Sobolev Error Estimates for Filtered Back Projection Reconstructions

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***Abstract*—This paper concerns the approximation of bivariate functions using the filtered back projection (FBP) formula from computerized tomography. To this end, we prove error estimates and convergence rates for the FBP reconstruction method for target functions f from a Sobolev space $\mathrm{H}^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$, where we bound the FBP reconstruction error with respect to the (weaker) norms of the (rougher) Sobolev spaces $\mathrm{H}^\sigma(\mathbb{R}^2)$, for $0 \leq \sigma \leq \alpha$. The results of this paper generalize previous of our findings in [2]–[4] for L^2-error estimates, i.e., for the case $\sigma = 0$, to Sobolev error estimates for all fractional orders $\sigma \in [0, \alpha]$ and provide criteria to assess the performance of the utilized low-pass filter by means of its window function.**

I. Introduction

The term *filtered back projection* (FBP) refers to a classical reconstruction technique in computerized tomography (CT), which deals with recovering the interior structure of a scanned object from X-ray scans. This X-ray data can be interpreted as a finite set of line integrals of the (unknown) *attenuation function* of the scanned object which describes the amount of energy that is absorbed by the medium.

We state the CT reconstruction problem as follows.

Problem 1 (Basic reconstruction problem): For $\Omega \subset \mathbb{R}^2$ reconstruct a bivariate function $f \in \mathrm{L}^1(\Omega)$ on its domain Ω from given Radon data

$$\big\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\big\},$$

where the *Radon transform* $\mathcal{R}f$ of f is defined as

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

for $(t,\theta) \in \mathbb{R} \times [0,\pi)$. ■

Thus, the CT reconstruction problem seeks for the inversion of the Radon transform $\mathcal{R}$. For a comprehensive mathematical treatment of $\mathcal{R}$ and its inversion, we refer to [6], [11].

In previous work [2]–[4] we derived L^2-error estimates and convergence rates for target functions f from fractional Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$, where $\alpha > 0$. More recently, we also proved Sobolev error estimates and convergence rates [1]. Although we use some of the results from [1] here, the primary goal of this paper is to generalize our previous results in [2] from L^2-error estimates to Sobolev error estimates in the rougher Sobolev spaces $\mathrm{H}^\sigma(\mathbb{R}^2)$, for $\sigma \in [0,\alpha]$.

The outline of this paper is as follows. In Section II, we consider the inversion of the Radon transform by the classical FBP formula. Further, we describe how the FBP can be stabilized by using suitable low-pass filters of finite bandwidth and with a compactly supported window function. This standard approach leads us to an approximate reconstruction formula, whose approximation quality will be evaluated in this paper. To this end, in Section III, we derive Sobolev error estimates for target functions from Sobolev spaces of fractional order. Additionally, we state asymptotic convergence rates as the bandwidth goes to infinity in Section IV. Asymptotic Sobolev error estimates with weaker assumptions are finally provided in Section V.

II. Filtered Back Projection

The inversion of the Radon transform $\mathcal{R}$ is well understood and given by the classical *filtered back projection formula*

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y), \qquad (1)$$

which holds for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathscr{C}(\mathbb{R}^2)$ (see [5, Theorem 6.2.]). Here, the univariate *Fourier transform* $\mathcal{F}$ applies to variable S and the *back projection* $\mathcal{B}h$ of $h \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$ is defined as

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta)+y\sin(\theta),\theta)\,\mathrm{d}\theta$$

for $(x,y) \in \mathbb{R}^2$. Note that, up to the constant $\frac{1}{\pi}$, the back projection operator $\mathcal{B}$ is the adjoint operator of the Radon transform $\mathcal{R}$.

We remark that the FBP formula is numerically *unstable*. By applying the filter $|S|$ to the Fourier transform $\mathcal{F}(\mathcal{R}f)$ in (1), especially the high frequency components of $\mathcal{R}f$ are amplified by the magnitude of $|S|$. Thus, the filtered back projection formula is in particular highly sensitive with respect to noise.

To reduce the noise sensitivity of the FBP formula, we follow a standard approach and replace the filter $|S|$ in (1) by a *low-pass filter* A_L of the form

$$A_L(S) = |S|\,W(S/L) \quad \text{for } S \in \mathbb{R}$$

with finite *bandwidth* $L > 0$ and an even *window function* $W \in \mathrm{L}^\infty(\mathbb{R})$ with compact support $\mathrm{supp}(W) \subseteq [-1,1]$.

