$$1 - \ ie_1 \wedge e_2, \ e_1 - ie_2$$

forms a basis of eigenvectors of $A$ on $\Lambda^-(\mathbb{R}^2)$, for the eigenvalues 0, resp. $-2\pi i x$. The Chern character power series, evaluated on $A$ yields

$$CH(\Lambda^+ - \Lambda^-)(A) = 1 + e^x - (1 + e^{-x}) = e^x - e^{-x},$$

thus the Chern character power series on $\Lambda^+ - \Lambda^-$, evaluated on a matrix $A \in \mathfrak{so}(2n)$, with eigenvalues $\{\pm 2\pi i x_k\}_{k=1,\dots,n}$, is equal to

$$CH(\Lambda^+ - \Lambda^-)(A) = \prod_{k=1}^{n} (e^{x_k} - e^{-x_k}),$$

and it clearly vanishes for $x_k = 0$, $\forall k = 1, \dots, n$, thus it is divisible by $Pf(A) = x_1 \dots x_n$ and we can compute

$$\frac{CH(\Lambda^+ - \Lambda^-) \cdot TD}{Pf}(A) = (-1)^n \prod_{k=1}^{n} \frac{x_k \left(e^{\frac{x_k}{2}} - e^{-\frac{x_k}{2}}\right)\left(e^{\frac{x_k}{2}} + e^{-\frac{x_k}{2}}\right)}{\left(e^{\frac{x_k}{2}} - e^{-\frac{x_k}{2}}\right)^2} = (-1)^n 2^n \prod_{k=1}^{n} \frac{\frac{x_k}{2}}{\tanh \frac{x_k}{2}}.$$

This power series will be denoted with $L \in \mathbb{R}[[\mathfrak{so}(2n)]]^{\mathrm{SO}(2n)}$ (it is an invariant power series, but we can truncate it to obtain invariant polynomials).

Now, the analytic function $\frac{x}{\tanh x}$ is even, and its Taylor expansion starts with

$$\frac{x}{\tanh x} = 1 + \frac{1}{3}x^2 + h.o.t.,$$

therefore we can conclude that, for a 4-dimensional , oriented manifold, its signature is

$$sign(M) = \int_M \frac{p_1(M)}{3},$$

because one-third of $P_1 = x_1^2 + x_2^2$ is the only term of order 2 (that corresponds to a 4-form via Chern-Weil theory) in the integrand

$$2^2 \frac{\frac{x_1}{2}}{\tanh \frac{x_1}{2}} \frac{\frac{x_2}{2}}{\tanh \frac{x_2}{2}} = 4(1 + \frac{x_1^2}{12} + h.o.t.)(1 + \frac{x_2^2}{12} + h.o.t.) = 4 + \frac{x_1^2 + x_2^2}{3} + h.o.t. \quad (16)$$

In general, one defines the *L-genus* of a manifold to be the (characteristic) cohomology class represented by $L(K^\nabla)$, for $\nabla$ an $\mathrm{SO}(2n)$-connection on $TM$. We have obtained

**Theorem 11.6** (Hirzebruch's signature Theorem)**.** *The signature of a $4k$-dimensional, oriented, compact manifold $M$ is equal to its $L$-genus integrated on $M$.*

**Remark 11.7.** It is clear that the $L$-genus is a polynomial expression in the Pontryagin classes of $M$, with rational coefficients. It is *a priori* not obvious that the $L$-genus is an integer cohomology class; for higher dimensions the denominators in the polynomial expressions of $L$ in terms of the Pontryagin classes grow very fast, see [4].

## 12. Moduli space of self-dual connections on a compact 4-manifold

We have seen in Proposition 8.20 that SD and ASD connections on a compact oriented Riemannian manifold $M$, if they exist, are absolute minimizers of the Yang-Mills functional. In this section we briefly describe a famous theorem of Atiyah-Hitchin-Singer [1] that describes, under some conditions, the *moduli space* of such connections:

The idea is to consider the space (in general, it is infinite-dimensional) of SD connections on a given principal $G$-bundle $P \to M$, then to consider its orbit space w.r.t. the action of the *gauge group*, which is the (infinite-dimensional) "Lie" group of gauge equivalences. This quotient space is a topological space called the *moduli space* $\mathcal{M}^{SD}(M, P)$ of SD connections on $P$. It is *a priori* not clear that it Hausdorff or if it is (at least piecewise) a smooth, finite-dimensional manifold.

