ZMP Seminar, Hamburg

2017-04-13

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An Introduction to Hitchin Systems

This talk is based on the first six chapters of

[Hit] Nigel Hitchin (1987), Self-Duality Equations on a Riemann Surface, Proceedings of the London Mathematical Society s3-55: 59-126

Section 1: A quick reminder of gauge theory

- Classical gauge theories can be formulated in the language of principal bundles.
- Recall that a principal bundle is a bundle $P \to M$ with a right group action R of a group G which is fibre preserving and fibrewise free and transitive.
- Physical interpretation: M is spacetime, P encodes "internal degrees of freedom"
- The group G is called the structure group of the principal bundle. We will denote its Lie algebra by $\mathfrak{g}$. (In this talk: $G := so(3)$)
- There is an important canonical action of G on $\mathfrak{g}$ which is given by

$$\mathrm{Ad}_g X := \frac{d}{dt}\Big|_{t=0} (g \exp(tX) g^{-1})$$

  This is called the adjoint action.
- There is an important vector bundle associated to P, called the adjoint bundle

$$\mathrm{Ad}P := (P \times \mathfrak{g})/\sim \quad \text{where} \quad (p \cdot g, X) \sim (p, \mathrm{Ad}_g X)$$

  The typical fibre of $\mathrm{Ad}P$ is $\mathfrak{g}$ therefore any form with values in $\mathrm{Ad}P$ can be described locally as a form with values in the Lie algebra $\mathfrak{g}$.
- The gauge group of P (or group of gauge transformations) is

$$\mathrm{Gau}(P) := \{\varphi : P \to P \text{ diff} \mid \pi \circ \varphi = \varphi,\ R_g \circ \varphi = \varphi \circ R_g\}$$

  This is a Lie group with Lie algebra $\mathfrak{gau}(P) = \Omega^0(M; \mathrm{Ad}P)$.
- Physical gauge fields are modelled using connections (connection 1-forms). A $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P; \mathfrak{g})$ is a connection 1-form if

  ① $R_g^* A = \mathrm{Ad}_{g^{-1}} \circ A$ for all $g \in G$

  ② $A(\bar{X}) = X$ for all $X \in \mathfrak{g}$ (here $\bar{X}_p := \frac{d}{dt}\big|_{t=0} (p \cdot \exp tX)$ denotes the fundamental vector field (or infinitesimal action) corresponding to the Lie algebra element X)

- Interpretation: 4-potential
- One can show that every principal bundle has a connection.
- Moreover, the set of connections $\mathrm{Conn}(P)$ is an affine space over $\Omega^1(M; \mathrm{Ad}\,P)$
- The most important quantity that can be derived from ∇ is its curvature

$$F^A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P; \mathfrak{g})$$

The physical interpretation is the Faraday field strength tensor field.
- We have the Bianchi identity $d_A F^A = 0$, where d_A is the exterior covariant derivative corresponding to A
- The dynamical equation is called the vacuum Yang-Mills equation:

$$d_A * F^A = 0 \quad \text{(in vacuum)} \qquad \text{(YM)}$$

Here $*$ is the Hodge star operator corresponding to the spacetime metric.

Section 2: The self-duality equations

- The goal of gauge theory is to classify solutions of (YM) up to gauge transformations since F^A is invariant under gauge transformations
- (YM) is a system of nonlinear PDEs $\rightsquigarrow$ this is complicated!
- Fruitful approach: look for special solutions. Assume $\dim M = 4$.
A connection A is called a self-dual connection if $*F^A = F^A$. $\quad F^{pq}\varepsilon_{pqik} = F_{ik}$
These solve (YM) automatically since $d_A * F^A = \underbrace{d_A F^A}_{=0 \text{ Bianchi}} = 0$.

- Further simplification: assume $M = \mathbb{R}^4$ with the standard Riemannian metric
Then P can be trivialized globally and we can write

$$A = A_1 dx^1 + A_2 dx^2 + A_3 dx^3 + A_4 dx^4$$

where $A_k : \mathbb{R}^4 \to \mathfrak{g}$.
- Assume furthermore that all coefficients are independent of x_3 and x_4.
Then we can do a kind of dimensional reduction from $M = \mathbb{R}^4$ to $C := \mathbb{R}^2$ and define a new connection $\tilde{A} := A_1 dx^1 + A_2 dx^2$. This is well defined by assumption.