When replacing the filter $|S|$ in (1) by the low-pass filter $A_L(S)$, the reconstruction of f is no longer exact and we only get an *approximate FBP reconstruction*, denoted by f_L.

However, for target functions $f \in \mathrm{L}^1(\mathbb{R}^2)$ the reconstruction f_L is for any $L > 0$ defined almost everywhere on $\mathbb{R}^2$ (see [1, Proposition 3.1]) and, moreover, the resulting *approximate FBP formula* can be simplified as

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big). \qquad (2)$$

Further, f_L belongs to $\mathrm{L}^2(\mathbb{R}^2)$ (see [1, Proposition 4.2]) and can be expressed in terms of the target function f via

$$f_L = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L * \mathcal{R}f\big) = f * K_L, \tag{3}$$

where we define the *convolution kernel* $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$K_L(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}A_L\big)(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

For the sake of brevity, we call any application of the approximate FBP formula (2) an *FBP method*. Therefore, each FBP method provides one approximation f_L to f, $f_L \approx f$, whose quality depends on the choice of the low-pass filter A_L.

In the following, we analyse the intrinsic error of the FBP method which is incurred by the use of the low-pass filter A_L, i.e., we wish to analyse the reconstruction error

$$e_L = f - f_L$$

with respect to the filter's window W and bandwidth L.

We remark that pointwise and L^∞-error estimates on e_L were proven by Munshi et al. in [8]. Their theoretical results were further supported by numerical experiments in [9]. Error bounds for the L^p-norm of e_L, in terms of an L^p-modulus of continuity of f, were proven by Madych in [7].

In [2]–[4] we derived L^2-error estimates and convergence rates for target functions from fractional Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^2)$. Let us recall that the *Sobolev space* $\mathrm{H}^\alpha(\mathbb{R}^2)$ of order $\alpha \in \mathbb{R}$ is defined as

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \big\{ f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty \big\},$$

where

$$\|f\|_\alpha^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \big(1 + x^2 + y^2\big)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y),$$

and where $\mathcal{S}'(\mathbb{R}^2)$ denotes the Schwartz space of tempered distributions on $\mathbb{R}^2$.

We remark that in relevant applications of (medical) image processing, Sobolev spaces of compactly supported functions,

$$\mathrm{H}_0^\alpha(\Omega) = \big\{ f \in \mathrm{H}^\alpha(\mathbb{R}^2) \mid \mathrm{supp}(f) \subseteq \overline{\Omega} \big\},$$

on an open and bounded domain $\Omega \subset \mathbb{R}^2$, and of fractional order $\alpha > 0$ play an important role (cf. [10]). In fact, we can consider the density of an image in $\Omega \subset \mathbb{R}^2$ as a function from the Sobolev space $\mathrm{H}_0^\alpha(\Omega)$ whose order α is close to $\frac{1}{2}$.

III. Error Analysis

In this section, we analyse certain Sobolev norms of the inherent FBP reconstruction error e_L for target functions f from the Sobolev space $\mathrm{H}^\alpha(\mathbb{R}^2)$ of fractional order $\alpha > 0$. To be more precise, we generalize our L^2-error estimates of [2] to H^σ-error estimates for $0 \leq \sigma \leq \alpha$. To this end, we partly rely on [1], as this is indicated in the following discussion.

Let us assume that $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$. We first show that the approximate FBP reconstruction f_L belongs to the Sobolev space $\mathrm{H}^\sigma(\mathbb{R}^2)$ for $0 \leq \sigma \leq \alpha$.

In [1, Proposition 4.1] we have proven that the convolution kernel K_L belongs to $\mathscr{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ and, moreover, that its Fourier transform is given by

$$\mathcal{F}K_L(x,y) = W_L(x,y) \quad \text{for almost all } (x,y) \in \mathbb{R}^2.$$