**Theorem 12.1.** (Atiyah-Hitchin-Singer [1]) *Let $M$ be a compact Riemannain manifold of* self-dual type *and with positive scalar curvature. Then the moduli space $\mathcal{M}_0^{SD}(M, P)$ of* irreducible *self-dual connections on a given $G$-bundle ($G$ is a compact semisimple Lie group) $P \to M$ is either empty, or it is a smooth, Hausdorff, manifold of real dimension*

$$\dim \mathcal{M}_0^{SD}(M, P) = \int_M p_1(\mathrm{Ad}P) - (\chi(M) - sign(M)) \dim G.$$

A compact semisimple Lie group is a compact Lie group with trivial, or discrete center (e.g. $\mathrm{SU}(k)$, whose center is given by the $k$-th rooths of unity, is seminismple, $\mathrm{U}(k)$ is not, because its center is $\mathrm{U}(1)$ acting on $\mathbb{C}^k$ diagonally).

In this case, a connection is irreducible iff $\mathrm{Ad}P$ admits no local $\nabla$-parallel section (if $G$ had a non-trivial center, then $\mathfrak{g}$ would contain non-trivial Ad-invariant elements, which would define non-trivial global $\nabla$-parallel sections for every $G$-connection).

Finally, recall that the Riemann curvature tensor $R$ of a manifold can be seen as a symmetric linear map from $\Lambda^2 M \to \Lambda^2 M$. If $\dim M = 4$, the 2-forms on $M$ decompose in SD, resp ADS parts, thus $R$ admits a block decomposition as a symmetric map

$$R : \Lambda^2_+ M \oplus \Lambda^2_- M \to \Lambda^2_+ M \oplus \Lambda^2_- M,$$

and the components in $\mathrm{End}(\Lambda^2_\pm)$ are symmetric, and the trace of each of these components is equal to $\frac{Scal}{6}$, where $Scal$ is the scalar curvature of $M$. By definition, a manifold is *self-dual* if the component of $R$ in $\mathrm{End}(\Lambda^2_+)$ reduces to this diagonal part (the algebraic components of $R$ depend, in general, on the Ricci tensor and the Weyl tensors of $M$, [1]).

*Proof.* The proof is involved and only few ideas will be explained here. For a well-prepared reader, the best reference for the proof is the original paper [1]. The proof has 3 main steps:

(1) infinitesimal (where a computation of the *virtual* tangent space of $\mathcal{M}_0^{SD}(M,P)$ at a point $\nabla \in \mathcal{M}_0^{SD}(M,P)$ is made)
(2) local (where it is shown that the *virtual* vectors from the first step correspond to local deformations (curves in $\mathcal{M}_0^{SD}(M,P)$ starting in $\nabla$), thus providing local smooth charts)
(3) global (where it is shown that the topological space $\mathcal{M}_0^{SD}(M,P)$ is Hausdorff and thus a smooth manifold)

We only give some details of the first step: for this, we fix $\nabla$ an SD connection and construct an elliptic complex

$$(D) \qquad 0 \to C^\infty(\mathrm{Ad}P \otimes \mathbb{C}) \xrightarrow{\nabla} C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P \otimes \mathbb{C}) \xrightarrow{d_-^\nabla} C^\infty(\Lambda^2_- M \otimes \mathrm{Ad}P \otimes \mathbb{C}) \to 0, \tag{17}$$

where $d_-^\nabla$ is the ASD-part of $d^\nabla$, the covariant exterior differential

$$d^\nabla : C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P) \to C^\infty(\Lambda^2 M \otimes \mathrm{Ad}P).$$

The fact that $(D)$ is indeed a complex is due to the fact that

$$d^\nabla \circ \nabla = K^\nabla \cdot, \text{ thus } d_-^\nabla(\nabla(s)) = K_-^\nabla \cdot s = 0,$$

because $\nabla$ is SD, thus the ASD-part of $K^\nabla$ vanishes.

It is easy to see that this complex is elliptic (exercise!) and that it is universal, so its index can be computed by integrating some characteristic classes on $M$.

Let us first see why this complex is relevant for the infinitesimal step of the proof: note that every $G$-connection can be written $\nabla' = \nabla + \theta$, where $\theta \in C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P)$. As in the proof of the Chern-Weil theorem, we see that, if $\nabla^t := \nabla + \theta^t$ is a 1–parameter family of connections passing with $\nabla^0 = \nabla$, then

$$\left.\frac{d}{dt}\right|_{t=0} K^{\nabla^t} = d^\nabla \theta, \text{ where } \theta = \left.\frac{d}{dt}\right|_{t=0} \theta^t \in C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P),$$

and if all $\nabla^t$ are SD, then $d_-^\nabla \theta = 0$. So the kernel of the operator $d_-^\nabla$ characterizes, on an infinitesimal level, the deformations of $\nabla$ through SD connections.