- Of course we don't want to lose the information that was encoded in A_3 and A_4. It will be useful to consider $\mathbb{R}^2 \simeq \mathbb{C}$ and define $dz := dx^1 + i\,dx^2$, $d\bar{z} := dx^1 - i\,dx^2$. Then we may define $\phi := \frac{1}{2}(A_3 - iA_4)\,dz$ and $\phi^* := \frac{1}{2}(A_3 + iA_4)\,d\bar{z}$. Higgs field

In terms of $\tilde{A}$ and ϕ, the equation $*\Phi^A = F^A$ can be written as

$$\left\{ \begin{aligned} F^{\tilde{A}} + [\phi \wedge \phi^*] &= 0 \\ \bar{\partial}_{\tilde{A}} \phi &= 0 \end{aligned} \right. \qquad (SD)$$

no tilde from now on

- Note that these equations are coordinate independent and they are also conformally invariant. Hence we can consider them on a Riemann surface C instead of $\mathbb{C}$. (Recall that a Riemann surface can be described using holomorphic coordinates and that conformal and holomorphic transformations coincide.)
- Finally, we can define the problem. Let $P \to C$ be a G-principal bundle over a Riemann surface C of genus $g > 1$. We are interested in the solutions of the self-duality equations (or Hitchin's equations)

$$\left. \begin{aligned} F^A + [\phi \wedge \phi^*] &= 0, \\ \bar{\partial}_A \phi &= 0, \end{aligned} \right\} \quad (SD)$$

where $A \in \operatorname{Conn} P$ is an irreducible connection and $\phi \in \Omega^{1,0}(C; \operatorname{Ad} P \otimes \mathbb{C})$ is a Higgs field.
- We take the adjoint w.r.t. a hermitian metric — more on that in the next talk
- We will use the notation $\operatorname{Higgs}(P) := \Omega^{1,0}(C; \operatorname{Ad} P \otimes \mathbb{C})$ for the set of Higgs fields

Section 3: The Hitchin moduli space $\mathcal{M}_H$

- We are interested in the set of solutions of (SD) up to gauge tr.
- This set will turn out to be astonishingly well-behaved.
- We will use the notation $\mathcal{H} := \operatorname{Conn} P \times \operatorname{Higgs}(P)$.

  This is an infinite dimensional affine space over

$$\Omega^1(C; \operatorname{Ad} P) \times \Omega^{1,0}(C; \operatorname{Ad} P \otimes \mathbb{C}) \simeq \Omega^1(C; \operatorname{Ad} P \otimes \mathbb{C}).$$

- Moreover, we will use the notation

$$\tilde{\mathcal{M}} := \{ (A, \phi) \in \mathcal{H} \mid (A, \phi) \text{ solves (SD)} \}$$

$\mathcal{M}_H := \tilde{\mathcal{M}} / \mathcal{G}au(P)$ Hitchin modulispace

- In order to determine a candidate for $\dim \mathcal{M}_H$, we linearize the (SD) equations

<u>lemma 1</u> A curve (A_t, ϕ_t) satisfies the SD equations in $(A_0, \phi_0) = (A, \phi)$ to first order if $\left(\dot{A} := \frac{d}{dt}\Big|_{t=0} A_t, \ \dot{\phi} := \frac{d}{dt}\Big|_{t=0} \phi_t \right)$

$$\begin{cases} d_A \dot{A} + [\dot{\phi}, \phi^*] + [\phi, \dot{\phi}^*] = 0 \\ \bar{\partial}_A \dot{\phi} + [\dot{A}^{0,1}, \phi] = 0 \end{cases} \qquad \text{(LSD)}$$

- Since we are interested in solutions only up to gauge transformation, we need to know when a curve comes from an infinitesimal gauge transformation

<u>lemma 2</u> A curve (A_t, ϕ_t) comes from an infinitesimal gauge transformation $\dot{\psi}: M \to \mathrm{Ad}P$ if

$$\begin{cases} \dot{A} = d_A \dot{\psi} \\ \dot{\phi} = [\phi, \dot{\psi}] \end{cases}$$

