The algebra of multiple divisor functions and applications to multiple zeta values

Henrik Bachmann - Universität Hamburg

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multiple zeta values

$$\frac{5197}{691}\zeta(12) - 168\zeta(5,7) - 150\zeta(7,5) - 28\zeta(9,3) = 0$$

$$\zeta(s_1,\dots,s_l) := \sum_{n_1>n_2>\dots>n_l>0} \frac{1}{n_1^{s_1}\dots n_l^{s_l}}$$

stuffle shuffle

$$\zeta(2,3)+\zeta(3,2)+\zeta(5) = \zeta(2)\cdot\zeta(3) = \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$

modular forms

$$G_k = \frac{\zeta(k)}{(-2\pi i)^k} + \underbrace{\frac{1}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n}_{[k]}$$

$$\Delta = \sum_{n>0}\tau(n)q^n$$

Periodpolynomials

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multiple Eisenstein series

$$G_{4,4} = \zeta(4,4) + 20\zeta(6)(2\pi i)^2[2] + 3\zeta(4)(2\pi i)^4[4] + (2\pi i)^8[4,4]$$

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multiple divisor functions

$$[s_1, \dots, s_l] \in \mathbb{Q}[[q]]$$

quasi-shuffle product

$$[2] \cdot [3] = [2,3] + [3,2] + [5] - \frac{1}{12}[3]$$

derivation

$$\mathrm{d} = q\frac{d}{dq}$$

$$[2] \cdot [3] = [2,3] + 3[3,2] + 6[4,1] + \mathrm{d}[3] - 3[4]$$

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kernel of Z_k

$$[3], \mathrm{d}[3], [4] \in \ker Z_5$$

$$\Delta \in \ker Z_{12}$$

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Z_k

Periodpolynomials

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Z_k

Periodpolynomials

multiple q-zeta values

$\lim_{q\to 1}$

multiple zeta values

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Multiple divisor functions

As a generalization of the classical divisor sums we define for $r_1, \dots, r_l \geq 0$ the **multiple divisor sum** by

$$\sigma_{r_1,\dots,r_l}(n) := \sum_{\substack{u_1 v_1 + \dots + u_l v_l = n \\ u_1 > \dots > u_l > 0}} v_1^{r_1} \dots v_l^{r_l} \,.$$

With this we define for $s_1, \dots, s_l > 0$ the **multiple divisor function** of weight $s_1 + \dots + s_l$ and length l by

$$[s_1, \dots, s_l] := \frac{1}{(s_1 - 1)! \dots (s_l - 1)!} \sum_{n>0} \sigma_{s_1-1,\dots,s_l-1}(n) q^n \; \in \mathbb{Q}[[q]] \,.$$

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Example I

For $l = 1$ these are the classical divisor functions

$$[2] = \sum_{n>0} \sigma_1(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \dots$$

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Example II

$$[4,2] = \frac{1}{6} \sum_{n>0} \sigma_{3,1}(n) q^n = \frac{1}{6} \left(q^3 + 3q^4 + \underbrace{15q^5} + 27q^6 + 78q^7 + \dots \right)$$

$\sigma_{3,1}(5) = 1^3 \cdot 1^1 + 2^3 \cdot 1^1 + 1^3 \cdot 1^1 + 1^3 \cdot 2^1 + 1^3 \cdot 3^1 = 15$, because

$$5 = 4 \cdot 1 + 1 \cdot 1 = 2 \cdot 2 + 1 \cdot 1 = 3 \cdot 1 + 2 \cdot 1 = 3 \cdot 1 + 1 \cdot 2 = 2 \cdot 1 + 1 \cdot 3$$

Multiple divisor functions

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Example III

$$[4,4,4] = \frac{1}{216}\left(q^6 + 9q^7 + 45q^8 + 190q^9 + 642q^{10} + 1899q^{11} + \dots\right) \,,$$

$$[3,1,3,1] = \frac{1}{4}\left(q^{10} + 2q^{11} + 8q^{12} + 16q^{13} + 43q^{14} + 70q^{15} + \dots\right)$$

Multiple divisor functions - Filtration

Weight and length filtration

We define the vector space $\mathcal{MD}$ to be the $\mathbb{Q}$ vector space generated by $[\emptyset] = 1 \in \mathbb{Q}[[q]]$ and all multiple divisor sums $[s_1, \dots, s_l]$.

