Multiple divisor functions, their algebraic structure and the relation to multiple zeta values

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joint work: H.B., Ulf Kühn, arXiv:1309.3920 [math.NT] multiple zeta values

$$\frac{5197}{691}\zeta(12) - 168\zeta(5,7) - 150\zeta(7,5) - 28\zeta(9,3) = 0$$

$$\zeta(s_1,\dots,s_l) := \sum_{n_1>n_2>\dots>n_l>0} \frac{1}{n_1^{s_1}\dots n_l^{s_l}}$$

$$\zeta(2,3)+\zeta(3,2)+\zeta(5) \overset{\text{stuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$

modular forms

$$G_k = \frac{\zeta(k)}{(-2\pi i)^k} + \underbrace{\frac{1}{(k-1)!}\sum_{n>0} \sigma_{k-1}(n) q^n}_{[k]}$$

$$\Delta = \sum_{n>0} \tau(n) q^n$$

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$$G_{4,4} = \zeta(4,4) + 20\zeta(6)(2\pi i)^2[2] + 3\zeta(4)(2\pi i)^4[4] + (2\pi i)^8[4,4]$$

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multiple divisor functions

$$[s_1,\dots,s_l] \in \mathbb{Q}[[q]]$$

quasi-shuffle product

$$[2]\cdot[3] = [2,3]+[3,2]+[5]-\frac{1}{12}[3]$$

derivation

$$\mathrm{d} = q\frac{d}{dq}$$

$$[2]\cdot[3] = [2,3]+3[3,2]+6[4,1]+\mathrm{d}[3]-3[4]$$

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kernel of Z_k

$$[3], \mathrm{d}[3], [4] \in \ker Z_5$$

$$\Delta \in \ker Z_{12}$$

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$$G_{4,4} = \underline{\zeta(4,4)} + 20\underline{\zeta(6)}(2\pi i)^2\underline{[2]} + 3\underline{\zeta(4)}(2\pi i)^4\underline{[4]} + (2\pi i)^8\underline{[4,4]}$$

Z_k

Periodpolynomials

multiple zeta values

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}}$$

$$Z_5([2,3]) = \zeta(2,3)$$

$$\frac{5197}{691}\zeta(12) - 168\zeta(5,7) - 150\zeta(7,5) - 28\zeta(9,3) = 0 = Z_{12}(\Delta)$$

stuffle | shuffle

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2) \cdot \zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

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Z_k

multiple q-zeta values

$\lim_{q \to 1}$

Periodpolynomials

multiple zeta values

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Multiple divisor functions

As a generalization of the classical divisor sums we define for $r_1, \dots, r_l \geq 0$ the **multiple divisor sum** by

$$\sigma_{r_1,\dots,r_l}(n) := \sum_{\substack{u_1 v_1 + \dots + u_l v_l = n \\ u_1 > \dots > u_l > 0}} v_1^{r_1} \dots v_l^{r_l} .$$

With this we define for $s_1, \dots, s_l > 0$ the **multiple divisor function** of weight $s_1 + \dots + s_l$ and length l by

$$[s_1, \dots, s_l] := \frac{1}{(s_1 - 1)! \dots (s_l - 1)!} \sum_{n>0} \sigma_{s_1-1,\dots,s_l-1}(n) q^n \ \in \mathbb{Q}[[q]] .$$

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Example I

For $l = 1$ these are the classical divisor functions

$$[2] = \sum_{n>0} \sigma_1(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \dots$$

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Example II

$$[4,2] = \frac{1}{6} \sum_{n>0} \sigma_{3,1}(n) q^n = \frac{1}{6} \left(q^3 + 3q^4 + \underbrace{15q^5} + 27q^6 + 78q^7 + \dots \right)$$

$\sigma_{3,1}(5) = 1^3 \cdot 1^1 + 2^3 \cdot 1^1 + 1^3 \cdot 1^1 + 1^3 \cdot 2^1 + 1^3 \cdot 3^1 = 15$, because

$$5 = 4 \cdot 1 + 1 \cdot 1 = 2 \cdot 2 + 1 \cdot 1 = 3 \cdot 1 + 2 \cdot 1 = 3 \cdot 1 + 1 \cdot 2 = 2 \cdot 1 + 1 \cdot 3$$

Multiple divisor functions

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Example III

$$[4,4,4] = \frac{1}{216} \left(q^6 + 9q^7 + 45q^8 + 190q^9 + 642q^{10} + 1899q^{11} + \dots \right) \,,$$

$$[3,1,3,1] = \frac{1}{4} \left(q^{10} + 2q^{11} + 8q^{12} + 16q^{13} + 43q^{14} + 70q^{15} + \dots \right)$$

Multiple divisor functions - Filtration

Weight and length filtration

We define the vector space $\mathcal{MD}$ to be the $\mathbb{Q}$ vector space generated by $[\emptyset] = 1 \in \mathbb{Q}[[q]]$ and all multiple divisor sums $[s_1, \dots, s_l]$.