- Using these results, we construct a complex

$$0 \to \mathcal{G}au(P) \xrightarrow{d_1} T_{(A,\phi)} \mathcal{H} \xrightarrow{d_2} \Omega^2(M, \mathrm{Ad}P) \times \Omega^2(M, \mathrm{Ad}P \otimes \mathbb{C}) \to 0$$

where $d_1 \dot{\psi} := (d_A \dot{\psi}, [\phi, \dot{\psi}])$ and

$$d_2(\dot{A}, \dot{\phi}) := (d_A \dot{A} + [\dot{\phi}, \phi^*] + [\phi, \dot{\phi}^*], \bar{\partial}_A \dot{\phi} + [\dot{A}^{0,1}, \phi]).$$

Note that $\ker d_2 / \operatorname{im} d_1$ is the set of solutions up to infinitesimal gauge transformations

$\Rightarrow$ The candidate for $\dim \mathcal{M}_H$ is $\dim H^1$.

Using the Atiyah-Singer index theorem, we obtain

<u>lemma 3</u> $\dim H^0 - \dim H^1 + \dim H^2 = 12(1-g)$

One can show (vanishing theorems) that at so called irreducible connections

$$H^0 = 0 \text{ and } H^2 = 0$$

Therefore $\dim H^2 = -12 \cdot (g-1)$.

The fact that $\mathcal{M}$ is a smooth manifold comes from a Banach space version of the implicit function theorem.

<u>Theorem 1</u> $\mathcal{M}_H$ is a $12 \cdot (g-1)$ dimensional smooth manifold

Section 4: The basics of Hyperkähler geometry

- Before we can discuss the geometry of $\mathcal{M}_H$ in its full glory, we need to introduce / recall certain terms
- Recall that a Riemannian metric is a smoothly varying scalar product on the tangent spaces: g_{ij}
- An important concept from classical mechanics is a symplectic form, which is a kind of "antisymmetric scalar product" $\omega_{ij} = -\omega_{ji}$ with $\det \omega_{ij} \neq 0$ and
  ($\omega \in \Omega^2(M)$, $d\omega = 0$, ω is nondegenerate) $\partial_k \omega_{ij} - \partial_i \omega_{kj} + \partial_j \omega_{ki} = 0$
  This plays an important role in the usual Hamiltonian formulation of classical mechanics.
- Certain real mfds are actually complex mfds in disguise. In this case, multiplication by i corresponds to a map $I_p : T_pM \to T_pM$ with $I_p^2 = -1$ on each real tangent space. If such a map satisfies a certain integrability condition (namely $[X,Y] + I[IX,Y] + I[X,IY] - [IX,IY] = 0 \ \forall X,Y \in \mathfrak{X}(M)$, vanishing Nijenhuis tensor), then the manifold is indeed a complex mfd in disguise. We call such a family of maps a complex structure.
- Of course, a manifold can have all three structures at the same time.
  If they "play well with each other", i.e. if $g(IX,Y) = \omega(X,Y) \ \forall X,Y \in \mathfrak{X}(M)$
  $g_{ik} I^k{}_j = \omega_{ij}$
  then we call (M, g, ω, I) a Kähler manifold.
  Kähler manifolds find important applications in mathematical physics (e.g. the space of vacua in 4 dimensional $N=1$ supersymmetric field theory is a Kähler manifold)
- Locally on a Kähler manifold, the symplectic form has a holomorphic potential f
$$g = i\partial\bar{\partial} f$$

- But why stop at complex numbers? For quaternions, we have $i^2 = j^2 = k^2 = -1$ and $ij = k$
  Analogously, if we have a manifold M with
  - a Riemannian metric g $\qquad g_{ij}$

- three symplectic forms $\omega_I, \omega_J, \omega_K$ $\qquad (\omega^I)_{ij}, (\omega^J)_{ij}, (\omega^K)_{ij}$
- three complex structures I, J, K $\qquad I^i{}_j, J^i{}_j, K^i{}_j$

such that (M, g, I, ω_I), (M, g, J, ω_J), (M, g, K, ω_K) are Kähler mfds and $IJ = K$, then we call $(M, g, \omega_I, \omega_J, \omega_K, I, J, K)$ a hyperkähler manifold.