On $\mathcal{MD}$ we have the increasing filtration $\operatorname{Fil}^{\mathrm{W}}_{\bullet}$ given by the weight and the increasing filtration $\operatorname{Fil}^{\mathrm{L}}_{\bullet}$ given by the length, i.e., we have for $A \subseteq \mathcal{MD}$

$$\operatorname{Fil}^{\mathrm{W}}_k(A) := \big\langle [s_1, \dots, s_l] \in A \,\big|\, 0 \leq l \leq k \,,\, s_1 + \dots + s_l \leq k \big\rangle_{\mathbb{Q}}$$
$$\operatorname{Fil}^{\mathrm{L}}_l(A) := \big\langle [s_1, \dots, s_r] \in A \,\big|\, r \leq l \big\rangle_{\mathbb{Q}} .$$

If we consider the length and weight filtration at the same time we use the short notation $\operatorname{Fil}^{\mathrm{W,L}}_{k,l} := \operatorname{Fil}^{\mathrm{W}}_k \operatorname{Fil}^{\mathrm{L}}_l$.

As usual $\operatorname{gr}^{\mathrm{W}}_k(A) = \operatorname{Fil}^{\mathrm{W}}_k(A) / \operatorname{Fil}^{\mathrm{W}}_{k-1}(A)$ will denote the graded part (similar $\operatorname{gr}^{\mathrm{L}}_l$ and $\operatorname{gr}^{\mathrm{W,L}}_{k,l}$).

Theorem

The $\mathbb{Q}$-vector space $\mathcal{MD}$ has the structure of a bifiltered $\mathbb{Q}$-Algebra $(\mathcal{MD}, \cdot, \mathrm{Fil}^{\mathrm{W}}_{\bullet}, \mathrm{Fil}^{\mathrm{L}}_{\bullet})$, where the multiplication is the natural multiplication of formal power series and the filtrations $\mathrm{Fil}^{\mathrm{W}}_{\bullet}$ and $\mathrm{Fil}^{\mathrm{L}}_{\bullet}$ are induced by the $weight$ and $length$, in particular

$$\mathrm{Fil}^{\mathrm{W,L}}_{k_1,l_1}(\mathcal{MD}) \cdot \mathrm{Fil}^{\mathrm{W,L}}_{k_2,l_2}(\mathcal{MD}) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k_1+k_2,l_1+l_2}(\mathcal{MD}).$$

It is a (homomorphic image of a) quasi-shuffle algebra in the sense of Hoffman.

The first products of multiple divisor functions are given by

$$\begin{aligned}
[1] \cdot [1] &= 2[1,1] + [2] - [1]\,, \\
[1] \cdot [2] &= [1,2] + [2,1] + [3] - \frac{1}{2}[2]\,, \\
[1] \cdot [2,1] &= [1,2,1] + 2[2,1,1] + [2,2] + [3,1] - \frac{3}{2}[2,1]\,.
\end{aligned}$$

Multiple divisor functions - Algebra structure

The structure of being a quasi-shuffle algebra is as follows:

Let $\mathbb{Q}\langle A\rangle$ be the noncommutative polynomial algebra over $\mathbb{Q}$ generated by words with letters in $A = \{z_1, z_2, \dots\}$. Let $\diamond$ be a commutative and associative product on $\mathbb{Q}A$. Define on $\mathbb{Q}\langle A\rangle$ recursively a product by $1 * w = w * 1 = w$ and

$$z_a w * z_b v := z_a(w * z_b v) + z_b(z_a w * v) + (z_a \diamond z_b)(w * v)\,.$$

The commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle A\rangle, *)$ is called a quasi-shuffle algebra.

- The harmonic algebra (Hoffman) is an example of a quasi-shuffle algebra with $z_a \diamond z_b = z_{a+b}$ which is closely related to MZV.