Here, the compactly supported bivariate window function $W_L \in \mathrm{L}^\infty(\mathbb{R}^2)$ is defined as

$$W_L(x,y) = W\Big(\frac{r(x,y)}{L}\Big) \quad \text{for } (x,y) \in \mathbb{R}^2,$$

where we let

$$r(x,y) = \sqrt{x^2 + y^2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

This in combination with representation (3) for f_L yields

$$\begin{aligned} \|f_L\|_\sigma^2 &= \|f * K_L\|_\sigma^2 \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \big(1 + r(x,y)^2\big)^\sigma \, |(W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y) \\ &\leq \Big(\sup_{r(x,y) \leq L} |W_L(x,y)|^2 \Big) \, \|f\|_\alpha^2 = \|W\|_{\infty,[-1,1]}^2 \, \|f\|_\alpha^2. \end{aligned}$$

Thus, for $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ with $\alpha > 0$, the approximate FBP reconstruction f_L belongs to $\mathrm{H}^\sigma(\mathbb{R}^2)$ for all $0 \leq \sigma \leq \alpha$.

Let us now turn to the analysis of the FBP reconstruction error $e_L = f - f_L$ with respect to the H^σ-norm. For $\gamma \geq 0$, we define

$$r_\gamma(x,y) = \big(1 + r(x,y)^2\big)^\gamma = \big(1 + x^2 + y^2\big)^\gamma \quad \text{for } (x,y) \in \mathbb{R}^2$$

so that the H^σ-norm of e_L can be expressed as

$$\begin{aligned} \|e_L\|_\sigma^2 &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\sigma(x,y) \, |\mathcal{F}(f - f_L)(x,y)|^2 \, \mathrm{d}(x,y) \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\sigma(x,y) \, |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y) \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\sigma(x,y) \, |1 - W_L(x,y)|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y)$$

with

$$B_L = \big\{ (x,y) \in \mathbb{R}^2 \mid r(x,y) \leq L \big\}$$

and where

$$I_2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus B_L} r_\sigma(x,y) \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

For $\gamma \geq 0$, we define

$$\Phi_{\gamma,W}(L) = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\gamma} \quad \text{for } L > 0$$

so that we can bound I_1 from above by

$$I_1 \leq \Bigg(\sup_{(x,y) \in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)} \Bigg) \|f\|_\alpha^2 = \Phi_{\alpha-\sigma,W}(L) \|f\|_\alpha^2,$$

since

$$\sup_{(x,y) \in B_L} \frac{(1 - W_L(x,y))^2}{r_{\alpha-\sigma}(x,y)} = \sup_{S \in [-L,L]} \frac{(1 - W(S/L))^2}{(1 + S^2)^{\alpha-\sigma}}.$$

For $0 \leq \sigma \leq \alpha$, we can bound I_2 by

$$I_2 \leq L^{2(\sigma-\alpha)} \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus B_L} r_\sigma(x,y) \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ \leq L^{2(\sigma-\alpha)} \, \|f\|_\alpha^2.$$

Combining the estimates for I_1 and I_2, we finally obtain

$$\|e_L\|_\sigma^2 \leq \left(\Phi_{\alpha-\sigma,W}(L) + L^{2(\sigma-\alpha)}\right) \|f\|_\alpha^2.$$

We can summarize the discussion of this section as follows.

Theorem 1 (H^σ-error estimate, see [1, Theorem 5.2]): Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$ and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even and compactly supported with $\mathrm{supp}(W) \subseteq [-1,1]$. Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \left(\Phi^{1/2}_{\alpha-\sigma,W}(L) + L^{\sigma-\alpha}\right) \|f\|_\alpha, \qquad (4)$$

where

$$\Phi_{\alpha-\sigma,W}(L) = \sup_{S \in [-1,1]} \frac{(1-W(S))^2}{(1+L^2S^2)^{\alpha-\sigma}} \quad \text{for } L > 0. \qquad \blacksquare$$

We remark that for the special case $\sigma = 0$, the bound in (4) agrees with the L^2-error estimate of [2, Theorem 4.1].

Like the L^2-error bound in [2, Theorem 4.1], the H^σ-error estimate (4) involves the error term $\Phi_{\gamma,W}(L)$, but now with $\gamma = \alpha - \sigma$ rather than $\gamma = \alpha$. Consequently, we can rely on the analysis in [2] concerning the properties of $\Phi_{\gamma,W}(L)$.

In [2, Theorem 4.2] we have proven that, if the window function W is continuous on $[-1,1]$ and $W(0) = 1$, the error term $\Phi_{\gamma,W}(L)$ converges to 0 as L goes to ∞ for all $\gamma > 0$. With this we get the following convergence result for the H^σ-norm of the FBP reconstruction error.

