# HITCHIN'S INTEGRABLE SYSTEMS

## 1. Motivation

Recall one of the central object we care about: the moduli space $\mathcal{M} := \mathcal{M}_{G,C}$ of stable Higgs $G$-bundle over a compact Riemann surface $C$, equipped with the complex structure we denoted $I$ in previous talk, i.e. the one induced by $C$. In previous lectures we've seen various identification of that spaces with other moduli spaces leading to the so-called non abelian Hodge correspondence. In this talk, we'll study $\mathcal{M}$ in the framework of symplectic geometry. Tohether with the NAHC this gives a nice relation between field theories and integrable systems, and it turns out that many well known and physically interesting integrable systems are particular cases of this construction.

For the rest of the talk we'll assume that $G = GL_n(\mathbb{C})$ for the sake of concretness.

## 2. Symplectic structure on $\mathcal{M}$

Recall that $\mathcal{M}$ is the moduli space of rank $n$ stable Higgs bundle, i.e. of pairs $(E, \phi)$ where $E$ is a degree 0 rank $n$ holomorphic bundle and $\phi$ is a global section of $K \otimes \mathrm{End}(E)$, $K$ being the canonical line bundle on $C$, satisfying the stability condition. Recall that if $E$ is a stable holomorphic bundle, then $(E, \phi)$ is a stable Higgs bundle for any $\phi$ (the converse is false, c.f. Florian's talk).

Let $\mathcal{B}$ be the moduli space of gauge equivalence classes of stable holomorphic degree 0 rank $n$ bundles on $C$. Since topologically a bundle over a Riemann surface is completely determined by its degree, we might as well fix the underlying bundle of $E$ once and for all and think of $\mathcal{B}$ as the moduli space of holomorphic structure on it, i.e as the quotient

$$\tilde{\mathcal{B}}/\mathcal{G}$$

where $\tilde{\mathcal{B}}$ is the space of holomorphic structures on $E$ and $\mathcal{G}$ its automorphism group (the gauge group).

The tangent space at $E \in \tilde{\mathcal{B}}$ is naturally identified with $H^1(C, \mathrm{End}(E))$ and so, by Serre duality, the cotangent space at $E$ is $H^0(C, K \otimes \mathrm{End}(E))$. Modding out by $\mathcal{G}$ this gives a natural embedding

$$T^*\mathcal{B} \hookrightarrow \mathcal{M}$$

realizing $\mathcal{M}$ as a partial compactification of $T^*\mathcal{B}$ (by which we basically mean that the set of stable Higgs bundle whose underlying bundle is not stable is of sufficiently high codimension). The canonical symplectic structure on $T^*\mathcal{B}$ extends to a symplectic form on $\mathcal{M}$ (which was denoted by $\Omega_I$ in previous talks).

## 3. Algebraically integrable systems

A central definition is the following:

**Definition 3.1.** Let $M$ be a (complex, algebraic) symplectic variety. An algebraically integrable system (AIS) is a map $H$ from $M$ to some affine space $A$ whose fibers are generically:

- Lagrangian
- isomorphic to an open subset of a complex torus.

The first condition essentially reproduces the classical definition of an integrable system as a maximal Poisson commutative algebra of $\mathcal{O}(M)$. The second condition should be thought of as a compatibility condition between the "action-angle" coordinates and the complex structure. One should also mention that in the real case, under mild assumption the first condition actually implies the second one (this is the so-called Arnold-Liouville Theorem) while in tthe complex case this has to be imposed. Observe that any choice of a basis of linear forms on $A$ induces a Hamiltonian system in the more usual sense: composing those with $H$ induces $k$ independant Poisson commuting functions $H_i$ on $M$ where $k$ is the dimension of $A$, i.e. half the dimension of $M$ by the Lagrangian assumption.