Multiple divisor functions - Algebra structure

In our case we consider the quasi-shuffle algebra with

$$z_a \diamond z_b = z_{a+b} + \sum_{j=1}^{a} \lambda_{a,b}^j z_j + \sum_{j=1}^{b} \lambda_{b,a}^j z_j \,,$$

where with Bernoulli numbers B_n

$$\lambda_{a,b}^j = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!} \,.$$

Proposition

The map $[\,.\,] : (\mathbb{Q}\langle A \rangle, *) \longrightarrow (\mathcal{MD}, \cdot)$ defined on the generators by

$$z_{s_1} \dots z_{s_l} \mapsto [s_1, \dots, s_l]$$

is a homomorphism of algebras, i.e. it fulfils

$$[w * v] = [w] \cdot [v].$$

Proposition

The ring of modular forms $M_{\mathbb{Q}}(SL_2(\mathbb{Z}))$ and the ring of quasi-modular forms $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z}))$ are subalgebras of $\mathcal{MD}$.

For example we have

$$G_2 = -\frac{1}{24} + [2] \,, \quad G_4 = \frac{1}{1440} + [4] \,, \quad G_6 = -\frac{1}{60480} + [6] \,.$$

The proposition follows from the fact that $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]$ and $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_2, G_4, G_6]$.

Remark

Due to an old result of Zagier we have $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k,2}(\mathcal{MD})$.

Theorem

The operator $\mathrm{d} = q\frac{d}{dq}$ is a derivation on $\mathcal{MD}$ and it maps $\mathrm{Fil}^{\mathrm{W,L}}_{k,l}(\mathcal{MD})$ to $\mathrm{Fil}^{\mathrm{W,L}}_{k+2,l+1}(\mathcal{MD})$.

Examples:

$$\begin{aligned}
\mathrm{d}[1] &= [3] + \frac{1}{2}[2] - [2,1]\,,\\
\mathrm{d}[2] &= [4] + 2[3] - \frac{1}{6}[2] - 4[3,1]\,,\\
\mathrm{d}[2] &= 2[4] + [3] + \frac{1}{6}[2] - 2[2,2] - 2[3,1]\,,\\
\mathrm{d}[1,1] &= [3,1] + \frac{3}{2}[2,1] + \frac{1}{2}[1,2] + [1,3] - 2[2,1,1] - [1,2,1]\,.
\end{aligned}$$

The second and third equation lead to the first linear relation between multiple divisor functions in weight 4:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2]\,.$$

Multiple divisor functions - Derivation

In the lowest length we get several expressions for the derivative:

Proposition

Let $k \in \mathbb{N}$, then for any $s_1, s_2 \geq 1$ with $k = s_1 + s_2 - 2$ we have the following expression for $\mathrm{d}[k]$:

$$\binom{k}{s_1 - 1} \frac{\mathrm{d}[k]}{k} - \binom{k}{s_1 - 1} [k+1] = [s_1] \cdot [s_2] - \sum_{a+b=k+2} \left(\binom{a-1}{s_1 - 1} + \binom{a-1}{s_2 - 1} \right) [a, b] \,.$$

Observe that the right hand side is the formal shuffle product. The possible choices for s_1 and s_2 give $\left\lfloor \frac{k}{2} \right\rfloor$ linear relations in $\mathrm{Fil}^{\mathrm{W,L}}_{k,2}(\mathcal{MD})$, which are conjecturally all relations in length two.

Multiple divisor functions - $\mathrm{q}\mathcal{MZ}$

We define the space of all admissible multiple divisor functions $\mathrm{q}\mathcal{MZ}$ as

$$\mathrm{q}\mathcal{MZ} := \left\langle [s_1, \dots, s_l] \in \mathcal{MD} \;\middle|\; s_1 > 1 \right\rangle_{\mathbb{Q}}.$$

Theorem

- The vector space $\mathrm{q}\mathcal{MZ}$ is a subalgebra of $\mathcal{MD}$.
- We have $\mathcal{MD} = \mathrm{q}\mathcal{MZ}[\,[1]\,]$.
- The algebra $\mathcal{MD}$ is a polynomial ring over $\mathrm{q}\mathcal{MZ}$ with indeterminate $[1]$, i.e. $\mathcal{MD}$ is isomorphic to $\mathrm{q}\mathcal{MZ}[\,T\,]$ by sending $[1]$ to T.

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Sketch of proof:

- The first two statements can be proven by induction using the quasi-shuffle product.
- The algebraic independency of $[1]$ for the third statements follows from the fact that near $q = 1$ one has $[1] \approx \frac{-\log(1-q)}{1-q}$ but $[s_1, \dots, s_l] \approx \frac{1}{(1-q)^{s_1 + \dots + s_l}}$ for $[s_1, \dots, s_l] \in \mathrm{q}\mathcal{MZ}$.