Corollary 1: Let the assumptions of Theorem 1 be satisfied and let $W \in \mathscr{C}([-1,1])$ with $W(0) = 1$. Then, for $0 \leq \sigma < \alpha$, the H^σ-norm of the reconstruction error $e_L = f - f_L$ converges to 0 as L goes to ∞, i.e.,

$$\|e_L\|_\sigma = o(1) \quad \text{for} \quad L \longrightarrow \infty. \qquad \blacksquare$$

We remark that the result in Corollary 1 is *not* covered by our previous paper [1]. Indeed, throughout [1], we only rely on the weaker assumption $W \in \mathrm{L}^\infty(\mathbb{R})$ for the filter's window function W rather than on $W \in \mathscr{C}([-1,1])$.

IV. Rate of Convergence

In this section we analyse the convergence rate of the FBP reconstruction error $\|e_L\|_\sigma$ as L goes to ∞. To this end, we partly rely on [1], as indicated in the following discussion.

Let $S^*_{\gamma,W,L} \in [0,1]$, for $\gamma \geq 0$, denote the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\gamma,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\gamma} \quad \text{for } S \in [-1,1].$$

To determine the rate of convergence for $\|e_L\|_\sigma$, we assume that $S^*_{\alpha-\sigma,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha-\sigma,W} > 0$ satisfying

$$S^*_{\alpha-\sigma,W,L} \geq c_{\alpha-\sigma,W} \quad \text{for all } L > 0. \qquad (5)$$

Then, the error term $\Phi_{\alpha-\sigma,W}(L)$ is bounded above by

$$\Phi_{\alpha-\sigma,W}(L) = \frac{\left(1 - W(S^*_{\alpha-\sigma,W,L})\right)^2}{\left(1 + L^2 (S^*_{\alpha-\sigma,W,L})^2\right)^{\alpha-\sigma}} \\ \leq c^{2(\sigma-\alpha)}_{\alpha-\sigma,W} \, \|1 - W\|^2_{\infty,[-1,1]} \, L^{2(\sigma-\alpha)}.$$

Consequently, we obtain

$$\|e_L\|_\sigma^2 \leq \left(c^{2(\sigma-\alpha)}_{\alpha-\sigma,W} \, \|1-W\|^2_{\infty,[-1,1]} + 1\right) L^{2(\sigma-\alpha)} \, \|f\|_\alpha^2$$

so that

$$\|e_L\|_\sigma = \mathcal{O}(L^{\sigma-\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

In summary, this yields the following result.

Theorem 2 (Rate of convergence, see [1, Theorem 5.4]): Let the assumptions of Theorem 1 as well as the assumption (5) be satisfied. Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \left(c^{\sigma-\alpha}_{\alpha-\sigma,W} \, \|1-W\|_{\infty,[-1,1]} + 1\right) L^{\sigma-\alpha} \, \|f\|_\alpha. \quad (6)$$

In particular,

$$\|e_L\|_\sigma = \mathcal{O}(L^{-(\alpha-\sigma)}) \quad \text{for} \quad L \longrightarrow \infty. \qquad \blacksquare$$

Note that the decay rate $\alpha - \sigma$ in (6) is determined by the difference between the smoothness α of the target function f and the order σ of the Sobolev norm $\|\cdot\|_\sigma$ in which the reconstruction error e_L is measured.

We remark that assumption (5) is satisfied for a large class of window functions. For example, let W satisfy

$$W(S) = 1 \quad \text{for all } S \in (-\varepsilon, \varepsilon)$$

for some $\varepsilon \in (0,1)$. Then, assumption (5) is fulfilled with the constant $c_{\alpha-\sigma,W} = \varepsilon$ for all $0 \leq \sigma \leq \alpha$.

