# Introduction

Given a hyperkaehler manifold $M^{4n}$ with metric $g$, complex structures $I$, $J$ and $K$, and symplectic forms $\omega_I$, $\omega_J$ and $\omega_K$ that are all compatible, in the sense that $IJ = K$ and

$$\begin{aligned} g(I-,-) &= \omega_I(-,-) \\ g(J-,-) &= \omega_J(-,-) \\ g(K-,-) &= \omega_K(-,-) \end{aligned}$$

we can define what is called the twistor space of $M$. The main observation that allows us to construct this space, is the following

$$(aI + bJ + cK)^2 = -(a^2 + b^2 + c^2)$$

for real numbers $a, b, c$.

In other words, if we have $a^2 + b^2 + c^2 = 1$, $aI + bJ + cK$ defines a new complex structure. Such $(a, b, c)$ form a sphere, which we can give a complex structure $I_0$ through the standard one-point compactification of the complex plane. The complex coordinate of this sphere will be $\zeta$. Then we are free to pick

$$(a, b, c) = \frac{1}{1 + |\zeta|^2}(1 - |\zeta|^2, 2\mathrm{Re}(\zeta), -2\mathrm{Im}(\zeta))$$

The twistor space $Z$ of $M$, will now be defined as $Z = M \times S^2$, which we give the almost complex structure $(I^{(\zeta)}, I_0)$ at a point $(m, \zeta) \in M \times S^2$. One can apply the Newlander-Nirenberg theorem to show that this almost complex structure is integrable and thus makes $Z$ into a complex manifold.

# Properties of twistor space

It turns out that the twistor data contain enough information to recover the hyperkaehler structure. In this way, describing the twistor data is an equivalent, more geometrical way, of describing a hyperkaehler metric. Therefore, the search for hyperkaehler metrics can equivalently be stated as the search for such data. Unfortunately, this construction only defines a metric locally. This means finding global properties of the metric can be quite difficult. For example, proving geodesic completeness of the metric can be tricky.

To complete our construction, we shall now look for some desirable properties of the twistor space, which shall form the basis for the reversal of the construction.

## Normal bundle of holomorphic sections

Since $Z = M \times S^2$, there exists an obvious map $p : M \times S^2 \to S^2$. Actually, this projection extends to a holomorphic map $p : Z \to \mathbb{C}P^1$, which makes $Z$ into a holomorphic fiber bundle over $\mathbb{C}P^1$. We shall not give the proof of that here, but it is closely related to the proof that $Z$ is a complex manifold. The bundle admits a family of holomorphic sections, namely $\sigma_m : \zeta \mapsto (m, \zeta)$, the standard embedding into $M \times S^2$. The sections of the map $p$ are called twistor lines, $P_m = \mathrm{im}(\sigma_m)$.

With respect to this embedding, we can define the normal bundle $N$ of $P_m \subset M \times S^2$. This bundle fits into the short exact sequence

$$0 \to T\mathbb{C}P^1 \hookrightarrow TZ|_{P_m} \to N \to 0$$

Topologically, it is clear that $TZ|_{P_m} = N \oplus T\mathbb{C}P^1 = T_m M \oplus T\mathbb{C}P^1$ with the normal bundle $N = T_m M$ over $P_m$ trivial. However, holomorphically it is not clear that the normal bundle trivializes. In fact, it is not holomorphically trivial, but forms the bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ over $P_m$ as we shall see.