Multiple divisor functions - Connections to MZV

For the natural map $Z : \mathrm{q}\mathcal{MZ} \to \mathcal{MZ}$ of vector spaces given by

$$Z([s_1, \dots, s_l]) = \zeta(s_1, \dots, s_l)$$

we have a factorization

$$\begin{tikzcd} \mathrm{q}\mathcal{MZ} \arrow[r] \arrow[dr, "Z"'] & \bigoplus_k \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}) \arrow[d, "\bigoplus_k Z_k"] \\ & \mathcal{MZ}, \end{tikzcd}$$

Remark

If we endow $\overline{\mathrm{q}\mathcal{MZ}} = \bigoplus_k \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ with the induced multiplication given on equivalence classes, then the products in $\overline{\mathrm{q}\mathcal{MZ}}$ satisfy the stuffle relations. Moreover the right hand arrow is a homomorphism of algebras.

Multiple divisor functions - Connections to MZV

Extending Z by setting

$$Z([s_1,\dots,s_l][1]^r) = \zeta(s_1,\dots,s_l)T^r$$

we obtain a homomorphisms of vector spaces

$$\begin{tikzcd}
\mathcal{MD} = \mathrm{q}\mathcal{MZ}[[1]] \arrow[r] \arrow[dr, "Z"'] & \overline{\mathrm{q}\mathcal{MZ}}[[1]] \arrow[d, "\oplus_k Z_k"] \\
& \mathcal{MZ}[T],
\end{tikzcd}$$

such that the image of a multiple divisor function $[s_1,\dots,s_l]$ with $s_1 = 1$ equals the stuffle regularised $\zeta(s_1,\dots,s_l)$ as given by Ihara, Kaneko and Zagier (2006).

Remark

Again the right hand arrow is a homomorphism of algebras.

All relations between stuffle regularised multiple zeta values of weight k correspond to elements in the kernel of Z_k. We approach $\ker(Z_k) \subseteq \mathcal{MD}$ by using an analytical description of Z_k.

Proposition

For $[s_1, \ldots, s_l] \in \mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ we have the alternative description

$$Z_k\left([s_1, \ldots, s_l]\right) = \lim_{q \to 1} (1-q)^k [s_1, \ldots, s_l].$$

In particular we have

$$Z_k\left([s_1, \ldots, s_l]\right) = \begin{cases} \zeta(s_1, \ldots, s_l)\,, & s_1 + \cdots + s_l = k, \\ 0\,, & s_1 + \cdots + s_l < k\,. \end{cases}$$

Sketch of proof: The multiple divisor functions can be written as

$$[s_1, \dots, s_l] = \frac{1}{(s_1-1)! \dots (s_l-1)!} \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^{l} \frac{q^{n_j} P_{s_j-1}(q^{n_j})}{(1-q^{n_j})^{s_j}} ,$$

where $P_k(t)$ is the k-th Eulerian polynomial. These polynomials have the property that $P_k(1) = k!$ and therefore

$$\lim_{q \to 1} \frac{q^n P_{k-1}(q^n)}{(k-1)!} \frac{(1-q)^k}{(1-q^n)^k} = \frac{1}{n^k} .$$

Now doing some careful argumentation to justify the interchange of limit and summation one obtains

$$Z_{s_1+\dots+s_l}([s_1, \dots, s_l]) = \lim_{q \to 1} (1-q)^{s_1+\dots+s_l} [s_1, \dots, s_l] = \zeta(s_1 + \dots + s_l) .$$

$\square$

Multiple divisor functions - Connections to other q-analogues

Another example of a **q-analogue of multiple zeta values** are the modified multiple q-zeta values. They are defined for $s_1 > 1$,$s_2, \dots, s_l \geq 1$ as

$$\overline{\zeta}_q(s_1, \dots, s_l) = \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^{n} \frac{q^{n_j(s_j-1)}}{(1-q^{n_j})^{s_j}} .$$

Similar to the multiple divisor functions one derives

$$\lim_{q \to 1} (1-q)^{s_1 + \dots + s_l} \overline{\zeta}_q(s_1, \dots, s_l) = \zeta(s_1, \dots, s_l) .$$

They seem to have a close connection to multiple divisor functions an if all entries $s_i > 1$ they are indeed elements of $\mathcal{MD}$, e.g.

$$\overline{\zeta}_q(4) = [4] - [3] + \frac{1}{3}[2] .$$

But if at least one entry is 1 this connection is conjectural.