However, there are commonly used window functions W which do not satisfy assumption (5). In fact, in [2] we investigated the behaviour of $S^*_{\gamma,W,L}$ and $\Phi_{\gamma,W}(L)$ for $\gamma > 0$ numerically for the following window functions of the filter $A_L(S) = |S| \, W(S/L)$:

Name	$W(S)$ for $\lvert S\rvert \leq 1$	Parameter
Shepp–Logan	$\mathrm{sinc}(\pi S/2)$	-
Cosine	$\cos(\pi S/2)$	-
Hamming	$\beta + (1-\beta)\cos(\pi S)$	$\beta \in [1/2, 1]$
Gaussian	$\exp\left(-(\pi S/\beta)^2\right)$	$\beta > 1$

We summarize our numerical results from [2] as follows. For $\gamma < 2$, we found that assumption (5) is fulfilled and

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-2\gamma}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\gamma \geq 2$, assumption (5) is not fulfilled, since

$$S^*_{\gamma,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty,$$

and the convergence rate of $\Phi_{\gamma,W}$ stagnates by

$$\Phi_{\gamma,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

Note that all window functions in the above table are twice continuously differentiable on $[-1,1]$ with

$$W(0) = 1 \quad \text{and} \quad W'(0) = 0.$$

This motivated us to analyse the convergence behaviour of the error term $\Phi_{\gamma,W}$ for the special case of $\mathscr{C}^k$-window functions whose first $k-1$ derivatives vanish at zero (cf. [2]).

Consequently, to continue our analysis, we now consider even window functions W with compact support in $[-1,1]$ that additionally satisfy $W \in \mathscr{C}^k([-1,1])$ for some $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Under this assumption, we have proven in [2, Theorem 6.1] that for $\gamma \leq k$ the error term $\Phi_{\gamma,W}(L)$ is bounded above by

$$\Phi_{\gamma,W}(L) \leq \frac{1}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2\gamma} \quad \text{for all } L > 0$$

and for $\gamma > k$ by

$$\Phi_{\gamma,W}(L) \leq \begin{cases} \frac{1}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2\gamma} & \text{for } L < \frac{\sqrt{k}}{\sqrt{\gamma-k}} \\ \frac{c^2_{\gamma,k}}{(k!)^2} \, \|W^{(k)}\|^2_{\infty,[-1,1]} \, L^{-2k} & \text{for } L \geq \frac{\sqrt{k}}{\sqrt{\gamma-k}}, \end{cases}$$

where the constant

$$c_{\gamma,k} = \Big(\frac{k}{\gamma-k}\Big)^{k/2} \Big(\frac{\gamma-k}{\gamma}\Big)^{\gamma/2} \tag{7}$$

is strictly monotonically decreasing in $\gamma > k$. In particular,

$$\Phi_{\gamma,W}(L) = \mathcal{O}\Big(L^{-2\min\{k,\gamma\}}\Big) \quad \text{for} \quad L \longrightarrow \infty.$$

We remark that this theoretical result complies with our numerical experiments in [2]. In particular, the saturation of the convergence rate at order $\mathcal{O}(L^{-2k})$ is observed in [2] and so the numerical experiments in [2] show that the stated convergence rate of $\Phi_{\gamma,W}(L)$ is optimal for the special case of $\mathscr{C}^k$-windows.

Using the above bound of $\Phi_{\gamma,W}(L)$ in Theorem 1 gives the following H^σ-error estimate for $\mathscr{C}^k$-window functions.

Theorem 3 (H^σ-error estimate for $\mathscr{C}^k$-windows): Let the assumptions of Theorem 1 be satisfied. In addition, let $W \in \mathscr{C}^k([-1,1])$ for $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \bigg(\frac{1}{k!} \, \|W^{(k)}\|_{\infty,[-1,1]} + 1\bigg) L^{\sigma-\alpha} \, \|f\|_\alpha$$

for $\alpha - \sigma \leq k$, and by

$$\|e_L\|_\sigma \leq \Big(\frac{c_{\alpha-\sigma,k}}{k!} \, \|W^{(k)}\|_{\infty,[-1,1]} \, L^{-k} + L^{\sigma-\alpha}\Big) \|f\|_\alpha$$

for $\alpha - \sigma > k$ and sufficiently large $L > 0$. In particular,

$$\|e_L\|_\sigma = \mathcal{O}\Big(L^{-\min\{k,\alpha-\sigma\}}\Big) \quad \text{for} \quad L \longrightarrow \infty. \qquad \blacksquare$$

Note that in Theorem 3 for $\alpha - \sigma \leq k$ the decay rate of $\|e_L\|_\sigma$ is determined by the difference between the smoothness α of the target function f and the order σ of the considered Sobolev norm, whereas for $\alpha - \sigma > k$ the decay rate saturates at $\mathcal{O}(L^{-k})$. Here, k denotes the differentiability order of the window function W, whose first $k-1$ derivatives are required to vanish at zero. However, in this case the error bound still decreases at increasing α, since the involved constant $c_{\alpha-\sigma,k}$ is strictly monotonically decreasing in $\alpha - \sigma > k$. Thus, a smoother target function still permits a better approximation, as expected. Nevertheless, the attainable convergence rate is limited by the differentiability order of the filter's window function.