### Gluing of the normal bundle

For the proof, we use the identification of the quaternion algebra $\mathbb{H}$ with the algebra $\mathfrak{su}(2)$. We represent the actions of $I$, $J$ and $K$ on $T_m M = N = \mathbb{C}^{2n}$ as follows

$$\begin{aligned} I &= \begin{pmatrix} i \cdot \mathrm{id}_n & 0 \\ 0 & -i \cdot \mathrm{id}_n \end{pmatrix} \\ J &= \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix} \\ K &= \begin{pmatrix} 0 & i \cdot \mathrm{id}_n \\ i \cdot \mathrm{id}_n & 0 \end{pmatrix} \end{aligned}$$

Then

$$I^{(\zeta)} = \frac{1}{1+|\zeta|^2} \begin{pmatrix} i(1-|\zeta|^2) \cdot \mathrm{id}_n & 2\bar{\zeta} \cdot \mathrm{id}_n \\ -2\zeta \cdot \mathrm{id}_n & -i(1-|\zeta|^2) \cdot \mathrm{id}_n \end{pmatrix}$$

with eigenvectors around $\zeta = 0$ of the form

$$\begin{pmatrix} v_1 \\ i\zeta v_1 \end{pmatrix}, \begin{pmatrix} i\bar{\zeta} v_2 \\ v_2 \end{pmatrix}$$

and thus around $\zeta = \infty$ of the form

$$\begin{pmatrix} -i\zeta^{-1} w_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} w_2 \\ -i\bar{\zeta}^{-1} w_2 \end{pmatrix}$$

The left eigenvectors have eigenvalue $i$, while the right vectors have eigenvalue $-i$. Therefore, $\zeta$ is the coordinate on $\mathbb{C}P^1$ that describes $Z \to \mathbb{C}P^1$ as a holomorphic bundle.

In any case, to transition from one set of (holomorphic) eigenvectors to the other, we have to multiply by $i\zeta \cdot \mathrm{id}_{2n}$. Therefore, $T_m M = N$ patches together by a transition function $i\zeta \cdot \mathrm{id}_{2n}$ on the two opens covering the sphere. Therefore, we have the bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ as the normal bundle.

## Holomorphic section of $\left(\bigwedge^2 T_F^*\right) \otimes \mathcal{O}(2)$ defining sympectic form on the fibres

We defined the normal bundle of a section to fit into a short exact sequence. We will do the same for the entire bundle $TM = T_F$. Here we use the notation $T_F$ for when we want to reverse the construction. Then we obtain the short exact sequence

$$0 \to T_F \to TZ \to T\mathbb{C}P^1 \to 0$$

The map $p_* : TZ \to T\mathbb{C}P^1$ is the push-forward of the projection map.

We now want to construct a holomorphic symplectic form on the fibres. To interpret this form, we can look at a Hermitian metric on a Kaehler manifold. Just like a Hermitian form splits into a real part, which forms the metric, and an imaginary part, which forms a symplectic form. Just as we can construct the metric from the symplectic form for a Kaehler manifold, the twistor construction allows us to determine the hyperkaehler metric through the holomorphic symplectic form.

Given the three symplectic forms, we form the following combination

$$\varpi^{(0)} = \omega_J + i\omega_K$$

It can be explicitly checked that this is the unique (up to multiplication by a constant) holomorphic form with respect to $I = I^{(0)}$. To define a holomorphic symplectic form with respect to $I^{(\zeta)}$, we use the fact that the eigenvectors with eigenvalue $i$ of $I^{(\zeta)}$ have the following form

$$\begin{pmatrix} v \\ i\zeta v \end{pmatrix} = (1 + \zeta K) \begin{pmatrix} v \\ 0 \end{pmatrix}$$

This transformation implies a transformation on $\varpi^{(0)}$. This will give us a form quadratic in $\zeta$, which can therefore be extended to a section over the sphere valued in its tangent bundle

$$\omega^{(\zeta)} = \left((\omega_J + i\omega_K) + 2\zeta\omega_I - \zeta^2(\omega_J - i\omega_K)\right) \frac{d}{d\zeta} \in \left(\bigwedge^2 T_F^*\right) \otimes \mathcal{O}(2)$$

One can check that the above form is holomorphic with respect to $I^{(\zeta)}$ and note that this is unique.