On $\mathcal{MD}$ we have the increasing filtration $\operatorname{Fil}^{\mathrm{W}}_{\bullet}$ given by the weight and the increasing filtration $\operatorname{Fil}^{\mathrm{L}}_{\bullet}$ given by the length, i.e., we have for $A \subseteq \mathcal{MD}$

$$\operatorname{Fil}^{\mathrm{W}}_k(A) := \left\langle [s_1, \dots, s_l] \in A \,\middle|\, 0 \leq l \leq k \,,\; s_1 + \dots + s_l \leq k \right\rangle_{\mathbb{Q}}$$

$$\operatorname{Fil}^{\mathrm{L}}_l(A) := \left\langle [s_1, \dots, s_r] \in A \,\middle|\, r \leq l \right\rangle_{\mathbb{Q}}.$$

If we consider the length and weight filtration at the same time we use the short notation $\operatorname{Fil}^{\mathrm{W},\mathrm{L}}_{k,l} := \operatorname{Fil}^{\mathrm{W}}_k \operatorname{Fil}^{\mathrm{L}}_l$.

As usual $\operatorname{gr}^{\mathrm{W}}_k(A) = \operatorname{Fil}^{\mathrm{W}}_k(A) / \operatorname{Fil}^{\mathrm{W}}_{k-1}(A)$ will denote the graded part (similar $\operatorname{gr}^{\mathrm{L}}_l$ and $\operatorname{gr}^{\mathrm{W},\mathrm{L}}_{k,l}$).

Multiple divisor functions - Algebra structure

Theorem

The $\mathbb{Q}$-vector space $\mathcal{MD}$ has the structure of a bifiltered $\mathbb{Q}$-Algebra $(\mathcal{MD}, \cdot, \mathrm{Fil}^{\mathrm{W}}_{\bullet}, \mathrm{Fil}^{\mathrm{L}}_{\bullet})$, where the multiplication is the natural multiplication of formal power series and the filtrations $\mathrm{Fil}^{\mathrm{W}}_{\bullet}$ and $\mathrm{Fil}^{\mathrm{L}}_{\bullet}$ are induced by the $weight$ and $length$, in particular

$$\mathrm{Fil}^{\mathrm{W,L}}_{k_1,l_1}(\mathcal{MD}) \cdot \mathrm{Fil}^{\mathrm{W,L}}_{k_2,l_2}(\mathcal{MD}) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k_1+k_2,l_1+l_2}(\mathcal{MD}).$$

It is a (homomorphic image of a) quasi-shuffle algebra in the sense of Hoffman.

The first products of multiple divisor functions are given by

$$\begin{aligned} [1]\cdot[1] &= 2[1,1]+[2]-[1]\,, \\ [1]\cdot[2] &= [1,2]+[2,1]+[3]-\frac{1}{2}[2]\,, \\ [1]\cdot[2,1] &= [1,2,1]+2[2,1,1]+[2,2]+[3,1]-\frac{3}{2}[2,1]\,. \end{aligned}$$

Multiple divisor functions - Algebra structure

To prove this theorem we need to rewrite the multiple divisor functions. For this we define a *normalized polylogarithm* by

$$\widetilde{\mathrm{Li}}_s(z) := \frac{\mathrm{Li}_{1-s}(z)}{\Gamma(s)},$$

where for $s, z \in \mathbb{C}$, $|z| < 1$ the polylogarithm $\mathrm{Li}_s(z)$ of weight s is given by

$$\mathrm{Li}_s(z) = \sum_{n>0} \frac{z^n}{n^s}.$$

Proposition

For $q \in \mathbb{C}$ with $|q| < 1$ and for all $s_1, \dots, s_l \in \mathbb{N}$ we can write the multiple divisor functions as

$$[s_1, \dots, s_l] = \sum_{n_1 > \cdots > n_l > 0} \widetilde{\mathrm{Li}}_{s_1}\left(q^{n_1}\right) \dots \widetilde{\mathrm{Li}}_{s_l}\left(q^{n_l}\right).$$

Multiple divisor functions - Algebra structure

The product of $[s_1]$ and $[s_2]$ can thus be written as

$$\begin{aligned}[s_1] \cdot [s_2] &= \sum_{n_1 > n_2 > 0} \dots + \sum_{n_2 > n_1 > 0} \dots + \sum_{n_1 = n_2 > 0} \widetilde{\mathrm{Li}}_{s_1}\left(q^{n_1}\right) \widetilde{\mathrm{Li}}_{s_2}\left(q^{n_1}\right) \\ &= [s_1, s_2] + [s_2, s_1] + \sum_{n>0} \widetilde{\mathrm{Li}}_{s_1}\left(q^{n}\right) \widetilde{\mathrm{Li}}_{s_2}\left(q^{n}\right) .\end{aligned}$$

In order to prove that this product is an element of $\mathrm{Fil}^{\mathrm{W}}_{s_1+s_2}(\mathcal{MD})$ we will show that the product $\widetilde{\mathrm{Li}}_{s_1}\left(q^{n}\right) \widetilde{\mathrm{Li}}_{s_2}\left(q^{n}\right)$ is a rational linear combination of $\widetilde{\mathrm{Li}}_{j}\left(q^{n}\right)$ with $1 \leq j \leq s_1 + s_2$.