V. Asymptotic Error Estimates

In this section, we finally derive an asymptotic H^σ-error estimate for the FBP method under weaker assumptions.

For this purpose, we consider even window functions $W \in \mathrm{L}^\infty(\mathbb{R})$ with compact support in $[-1,1]$ that are k-times differentiable only at the origin for some $k \geq 2$ with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

As in the previous error estimates of this paper, we consider target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ for some $\alpha > 0$ and analyse the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ for $0 \leq \sigma \leq \alpha$.

We again start with splitting the H^σ-norm of e_L into the sum of two integrals

$$\begin{aligned} \|e_L\|^2_\sigma &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} r_\sigma(x,y) \, |\mathcal{F}(f - f_L)(x,y)|^2 \, \mathrm{d}(x,y) \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\sigma(x,y) \, |1 - W_L(x,y)|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y)$$

with

$$B_L = \big\{(x,y) \in \mathbb{R}^2 \mid r(x,y) \leq L\big\}$$

and where

$$I_2 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2 \setminus B_L} r_\sigma(x,y) \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

As before, the integral I_2 can be bounded above by

$$I_2 \leq L^{2(\sigma-\alpha)} \, \|f\|^2_\alpha.$$

The integral I_1 can be expressed as

$$I_1 = \frac{1}{4\pi^2} \int_{B_L} r_\sigma(x,y) \left| 1 - W\Big(\frac{r(x,y)}{L}\Big)\right|^2 |\mathcal{F}f(x,y)|^2 \mathrm{d}(x,y).$$

Because $W \in \mathrm{L}^\infty(\mathbb{R})$ is k-times differentiable at zero, we can apply Taylor's theorem and, thus, there exists a function $h_k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$W(S) = \sum_{j=0}^{k} \frac{W^{(j)}(0)}{j!} \, S^j + h_k(S) \, S^k \quad \text{for all } S \in \mathbb{R}$$

and

$$\lim_{S \to 0} h_k(S) = 0.$$

By assumption, the window W satisfies

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Hence, for $(x,y) \in \mathbb{R}^2$ follows that

$$W\Big(\frac{r(x,y)}{L}\Big) = 1 + \Big(\frac{W^{(k)}(0)}{k!} + h_k\Big(\frac{r(x,y)}{L}\Big)\Big)\Big(\frac{r(x,y)}{L}\Big)^k$$

for all $L > 0$. If we now define, for $\gamma \geq 0$,

$$\phi^*_{\gamma,L,k} = \max_{(x,y)\in B_L} \frac{\Big(\frac{r(x,y)}{L}\Big)^{2k}}{r_\gamma(x,y)} = \max_{S\in[0,1]} \frac{S^{2k}}{(1+L^2\,S^2)^\gamma},$$

then integral I_1 can be bounded by

$$I_1 \leq 2\,\phi^*_{\alpha-\sigma,L,k}\,(I_3 + I_4),$$

where we let

$$I_3 = \frac{1}{4\pi^2}\int_{B_L} \Big(\frac{W^{(k)}(0)}{k!}\Big)^2 r_\alpha(x,y)\,|\mathcal{F}f(x,y)|^2\,\mathrm{d}(x,y)$$

and

$$I_4 = \frac{1}{4\pi^2}\int_{B_L} h_k\Big(\frac{r(x,y)}{L}\Big)^2 r_\alpha(x,y)\,|\mathcal{F}f(x,y)|^2\,\mathrm{d}(x,y).$$

We have

$$I_3 \leq \Big(\frac{W^{(k)}(0)}{k!}\Big)^2 \|f\|_\alpha^2$$

and, by using Lebesgue's theorem on dominated convergence, we obtain

$$I_4 = o(1) \quad \text{for} \quad L \longrightarrow \infty.$$

This leads to

$$I_1 \leq 2\,\phi^*_{\alpha-\sigma,L,k}\Big(\frac{W^{(k)}(0)}{k!}\Big)^2 \|f\|_\alpha^2 + \phi^*_{\alpha-\sigma,L,k}\,o(1).$$