Multiple divisor functions - Connections to MZV

Theorem

For the kernel of $Z_k \in \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ we have

- If $s_1 + \cdots + s_l < k$ then $Z_k([s_1, \ldots, s_l]) = 0$.
- For any $f \in \mathrm{Fil}_{k-2}^{\mathrm{W}}(\mathcal{MD})$ we have $Z_k(\mathrm{d}(f)) = 0$.
- If $f \in \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$, then $Z_k(f) = 0$.

Sketch of proof: First we extend the Z_k to a larger space. We define for $\rho \in \mathbb{R}$

$$\mathcal{Q}_\rho = \left\{ \sum_{n>0} a_n q^n \in \mathbb{R}[[q]] \mid a_n = O(n^{\rho-1}) \right\}$$

and

$$\mathcal{Q}_{<\rho} = \left\{ \sum_{n>0} a_n q^n \in \mathbb{R}[[q]] \mid \exists\, \varepsilon > 0 \text{ with } a_n = O(n^{\rho-1-\varepsilon}) \right\}.$$

For $\rho > 1$ define the map Z_ρ for a $f = \sum_{n>0} a_n q^n \in \mathbb{R}[[q]]$ by

$$Z_\rho(f) = \lim_{q \to 1} (1-q)^\rho \sum_{n>0} a_n q^n .$$

Lemma

- Z_ρ is a linear map from $\mathcal{Q}_\rho$ to $\mathbb{R}$
- $\mathcal{Q}_{<\rho} \subset \ker Z_\rho$.
- $\mathrm{d}\, \mathcal{Q}_{<\rho-1} \subset \ker(Z_\rho)$, where as before $\mathrm{d} = q\frac{d}{dq}$.
- For any $s_1, \dots, s_l \geq 1$ we have $[s_1, \dots, s_l] \in \mathcal{Q}_{<s_1+\dots+s_l+1}$.

Example I: We have seen earlier that the derivative of $[1]$ is given by

$$\mathrm{d}[1] = [3] + \frac{1}{2}[2] - [2,1]$$

and because of the theorem it is $\mathrm{d}[1], [2] \in \ker Z_3$ from which $\zeta(2,1) = \zeta(3)$ follows.

Example II: (Shuffle product) We saw that for $s_1 + s_2 = k+2$

$$\binom{k}{s_1-1}\frac{d[k]}{k} = [s_1]\cdot[s_2] + \binom{k}{s_1-1}[k+1] - \sum_{a+b=k+2}\left(\binom{a-1}{s_1-1}+\binom{a-1}{s_2-1}\right)[a,b]$$

Applying Z_{k+2} on both sides we obtain the shuffle product for single zeta values

$$\zeta(s_1)\cdot\zeta(s_2) = \sum_{a+b=k+2}\left(\binom{a-1}{s_1-1}+\binom{a-1}{s_2-1}\right)\zeta(a,b)\,.$$

Example III: For the cusp form $\Delta \in S_{12} \subset \ker(Z_{12})$ we derived the representation

$$\frac{1}{2^6 \cdot 5 \cdot 691}\Delta = 168[5,7] + 150[7,5] + 28[9,3] + \frac{1}{1408}[2] - \frac{83}{14400}[4] + \frac{187}{6048}[6] - \frac{7}{120}[8] - \frac{5197}{691}[12]\,.$$

Letting Z_{12} act on both sides one obtains the relation

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)\,.$$

Multiple divisor functions - Connections to MZV

To summarize we have the following objects in the kernel of Z_k, i.e. ways of getting relations between multiple zeta values using multiple divisor functions.

- Elements of lower weights, i.e. elements in $\mathrm{Fil}^{\mathrm{W}}_{k-1}(\mathcal{MD})$.
- Derivatives
- Modular forms, which are cusp forms
- Since $0 \in \ker Z_k$, any linear relation between multiple divisor functions in $\mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ gives an element in the kernel.