### Antiholomorphic involution on $Z$

The last property we will need is an antiholomorphic involution. It should be noted that $I^{(-\bar{\zeta}^{-1})} = -I^{(\zeta)}$, as can be easily calculated. On top of that, the antipodal map on the sphere induces the change of complex structure $I_0 \to -I_0$. Therefore, the map $\tau : (m, \zeta) \mapsto (m, -\bar{\zeta}^{-1})$ defines an antiholomorphic involution for us. It also induces the antipodal map (which we shall also denote $\tau$) on the sphere, since the following diagram clearly commutes

$$\begin{array}{ccc} Z & \to^{\tau} & Z \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 & \to^{\tau} & \mathbb{C}P^1 \end{array}$$

The map $\tau$ behaves in a particularly nice way with the other properties. As is easily checked $\tau^*\varpi^{(\tau(\zeta))} = \bar{\varpi}^{(\zeta)}$. On top of that, any holomorphic section is invariant under $\tau$, in the sense that $\tau(P_m) = P_m$. This is equivalent to the fact that $\tau$ does not act on the fibres, so that the structure of a fibre bundle is retained under the action of $\tau$. It also means that the normal bundle keeps the form $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ under the action of $\tau$.

## Reversing the twistor construction

It turns out we now have enough information to reconstruct the hyperkaehler metric. Let us therefore define a twistor space more abstractly as a space $Z$ of complex dimension $2n + 1$, such that

- There exists a map $p : Z \to \mathbb{C}P^1$ that makes $Z$ into a holomorphic fibre bundle over $\mathbb{C}P^1$
- The bundle admits a family of holomorphic sections, which all have normal bundle $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$ in $Z$
- There exists a holomorphic form in $\left(\bigwedge^2 T_F^*\right) \otimes \mathcal{O}(2)$, which defines a symplectic form on each fibre
- There exists an antiholomorphic involution $\tau$, compatible with the above, and which induces the anti-podal map on $\mathbb{C}P^1$

Somewhat surprisingly, it is not necessary to define $Z$ as a direct product $M \times S^2$, where $M$ is a complex manifold. This actually follows from the above definition.

## Smoothness of space of twistor lines (parameter space)

To study the space of twistor lines, we are first interested in understanding this space locally. If we have a section $P_m$, one can think of a holomorphic vector field pointed along the fibres as an infinitesimal deformation of $P_m$. This would be a holomorphic section of the normal bundle.

Kodaira's theorem tells us that an infinitesimal deformation can be integrated because the obstruction $H^1(\mathbb{C}P^1, \mathcal{O}(1))$ vanishes. This is closely related to the classical deformation theory, where deformations of complex structures on a Riemann surface $\Sigma$ are classified by the quadratic differentials. Note that the tangent bundle of our parameter space is defined by the global holomorphic sections of the normal bundle, so $H^0(P_m, N) = \mathbb{C}^{2n} \otimes H^0(\mathbb{C}P^1, \mathcal{O}(1))$ (where we used Kunneth's theorem). However, since the space $H^0(\mathbb{C}P^1, \mathcal{O}(1))$ has complex dimensions two, the parameter space has complex dimension $4n$.

To reduce this complex space to a real space, we need the antiholomorphic involution $\tau$. The real twistor lines, i.e. the ones invariant under $\tau$, now form a $4n$-real dimensional submanifold $M$. Then due to the existence of $\tau$, we can write the tangent bundle of the parameter space $H^0(P_m, N)$ as the complexification of $T_m M$, so that

$$T_m M \otimes \mathbb{C} = H^0(P_m, N) = H^0(P_m, T_F|_{P_m})$$

## Constructing the metric

The metric will be constructed locally by multiplying to symplectic structures together. For this, we first use Kunneth's theorem to split

$$H^0(P_m, T_F|_{P_m}) = H^0(P_m, T_F|_{P_m} \otimes \mathcal{O}(-1)) \otimes H^0(P_m, \mathcal{O}(1))$$