In [2], we have shown that the maximum $\phi^*_{\gamma,L,k}$ is bounded by

$$\phi^*_{\gamma,L,k} \leq L^{-2\gamma}$$

for $\gamma \leq k$ and by

$$\phi^*_{\gamma,L,k} \leq \begin{cases} L^{-2\gamma} & \text{for } L < \frac{\sqrt{k}}{\sqrt{\gamma-k}} \\ c^2_{\gamma,k}\,L^{-2k} & \text{for } L \geq \frac{\sqrt{k}}{\sqrt{\gamma-k}} \end{cases}$$

for $\gamma > k$ with the strictly decreasing constant $c_{\gamma,k}$ from (7). Thus, we obtain

$$I_1 \leq \frac{2}{(k!)^2}\,|W^{(k)}(0)|^2\,L^{2(\sigma-\alpha)}\,\|f\|_\alpha^2 + o\Big(L^{2(\sigma-\alpha)}\Big)$$

for $\alpha - \sigma \leq k$, or $\alpha - \sigma > k$ and $L < L^*$, as well as

$$I_1 \leq \frac{2}{(k!)^2}\,c^2_{\alpha-\sigma,k}\,|W^{(k)}(0)|^2\,L^{-2k}\,\|f\|_\alpha^2 + o\big(L^{-2k}\big)$$

for $\alpha - \sigma > k$ and $L \geq L^*$ with the critical bandwidth

$$L^* = \frac{\sqrt{k}}{\sqrt{\alpha-\sigma-k}}$$

and the strictly monotonically decreasing constant

$$c_{\alpha-\sigma,k} = \Big(\frac{k}{\alpha-\sigma-k}\Big)^{k/2}\Big(\frac{\alpha-\sigma-k}{\alpha-\sigma}\Big)^{(\alpha-\sigma)/2}$$

for $\alpha - \sigma > k$.

In summary, we have proven the following asymptotic H^σ-error estimate for the approximate FBP reconstruction f_L.

Theorem 4 (Asymptotic H^σ*-error estimate):* Let the assumptions of Theorem 1 be satisfied. Moreover, let W be k-times differentiable at the origin, $k \geq 2$, with

$$W(0) = 1, \quad W^{(j)}(0) = 0 \quad \text{for all } 1 \leq j \leq k-1.$$

Then, for $0 \leq \sigma \leq \alpha$, the H^σ-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_\sigma \leq \bigg(\frac{\sqrt{2}}{k!}|W^{(k)}(0)| + 1\bigg) L^{\sigma-\alpha}\,\|f\|_\alpha + o(L^{\sigma-\alpha})$$

for $\alpha - \sigma \leq k$, and by

$$\|e_L\|_\sigma \leq \bigg(\frac{\sqrt{2}}{k!}c_{\alpha-\sigma,k}|W^{(k)}(0)|L^{-k} + L^{\sigma-\alpha}\bigg)\|f\|_\alpha + o(L^{-k})$$

for $\alpha - \sigma > k$ and sufficiently large $L > 0$. In particular,

$$\|e_L\|_\sigma = \mathcal{O}\Big(L^{-\min\{k,\alpha-\sigma\}}\Big) \quad \text{for} \quad L \longrightarrow \infty. \qquad \blacksquare$$

VI. Conclusion

We conclude that the *flatness* of the window W determines the convergence rate of the H^σ-error bounds for the inherent FBP reconstruction error. Indeed, if the first $k-1$ derivatives of W vanish at zero, the convergence rate saturates at $\mathcal{O}(L^{-k})$ for $\alpha - \sigma > k$. Further, the quantity $|W^{(k)}(0)|$ dominates the error bounds and can be used as an indicator to predict the approximation quality of the FBP method in the H^σ-norm. We remark that the windows we considered earlier satisfy the assumptions of our theory with $k = 2$ so that Theorem 4 predicts an affine-linear behaviour of the H^σ-error with respect to $|W''(0)|$. For $\sigma = 0$, this was observed numerically in [9].

In practice, the FBP method has to be discretized, leading to inevitable discretization errors. However, the analysis of these is beyond the aims and scopes of this paper. Instead, we provide quantitative criteria to a priori evaluate the performance of the chosen low-pass filter in the continuous setting.

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