- Hyperkähler manifolds are important in mathematical physics (e.g. the space of vacua in 3 dimensional $N=4$ supersymmetric field theory is a hyperkähler mfd)
- Hyperkähler manifolds are rigid in the sense that (I, J, K) determine the Riemannian metric uniquely.

Section 5: Geometry on the Hitchin moduli space

- Now we are going to show that $\mathcal{M}_H$ admits a hyperkähler structure
- Recall that $\mathcal{H}$ is an affine space, and its tangent space is at any point (A, ϕ)

$$T_{(A,\phi)}\mathcal{H} \simeq \Omega^1(M; \mathrm{Ad}P \otimes \mathbb{C})$$

- Our general strategy is to define geometric structures on $\mathcal{H}$, show that they restrict well to $\tilde{\mathcal{M}}_H$ and then show that they are invariant under gauge transformations. Then they define a structure on $\mathcal{M}_H$
- Note that we can define $\mathrm{Tr}: \Omega^1(M; \mathrm{Ad}P \otimes \mathbb{C}) \to \Omega^1(M; \mathbb{C})$ by acting only on the Lie algebra part: $\mathrm{Tr}(A\, dz) := (\mathrm{Tr} A)\, dz$
- We start with the Riemannian metric. For $X = \dot{\phi}\, dz + \dot{A}\, d\bar{z} =: X_1\, dz + X_2\, d\bar{z}$, we define $\tilde{g}(X, X) := \int_M \mathrm{Tr}(X_1^* X_1 - X_2 X_2^*)\, d\bar{z} \wedge dz$.

  This induces the Riemannian metric g on $\mathcal{M}_H$.

- Since the tangent space of $\mathcal{H}$ is $\Omega^1(M; \mathrm{Ad}P \otimes \mathbb{C})$, the pointwise multiplication by i defines a complex structure $\tilde{I}$ on $\mathcal{H}$. This induces the complex structure I on $\mathcal{M}_H$.

- Using I and g we can define a form ω_I by $\omega_I(X, Y) := g(IX, Y)$. It can be shown that ω_I is a symplectic form, thus (M, g, ω_I, I) is a Kähler manifold.

- To construct the other two Kähler structures, we take a different approach. Define a complex symplectic form on H by

$$\Omega(X,Y) := \int_M \mathrm{Tr}(Y_2 X_1 - Y_1 X_2)\, d\bar{z} \wedge dz$$

and let $\omega_J := \mathrm{Re}\,\Omega$ and $\omega_K := \mathrm{Im}\,\Omega$.

- This way we obtain two symplectic forms on H. Those induce complex structures J and K via $g(J\cdot,\cdot) := \omega_J$ and $g(K\cdot,\cdot) := \omega_K$.

- These can be shown to satisfy

$$I(X,Y) = (iX, iY)$$

$$J(X,Y) = (iY^*, -iX^*)$$

$$K(X,Y) = (-Y^*, X^*)$$

Now $IJ(X,Y) = I(iY^*, -iX^*) = (-Y^*, X^*) = K(X,Y)$. Thus we have defined a hyperkähler structure on $\mathcal{M}_H$.

- One final piece of structure: if $(A,\phi) \in \tilde{\mathcal{M}}_H$ satisfy the SD equations

$$F^A + [\phi \wedge \phi^*] = 0$$

$$\bar{\partial}_A \phi = 0,$$

then $e^{i\theta}\phi$ also satisfies the SD equations for $e^{i\theta} \in S^1$. This way

$$e^{i\theta}.(A,\phi) := (A, e^{i\theta}\phi) \qquad (A,\phi) \in \tilde{\mathcal{M}}_H$$

is an S^1-action on $\tilde{\mathcal{M}}_H$. This action commutes with gauge transformations, thus $e^{i\theta}.[(A,\phi)] := [e^{i\theta}.(A,\phi)]$ defines an S^1-action on the Hitchin moduli space.

- This action is Hamiltonian with moment map $\mu(A,\phi) := -\frac{1}{2}\|\phi\|_{L^2}^2$. This map is a Morse function and it can be used to study the topology of the Hitchin moduli space.

[There are probably several typos and errors in this script. Please consult Hitchin's article.]