Now the compatibility of $\tau$ with the fibre bundle implies that this splitting remains well-defined and invariant under the action of $\tau$. Since $\varpi$ is a global holomorphic vector field valued in $\bigwedge^2 T_F|_{P_m}$ over $P_m$, so that it is an element of $H^0(P_m, \left(\bigwedge^2 T_F^*|_{P_m}\right) \otimes \mathcal{O}(2))$, it defines a symplectic form over $H^0(P_m, T_F|_{P_m} \otimes \mathcal{O}(-1)) = \mathbb{C}^{2n}$. The determinant defines a symplectic structure over the second factor, $H^0(P_m, \mathcal{O}(1)) = \mathbb{C}^2$ as follows

$$\langle a + b\zeta, c + d\zeta \rangle = ad - bc$$

We define the metric as $g = \varpi \otimes \langle -, - \rangle$. For a given $X \in H^0(P_m, T_F|_{P_m})$ and a splitting $X = X_1 + X_2\zeta$, we obtain

$$\begin{aligned}
g(X, X) &= (\varpi \otimes \langle -, - \rangle)(X_1 + X_2\zeta, X_1 + X_2\zeta) \\
&= (\varpi \otimes \langle -, - \rangle)(X_1, X_2\zeta) + (\varpi \otimes \langle -, - \rangle)(X_2\zeta, X_1) \\
&= \varpi(X_1, X_2)\langle 1, \zeta \rangle + \varpi(X_2, X_1)\langle \zeta, 1 \rangle \\
&= 2\varpi(X_1, X_2)
\end{aligned}$$

This construction makes $g$ into a symmetric complex bilinear form.

**The metric restricted to real tangent vectors**

The next step is to construct the real tangent vectors. We shall do this procedure by introducing quaternionic structures on $H^0(P_m, T_F|_{P_m} \otimes \mathcal{O}(-1))$ and $H^0(P_m, \mathcal{O}(1))$ and tensoring these together to form a real structure.

On the space $H^0(P_m, \mathcal{O}(1)) = H^0(\mathbb{C}P^1, \mathcal{O}(1))$ there exists a unique (up to sign) quaternionic structure that is induced by the antiholomorphic involution $\tau$. Given a section $s : \mathbb{C}P^1 \to \mathcal{O}(1)$, then this structure is defined by $j(s(\zeta)) = -\bar{s}(-\bar{\zeta}^{-1})$. We indeed have a quaterionic structure $j^2 = -1$, precisely because we work with the antipodal map.

Now we have to define a quaternionic structure $j$ (we will use the same letter) on $H^0(P_m, T_F|_{P_m} \otimes \mathcal{O}(-1))$ in such a way that it is compatible with the real structure defined from the involution $\tau$. The assumption that $\tau$ induces the anti-podal map, allows us to uniquely reconstruct $j$ by the tensor product

$$\tau = j \otimes j$$

The real structure defined on $H^0(P_m, T_F|_{P_m})$ then allows us to define real vectors in $H^0(P_m, T_F|_{P_m})$. A real vector has the form $X = X_1 - j(X_1)\zeta$, so that restricting our metric to these vectors, gives it the following form

$$g(X, X) = -2\varpi(X_1, j(X_1))$$

This turns out to define a metric. For example, positive-definiteness follows from the compatibility equation $\tau^*\varpi^{(\zeta)} = \varpi^{(-\bar{\zeta}^{-1})}$.

There is actually a short-cut to get to the metric that does not require to do these calculations. We will show this in the example at the end.

## Identifying fibres with the hyperkaehler manifold

From the non-degenerate property of the metric, we see that if $X = X_1 - j(X_1)\zeta$ vanishes at some $\zeta = \zeta_0$, so that $X_1 = j(X_1)\zeta_0$, then

$$g^{(\zeta_0)}(X, X) = -2\varpi^{(\zeta_0)}(j(X_1)\zeta_0, j(X_1)) = -2\zeta_0\varpi^{(\zeta_0)}(j(X_1), j(X_1)) = 0$$

Since $g$ is positive definite, $X = 0$ identically. Thus, the infinitesimal deformations generated by real tangent vectors on the space of real twistor lines, do not vanish anywhere along the twistor line. Therefore, any two different real twistor lines intersect the fibres in different points of $M$ everywhere along the twistor line. In other words, any point $z \in Z$ over $\zeta \in \mathbb{C}P^1$ has a unique twistor line going through $z$. Thus the set of twistor lines, $M$, identifies with the fibres of $Z \to \mathbb{C}P^1$. So for simplicity let us identify the fibre $Z_{\zeta=0} = M$.