On the other hand, if such a deformation is induced by a family of gauge equivalences $\phi^t : P \to P$, then the Ad$P$-valued 1-forms (or, equivalently, the Ad-equivariant horizontal 1-forms on $P$) $\theta^t$ are determined by (5)

$$\theta^t(X) = Ad_{\bar{\phi}^{t^{-1}}}\omega(X) - \omega(X) + \omega^{MC}(\bar{\phi}^t{}_*(X)), \ \forall X \in TP, \tag{18}$$

where $\omega$ is the connection 1–form associated to $\nabla$ and $\bar{\phi}^t : P \to G$ is the AD-equivariant function that corresponds to the gauge transformation $\phi^t$. As we know that $\theta^t$ is a horizontal 1-form on $P$ which, because of its equivariance properties, induces a section in $\Lambda^1 M \otimes \mathrm{Ad}P$, it suffices to fix a point $p \in P_x$, $x \in M$, and consider $X \in \ker \omega_p$ in (18).

We have then $\bar{\phi}^t(p)$ a curve in $G$, with $\bar{\phi}^0 = e$, the neutral element of $G$, and $\theta^t_P(X) = \omega^{MC}(\bar{\phi}^t{}_*(X))$ is a family of vectors in $\mathfrak{g}$. If we set

$$X := \left.\frac{d}{ds}\right|_{s=0} c_s, \text{ where } c_s \in P$$

is a $\nabla$-horizontal curve, with $c_0 = p$, then

$$\left.\frac{d}{dt}\right|_{t=0} \theta^t_p(X) = \left.\frac{d}{ds}\right|_{s=0} \omega^{MC}\left(\left.\frac{d}{dt}\right|_{t=0} \bar{(}\phi^t)(c^s)\right).$$

Denote by $\phi'(q) \in \mathfrak{g}$ the derivative in $t$, for $t = 0$, of the map

$$t \mapsto \omega^{MC}(\bar{\phi}^t(q)),$$

for all $q \in P$. We obtain a smooth map $\phi' : P \to \mathfrak{g}$ which induces a section in Ad$P$ (because it is equivariant – exercise!). We conclude that

$$\left.\frac{d}{dt}\right|_{t=0} \theta^t_p(X) = d\phi'(X).$$

This implies that the space of infinitesimal deformations of $\nabla$ through gauge-equivalent connections is the space Im($\nabla$), for

$$\nabla : C^\infty(\mathrm{Ad}P) \to C^\infty(\Lambda^1 M \otimes \mathrm{Ad}P),$$

thus the *virtual* tangent space to $\mathcal{M}_0^{SD}(M, P)$ is the vector space

$$\ker d_-^\nabla / \mathrm{Im}\nabla,$$

i.e. the middle cohmology term of (17). In order to show that its dimension (which is finite because the complex (17) is elliptic) is exactly equal to minus the index of $(D)$, we need to show that the other two cohomology terms vanish.

For the term in $H^0$, which is simply $\ker \nabla$, this is implied by the fact that $\nabla$ is irreducible, because $\ker \nabla$ is exactly the space of $\nabla$-parallel sections of Ad$P$.

For the term in $H^2$, which is coker$d_-^\nabla$, this is implied by the conditions on the Riemannian manifold $M$: self-dual and with positive scalar curvature. We refer the reader to [1] for details.

We still need to compute the index of $(D)$; this is done with the Index Theorem: we need to compute the quotient of the Chern character polynomial of $(\Lambda^0 - \Lambda^1 + \Lambda^2_-)$, which is one half of the difference of

$$CH(\Lambda^0 - \Lambda^1 + (\Lambda^2_- + \Lambda^2_+) - \Lambda^3 - \Lambda^4) = CH(2\Lambda^0 - 2\Lambda^1 + (\Lambda^2_- + \Lambda^2_+)) \quad \text{and}$$

$$CH((\Lambda^0 + \Lambda^4)_+ + (\Lambda^1 + \Lambda^3)_+ + \Lambda^2_+ - (\Lambda^0 + \Lambda^4)_- - (\Lambda^1 + \Lambda^3)_- - \Lambda^2_- = CH(\Lambda^2_+ - \Lambda^2_-).$$