This already implies that locally $M$ can be identified with the product of $A$ by a complex torus, and since by construction the $H_i$'s are constant on the fiber over each point they provide globally defined coordinates for the "$A$ part", and can then be completed by choosing locally standard angle coordinates $\theta_i$ on the torus fibers. This doesn't, however, take the symplectic structure into account. Hence one can go onse step further: it is possible to define action coordinates $a_i$ depending only on the $H_i$ which satisfies the canonical commutation relations

$$\{a_i, a_j\} = 0 \qquad \{\theta_i, \theta_j\} = 0$$
$$\{a_i, \theta_j\} = \delta_{i,j}$$

This is the obvious analog of the real case, although it should be noted that the action coordinates are less canonical in the complex case, since they depend on the additional choice of a Lagrangian subspace in the first homology group of the $k$-dimensional torus.

Roughly speaking, we claim that there is a natural AIS on $\mathcal{M}$ obtained by mapping a Higgs bundle $(E, \phi)$ to the coefficients of the characteristic polynomial of $\phi$. Since $\phi$ is not quite an endomorphism, those will not be numbers but rather global sections of the bundle $K^{\otimes i}$ for some $i$. Then the fibers of this can be described in terms of the eigenvalue problem for $\phi$, again suitably addapted to the situation. Recall that if $V$ is an $n$-dimensional vector space, then taking either the coefficients of the characteristic polynomial, or the maps $f \mapsto \mathrm{tr}\,(f^n)$ provides a system of homogeneous generators for the algebra of invariants $\mathcal{O}(\mathrm{End}(V))^{GL(V)}$. We want something similar for $\phi$.

More preciesely, $\phi$ can be thought of as a twisted endomorphism of $E$, ie as a bundle map

$$E \longrightarrow K \otimes E.$$

Hence it makes sense to define $\phi^k$ by composing this map $k$ times leading to a map

$$E \longrightarrow K^{\otimes k} \otimes E.$$

Finally, taking the trace w.r.t $E$ leads to an element $\mathrm{tr}\,(\phi^k)$ in $H^0(C, K^{\otimes k})$. Since the target of this map is indepenant of $(E, \phi)$ and since those are gauge invariant by construction, we can then assemble those into a single map

$$H : \mathcal{M} \longrightarrow \mathcal{A} := \bigoplus_{k=1}^{n} H^0(C, K^{\otimes k}).$$

**Theorem 3.2** (Hitchin). *The map $H$ induces an AIS on $\mathcal{M}$.*

As explained above, to actually get a set of function on $\mathcal{M}$ defining an integrable system we only need to choose a basis of the dual space $\mathcal{A}^*$. Using again Serre duality, this is identified with a space of $(0,1)$ forms

$$\bigoplus_{k=1}^{n} H^1(C, K^{\otimes(1-k)}).$$

Hence, any basis $\{\alpha_{k,i}\}$ of $H^1(C, K^{\otimes(1-k)})$ induces a set of functions on $\mathcal{M}$ defined by

$$H_{k,i} := (E, \phi) \longmapsto \int_C \alpha_{k,i} \mathrm{tr}\,(\phi^k).$$

We shall see that this map is not quite surjective, however there is a further compactification of $\mathcal{M}$ relying on the notion of *semi-stable* Higgs bundle. This space is again symplectic, the map $H$ extends and becomes a fibration, known as the Hitchin fibration.

It follows from a standard result on simple Lie group (and is easily checked in the case at hand) that

$$\sum_{k=1}^{n} (2k-1) = n^2 = \dim GL_n$$

where in this formula $k$ appears as the degree of the $k$th generator of the algebra of invariant polynomial on $\mathfrak{gl}_n$. It then follows from Riemann-Roch that

$$\dim H^0(C, K^{\otimes k}) = (g-1)(2k-1)$$

and therefore

$$\dim \mathcal{A} = (g-1) \dim GL_n = \frac{1}{2} \dim \mathcal{M}$$

where $g$ is the genus of $C$, as required.