A tangent vector in $T_m M$ then identifies with an $X_1 \in H^0(P_m, T_F|_{P_m} \otimes \mathcal{O}(-1))$. Actually, since $T_F \otimes \mathcal{O}(-1) = \mathbb{C}^{2n}$ is trivial over $P_m$ and the only globally defined functions on the sphere are constant, picking $X_1$ defines the tangent vector for any $\zeta \in P_m$, not just $\zeta = 0$.

By definition, the fibre $Z_0$ is a complex manifold, so this identification gives $M$ a complex structure $I$. Picking any other fibre, defines a different complex structure on $M$. By some simple calculations we can show that the metric defined above is in fact Kaehler with respect to the complex structure $I^{(\zeta)}$ and the quaternionic structure $j$ and satisfies all necessary properties to be turned into a hyperkaehler metric.

## Baby example: standard metric on $\mathbb{R}^4$

We consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates $(z, \bar{w})$ and the standard flat metric

$$g = \frac{1}{2}(dz d\bar{z} + dw d\bar{w})$$

We use the complex structures $I$, $J$ and $K$ as previously defined:

$$\begin{aligned} I &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\ J &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ K &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \end{aligned}$$

We can calculate the symplectic forms and combine them into the holomorphic symplectic form as follows

$$\varpi^{(\zeta)} = -dz \wedge dw + i\zeta(dz \wedge d\bar{z} + dw \wedge d\bar{w}) + \zeta^2 d\bar{z} \wedge d\bar{w}$$

We will pick the real structure to be of the form

$$\tau(\eta_1, \eta_2, \zeta) = \left( \frac{\bar{\eta}_2}{\bar{\zeta}}, -\frac{\bar{\eta}_1}{\bar{\zeta}}, -\frac{1}{\bar{\zeta}} \right)$$

It can be checked that this structure is compatible with $\varpi$. Our twistor space will then have the form $Z = \mathbb{C}^2 \otimes \mathcal{O}(1) = \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathbb{C}P^1$. We consider twistor lines $(a_1 + a_2\zeta, b_1 + b_2\zeta)$. The reality condition forces the real twistor lines to be of the form $(z - \bar{w}\zeta, w + \bar{z}\zeta)$. This defines the twistor space of the standard metric.

To reverse the construction, we simply need to know what the quaternionic structure does on the space of real twistor lines. We restrict to the fibre over $\zeta = 0$, so that the quaternionic structure simplifies to

$$J(z, w) = (\bar{w}, -\bar{z})$$

The holomorphic symplectic form takes the form

$$\varpi^{(0)} = -dz \wedge dw$$

over the fibre. Therefore, the metric is given by

$$\begin{aligned} g((z,w),(z,w)) &= \varpi^{(0)}((z,w), J(z,w)) \\ &= \varpi^{(0)}((z,w),(-\bar{w},\bar{z})) \\ &= z\bar{z} + w\bar{w} \end{aligned}$$

and we recover the metric we started with.

## Another quick trick

Note that $\varpi^{(\zeta)} \wedge ... \wedge \varpi^{(\zeta)} = \left(\varpi^{(\zeta)}\right)^{r+1} = 0$, since $\varpi$ is a holomorphic symplectic form. If we expand this equation in $\zeta$, we find

$$(\omega_J + i\omega_K)^{r+1} \pm 2\zeta(\omega_J + i\omega_K)^r \wedge \omega_I + ... \pm 2\zeta^{2r+1}(\omega_J - i\omega_K)^r \wedge \omega_I + (\omega_J - i\omega_K)^{r+1} = 0$$