The first line above corresponds to the computation for the de Rham complex and the second for the signature operator. Therefore, the integrand (recall that the integrand also contains the Todd class) for the non-twisted elliptic complex

$$(D_0) \qquad 0 \to C^\infty(M) \xrightarrow{d} C^\infty(\Lambda^1 M)C^\infty(M) \xrightarrow{d_-} C^\infty(\Lambda^2_- M) \to 0$$

is half of the difference of the integrands for the de Rham complex and for the signature operator, i.e.,

$$\frac{1}{2}(Pf - L) = -2 + \frac{1}{2}(\chi(M) - sign(M))vol_M,$$

where we have only written the 0 degree term, which is $\frac{1}{2}(0-4) = -2$, and the term of degree 4, which is the result of the integral (hence the index of $(D_0)$) times the volume element of $M$ (chosen such that $\int_M vol_M = 1$). Recall the formula for $L$ from (16).

For the twisted complex (17), we need to multiply this integrand with the Chern character of $\mathrm{Ad}P \otimes \mathbb{C}$, which is

$$ch(\mathrm{Ad}P\otimes\mathbb{C}) = \mathrm{rank}(\mathrm{Ad}P\otimes\mathbb{C}) + c_1(\mathrm{Ad}P\otimes\mathbb{C}) + \frac{1}{2}(c_1^2 - c_2)(\mathrm{Ad}P\otimes\mathbb{C}) = \dim G + \frac{1}{2}p_1(\mathrm{Ad}P).$$

Here again, we have only considered the terms of degree at most 4.

We conclude

$$\mathrm{ind}(D) = -\int_M p_1(\mathrm{Ad}P) + \frac{1}{2}\dim G(\chi(M) - sign(M)).$$

As the index of $D$ is equal to minus the dimension of the virtal tangent space, we conclude (assuming the steps 2 and 3 of the proof – see [1]) that $\mathcal{M}_0^{SD}(M,P)$ is a smooth manifold of dimension

$$\int_M p_1(\mathrm{Ad}P) - \frac{1}{2}\dim G(\chi(M) - sign(M)),$$

as claimed. □

In particular, there exist SD connections on $P$ only if

$$\int_M p_1(\mathrm{Ad}P) \geq \frac{1}{2}\dim G(\chi(M) - sign(M)).$$

In the case of the sphere and $G := \mathrm{SU}(2)$, one can show that $p_1(\mathrm{Ad}P) = 8k\, vol_M$, with $k \in \mathbb{Z}$. The isomorphism class of a SU(2)-bundle on $S^4$ is, in fact, determined by this number $k$. If we denote $\mathcal{M}_0^{SD}(M,P)$ by $\mathcal{M}_0^{SD}(M,k)$, we can state:

**Corollary 12.2.** *The moduli space of irreducible self-dual connections on an* SU(2)*-principal bundle of type $k \in \mathbb{N}$ over the round sphere $S^4$ is a smooth manifold of dimension $8k-3$.*

Moreover, the round $S^4$ is not only seld-dual, but anti-self-dual as well, i.e. it it also self-dual for the opposite orientation (in fact, $S^4$ is the only compact orientable manifold with positive scalar curvature being at the same time self-dual and anti-self-dual), so the conditions of the AHS theorem also apply for $S^4$ with opposite orientation. But SD connections for this opposite orientation are ASD for the standard orientation, thus

**Corollary 12.3.** *The moduli space of irreducible anti-self-dual connections on an* SU(2)*-principal bundle of type $k \in \mathbb{N}$ over the round sphere $S^4$ is a smooth manifold of dimension $-8k-3$.*

In fact, all these moduli spaces can be shown to be non-empty, if their dimension, as given by the AHS theorem, is positive. For $k = 1$ there are explicit solutions of the Yang-Mills self-dual equations covering all the moduli space.

Note that for $k = 0$ (the trivial bundle), all SD connections are automatically also ASD, thus *flat* (the curvature vanishes identically). However, since $S^4$ is simply-connected, this implies that it is a trivial connection, therefore non irreducible. So $\mathcal{M}_0^{SD}(S^4, 0) = \mathcal{M}_0^{ASD}(S^4, 0) = \emptyset$.

For $k > 0$, $\mathcal{M}_0^{SD}(S^4, k)$ is a smooth manifold as above and $\mathcal{M}_0^{ASD}(S^4, k) = \emptyset$ and for $k < 0$, $\mathcal{M}_0^{ASD}(S^4, k)$ is a smooth manifold as above and $\mathcal{M}_0^{SD}(S^4, k) = \emptyset$.

## References

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