Since the fibers have the correct dimension, to check the first defining condition of an AIS it remains to show that the $H_{k,i}$ Poisson commute. For the sake of simplicity we'll show this for $T^*\mathcal{B}$ rather than for $\mathcal{M}$. This relies on the standard fact that the cotangent bundle of a quotient has a standard description via Hamiltonian reduction. The gauge group $\mathcal{G}$ acts naturally on $T^*\tilde{\mathcal{B}}$ and there is a canonical moment map

$$\mu : T^*\tilde{\mathcal{B}} \longrightarrow \mathfrak{gl}_n^*$$

for this action, and one can show that

$$T^*\mathcal{B} = \mu^{-1}(\{0\})/\mathcal{G}.$$

Now $\tilde{\mathcal{B}}$ is an (infinite dimensional) affine space, so its cotangent bundle is trivial

$$T^*\tilde{\mathcal{B}} = \tilde{\mathcal{B}} \times H^0(C, K \otimes \mathrm{End}(E)).$$

The map $H$ clearly extends to this space and by definition factor through the projection on the second factor. This implies that the lift of the $H_{k,i}$ Poisson commute, and this is a standard exercise to show that this implies that they also commutes after taking Hamiltonian reduction.

## 4. Description of the fibers of $H$

Roughly speaking, we will associate to $\phi$ an $n$-fold branched covering $S$ of $C$ whose fiber over $x$ is the set of eigenvalues of $\phi(x)$, and then a line bundle on $S$, that is a point of the so-called Jacobian $Jac(S)$ of $S$. Utimately this will provide an identification of a generic fiber of $H$ with an open subset of $Jac(S)$ which is a complex torus.

Now that we have an analog of the characterisic polynomial, we can talk about eigenvalues: let $W$ be the total space of the line bundle $K$ and for $(E, \phi) \in \mathcal{M}$ let $S := S_{E,\phi} \subset W$ be the curve defined by

$$S = \{(x, y) \in T^*C, \det(\phi(x) - y) = 0\}.$$

This is the so-called spectral curve and defines a $n$-fold branched covering over $C$ whose fiber over $x$ is the set of eigenvalues

$$\{y \in K_x, \exists e \in E_x, \phi(x) \cdot e = ye\}.$$

If $x \in C$ is away from the branching points (i.e. if $\phi(x)$ has $n$ distinct eigenvalues), then to each point of the fiber over $x$ one can attach the corresponding one dimensional eigenspace. This assemble into a line bundle $L$ which can then be extended to the whole of $S$ (i.e. including the branching points).

By definition, $S = S_a$ depends only on the characteristic polynomial of $\phi$, hence on a choice of a (again generic) point $a$ in $\mathcal{A}$, and any pair $(E, \phi)$ in the fiber over $a$ induces a line bundle on $S_a$, hence a point in $Jac(S_a)$.

We claim that this map is injective: fix some $n$-fold covering $\pi : S \to C$ and let $L$ be a line bundle over $S$. Roughly speaking, we construct from that a pair $(E, \phi)$ by declaring that over generic points $x$ of $C$, the fiber of $E$ at $x$ is a direct sum of the fibers of $L$ at the fiber of $x$, i.e.

$$E_x := \bigoplus_{y \in \pi^{-1}(x)} L_y.$$

Then we recover $\phi$ by declaring that it's diagonal with respect to this direct sum decomposition, with the prescribed eigenvalues of the diagonal.

More precisely, a bundle over $C$ is completely determined by the data of its sections over all open subsets of $C$. Hence, being given a line bundle $L$ on $S$ we define a bundle $E$ on $C$ by declaring that its sections over any open $U \subset C$ are the sections of $L$ over $\pi^{-1}(U)$.

Therefore, we have identified a generic fiber of the map $H$ with an open subset of the Jacobian of some curve. If we allow semi-stable bundle instead of just stable one, then this construction turns out to be generically surjective as well, i.e. the fiber over a generic point $a$ can be identified with the whole of $Jac(S_a)$. This completes the proof of algebraic integrability.