This implies both

$$\begin{aligned} (\omega_J + i\omega_K)^r \wedge \omega_I &= 0 \\ (\omega_J - i\omega_K)^r \wedge \omega_I &= 0 \end{aligned}$$

With respect to the complex structure $I$ on the fibre $Z_{\zeta=0}$ , $(\omega_J + i\omega_K)^r$ is a $(r,0)$-form, while $(\omega_J - i\omega_K)^r$ is a $(0,r)$-form. Therefore $\omega_I$ must be a $(1,1)$-form, otherwise these equations would not hold in general. From this we conclude that

$$\omega_I(I^2-,-) = -\omega_I(I-,I-)$$

We can therefore define a compatible Kaehler metric $g(I-,-) = \omega_I(-,-)$. It turns out this is actually a hyperkaehler metric and it will be precisely the metric defined through the twistor construction. For example, for the twistor space of the flat metric, we find

$$\begin{aligned} g(-,-) &= -\omega_I(I-,-) \\ &= -\frac{i}{2}(idzd\bar{z} + idwd\bar{w}) \\ &= \frac{1}{2}(dzd\bar{z} + dwd\bar{w}) \end{aligned}$$

## Application to Hitchin's moduli space

From Arpan and Florian's talk, we know that there exists an equivalence between flat bundles with harmonic metric and Higgs bundles with harmonic metric. If the self-duality equations hold

$$\begin{aligned} F_E + [\phi, \bar{\phi}] &= 0 \\ \bar{\partial}_E \phi &= 0 \end{aligned}$$

and we know that under this harmonic metric $K_h$, we have the equality

$$K_h(\phi -, -) = K_h(-, \bar{\phi} -)$$

we can define a connection $\nabla = d_E + \phi + \bar{\phi}$ that turns out to be flat, so $\nabla^2 = 0$. In the context of Hitchin's original paper, this implies that from a SO(3)-valued Higgs bundle, we recover a $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SO}(3)_{\mathbb{C}}$-valued flat connection. Since the self-duality equations are invariant under the circle action $\phi \to e^{i\theta}\phi$, it will still be true that $\nabla_{e^{i\theta}} = d_E + e^{i\theta}\phi + e^{-i\theta}\bar{\phi}$ defines a flat connection. In fact, we can extend the circle bundle action of U(1) to its complexification $\mathrm{U}(1)_{\mathbb{C}} = \mathrm{GL}(1,\mathbb{C})$, and it will still be true. The parameter we use to describe $\mathrm{GL}(1,\mathbb{C})$ is $\zeta \in \mathbb{C}P^1 - \{0, \infty\}$. We conclude that $\nabla_\zeta = \frac{1}{\zeta}\phi + d_E + \zeta\bar{\phi}$ defines a flat $\mathrm{PSL}(2,\mathbb{C})$-connection. We define $A_\zeta = \frac{1}{\zeta}\phi + A + \zeta\bar{\phi} = \frac{1}{\zeta}\phi_z + A_{\bar{z}} + A_z + \zeta\phi^\dagger_{\bar{z}}$. It should be noted that we shall also use the notation $\phi^\dagger_{\bar{z}} = \bar{\phi}$.

From Aron's talk, we know that the space of solutions $(A_{\bar{z}}, \phi_z)$ to the self-duality equations, defines a hyperkaehler manifold. The tangent space is denoted by $(\dot{A}_{\bar{z}}, \dot{\phi}_z)$. We can naturally define the three complex structures

$$\begin{aligned} I(\dot{A}_{\bar{z}}, \dot{\phi}_z) &= (i\dot{A}_{\bar{z}}, i\dot{\phi}_z) \\ J(\dot{A}_{\bar{z}}, \dot{\phi}_z) &= (i\dot{\phi}^\dagger_{\bar{z}}, -i\dot{A}_z) \\ K(\dot{A}_{\bar{z}}, \dot{\phi}_z) &= (-\dot{\phi}^\dagger_{\bar{z}}, \dot{A}_z) \end{aligned}$$