Remark

The number of admissible multiple divisor functions $[s_1, \ldots, s_l]$ of weight k and length l equals $\binom{k-2}{l-1}$. Since we have $\mathcal{MD} = \mathrm{q}\mathcal{MZ}\,[[1]]$, it follows that knowing the dimension of $\mathrm{gr}^{\mathrm{W,L}}_{k,l}(\mathrm{q}\mathcal{MZ})$ is sufficient to know the number of independent relations.

Multiple divisor functions - Dimensions

$k \diagdown l$	0	1	2	3	4	5	6	7	8	9	10	11
0	**1**	**0**										
1	**0**	**0**										
2	**0**	**1**	**0**									
3	**0**	**1**	**1**	**0**								
4	**0**	**1**	**1**	**1**	**0**							
5	**0**	**1**	**2**	**2**	**1**	**0**						
6	**0**	**1**	**2**	**3**	3	1	**0**					
7	**0**	**1**	**3**	4	5	4	1	**0**				
8	**0**	**1**	**3**	6	8	8	5	1	**0**			
9	**0**	**1**	**4**	7	11	14	12	6	1	**0**		
10	**0**	**1**	**4**	10	16	21	23	17	7	1	**0**	
11	**0**	**1**	**5**	11	21	32	38	36	23	8	1	**0**
12	**0**	**1**	**5**	14	28	44	60	?	?	30	9	1
13	**0**	**1**	**6**	16	35	?	?	?	?	?	38	10
14	**0**	**1**	**6**	20	43	?	?	?	?	?	?	47
15	**0**	**1**	**7**	21	?	?	?	?	?	?	?	?

$\dim_{\mathbb{Q}} \mathrm{gr}^{\mathrm{W,L}}_{k,l}(\mathrm{q}\mathcal{MZ})$: **proven**, conjectured.

Multiple divisor functions - Dimensions

We observe that $d'_k := \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ satisfies: $d'_0 = 1$, $d'_1 = 0$, $d'_2 = 1$ and

$$d'_k = 2d'_{k-2} + 2d'_{k-3}, \quad \text{for } \ 5 \leq k \leq 11.$$

We see no reason why this shouldn't hold for all $k > 11$ also, i.e. we ask whether

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}) X^k = \sum_{k \geq 0} d'_k x^k \stackrel{?}{=} \frac{1 - X^2 + X^4}{1 - 2X^2 - 2X^3}. \tag{1}$$

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Compare this to the Zagier conjecture for the dimension d_k of $\mathcal{MZ}_k$

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathcal{MZ}) X^k = \sum_{k\geq 0} d_k X^k \stackrel{?}{=} \frac{1}{1 - X^2 - X^3}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21
d'_k	1	0	1	2	3	6	10	18	32	56	100	176	312	552	976

Multiple divisor functions - Summary

- Multiple divisor functions are formal power series in q with coefficient in $\mathbb{Q}$ coming from the calculation of the fourier expansion of multiple Eisenstein series.
- The space spanned by all multiple divisor functions form an differential algebra which contains the algebra of (quasi-) modular forms.
- A connection to multiple zeta values is given by the map Z_k whose kernel contains all relations between multiple zeta values of weight k.
- Questions:
  - What is the kernel of Z_k ?
  - Is there an analogue of the Broadhurst-Kreimer conjecture?
  - Is there a geometric/motivic interpretation of the multiple divisor functions?

- In the context of a generalisation of the Broadhurst-Kreimer conjecture in our context one may ask whether the modular forms replace the even zeta values. Both may be the only subalgebras which are nilpotent with respect to the length, this is due to the following reformulation of an old result by Zagier.

Proposition

As a subalgebra of $\mathcal{MD}$ the algebra of modular forms is graded with respect to the weight and filtered with respect to the length. We have

$$\sum_{k} \dim_{\mathbb{Q}} \mathrm{gr}_{k,l}^{\mathrm{W,L}} M(SL_2(\mathbb{Z}))\, x^k y^l = 1 + \frac{x^4}{1-x^2} y + \frac{x^{12}}{(1-x^4)(1-x^6)} y^2 \,,$$

in particular

$$\sum_{k} \dim_{\mathbb{Q}} M_k(SL_2(\mathbb{Z}))\, x^k = \frac{1}{(1-x^4)(1-x^6)} \,.$$