There exists a natural integration pairing, which defines a metric

$$\tilde{g} = \int_M \mathrm{Tr}(\delta A_z \delta A_{\bar{z}} + \delta\phi_z \delta\phi^\dagger_{\bar{z}})$$

Here $\delta A$ is dual to $\dot{A}$ and $\delta\phi$ is dual to $\dot{\phi}$. With respect to this metric, the space of solutions becomes a hyperkaehler manifold. As seen above, we picked a preferred complex structure $I$. We will therefore consider the combination

$$\begin{aligned} \omega_J + i\omega_K &= i\int_M \mathrm{Tr}(\delta\phi_z \wedge \delta A_{\bar{z}} + \delta A_z \wedge \delta\phi^\dagger_{\bar{z}}) + i\int_M \mathrm{Tr}(\delta\phi_z \wedge \delta A_{\bar{z}} - \delta A_z \wedge \delta\phi^\dagger_{\bar{z}}) \\ &= 2i\int_M \mathrm{Tr}(\delta\phi_z \wedge \delta A_{\bar{z}}) \end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}
\omega_J - i\omega_K &= -2i\int_M \mathrm{Tr}(\delta\phi_{\bar{z}}^\dagger \wedge \delta A_z) \\
\omega_I &= i\int_M \mathrm{Tr}(\delta A_{\bar{z}} \wedge \delta A_z + \delta\phi_{\bar{z}}^\dagger \wedge \delta\phi_z)
\end{aligned}$$

We can put these symplectic forms in the combination

$$\begin{aligned}
\varpi^{(\zeta)} &= \frac{1}{2\zeta}(\omega_J + i\omega_K) + \omega_I - \frac{1}{2}\zeta(\omega_J - i\omega_K) \\
&= i\int_M \mathrm{Tr}\left(\frac{1}{\zeta}\delta\phi_z \wedge \delta A_{\bar{z}} + (\delta A_z \wedge \delta A_{\bar{z}} + \delta\phi_{\bar{z}}^\dagger \wedge \delta\phi_z) + \zeta\delta\phi_{\bar{z}}^\dagger \wedge \delta A_z\right)
\end{aligned}$$

This combination is slightly different from the holomorphic symplectic form in the twistor section by a redefinition of $\zeta$ and a factor. We use this form, because it leads to an interesting identification between the parameter $\zeta$ from the twistor construction and the parameter used for the deformation $\nabla_\zeta$, in the following way:

$$\begin{aligned}
\int_M \mathrm{Tr}(\delta A_\zeta \wedge \delta A_\zeta) &= \int_M \mathrm{Tr}\left(\left(\frac{1}{\zeta}\delta\phi_z + \delta A_z + \delta A_{\bar{z}} + \zeta\delta\phi_{\bar{z}}^\dagger\right) \wedge \left(\frac{1}{\zeta}\delta\phi_z + \delta A_z + \delta A_{\bar{z}} + \zeta\delta\phi_{\bar{z}}^\dagger\right)\right) \\
&= 2\int_M \mathrm{Tr}\left(\frac{1}{\zeta}\delta\phi_z \wedge \delta A_{\bar{z}} + (\delta A_z \wedge \delta A_{\bar{z}} + \delta\phi_{\bar{z}}^\dagger \wedge \delta\phi_z) + \zeta\delta\phi_{\bar{z}}^\dagger \wedge \delta A_z\right)
\end{aligned}$$

So we see that

$$\varpi^{(\zeta)} = \frac{i}{2}\int_M \mathrm{Tr}(\delta A_\zeta \wedge \delta A_\zeta)$$

Another nice property is shown by the equality

$$A_\zeta(-\phi)^* = -\frac{1}{\bar{\zeta}}\phi^\dagger + A - \bar{\zeta}\phi = A_{-\frac{1}{\bar{\zeta}}}(\phi)$$

Under the quotient by the gauge action, the above construction, follows through and on this space the above equality allows us to define a real structure.

# References


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