Multiple divisor functions - Algebra structure

Lemma

For $a, b \in \mathbb{N}$ we have

$$\widetilde{\mathrm{Li}}_a(z) \cdot \widetilde{\mathrm{Li}}_b(z) = \sum_{j=1}^{a} \lambda_{a,b}^{j} \widetilde{\mathrm{Li}}_j(z) + \sum_{j=1}^{b} \lambda_{b,a}^{j} \widetilde{\mathrm{Li}}_j(z) + \widetilde{\mathrm{Li}}_{a+b}(z)\,,$$

where the coefficient $\lambda_{a,b}^{j} \in \mathbb{Q}$ for $1 \leq j \leq a$ is given by

$$\lambda_{a,b}^{j} = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!}\,,$$

with the Seki-Bernoulli numbers B_n.

Proof: For the proof we use the generating function

$$L(X) := \sum_{k>0} \widetilde{\mathrm{Li}_k}(z) X^{k-1} = \sum_{k>0} \sum_{n>0} \frac{n^{k-1} z^n}{(k-1)!} X^{k-1} = \sum_{n>0} e^{nX} z^n = \frac{e^X z}{1 - e^X z} \, .$$

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By direct calculation one can show that

$$L(X) \cdot L(Y) = \frac{1}{e^{X-Y} - 1} L(X) + \frac{1}{e^{Y-X} - 1} L(Y)\,.$$

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With the definition of Seki-Bernoulli numbers

$$\frac{X}{e^X - 1} = \sum_{n \geq 0} \frac{B_n}{n!} X^n$$

we find

$$L(X) \cdot L(Y) = \sum_{n>0} \frac{B_n}{n!} (X - Y)^{n-1} L(X) + \sum_{n>0} \frac{B_n}{n!} (Y - X)^{n-1} L(Y) + \frac{L(X) - L(Y)}{X - Y}\,.$$

Considering the coefficient of $X^{a-1} Y^{b-1}$ we get the result.

Multiple divisor functions - Algebra structure

The structure of being a quasi-shuffle algebra is as follows:

Let $\mathbb{Q}\langle A\rangle$ be the noncommutative polynomial algebra over $\mathbb{Q}$ generated by words with letters in $A = \{z_1, z_2, \dots\}$. Let $\diamond$ be a commutative and associative product on $\mathbb{Q}A$. Define on $\mathbb{Q}\langle A\rangle$ recursively a product by $1 * w = w * 1 = w$ and

$$z_a w * z_b v := z_a(w * z_b v) + z_b(z_a w * v) + (z_a \diamond z_b)(w * v)\,.$$

The commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle A\rangle, *)$ is called a quasi-shuffle algebra.

- The harmonic algebra (Hoffman) is an example of a quasi-shuffle algebra with $z_a \diamond z_b = z_{a+b}$ which is closely related to MZV.

Multiple divisor functions - Algebra structure

In our case we consider the quasi-shuffle algebra with

$$z_a \diamond z_b = z_{a+b} + \sum_{j=1}^{a} \lambda_{a,b}^j z_j + \sum_{j=1}^{b} \lambda_{b,a}^j z_j \,,$$

where

$$\lambda_{a,b}^j = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!} \,.$$

Proposition

The map $[\,.\,] : (\mathbb{Q}\langle A \rangle, *) \longrightarrow (\mathcal{MD}, \cdot)$ defined on the generators by

$$z_{s_1} \dots z_{s_l} \mapsto [s_1, \dots, s_l]$$

is a homomorphism of algebras, i.e. it fulfils

$$[w * v] = [w] \cdot [v].$$

Multiple divisor functions - Modular forms

Proposition

The ring of modular forms $M_{\mathbb{Q}}(SL_2(\mathbb{Z}))$ and the ring of quasi-modular forms $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z}))$ are subalgebras of $\mathcal{MD}$.

For example we have

$$G_2 = -\frac{1}{24} + [2]\,, \quad G_4 = \frac{1}{1440} + [4]\,, \quad G_6 = -\frac{1}{60480} + [6]\,.$$

The proposition follows from the fact that $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_4, G_6]$ and $\widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) = \mathbb{Q}[G_2, G_4, G_6]$.

Remark

Due to an old result of Zagier we have $M_{\mathbb{Q}}(SL_2(\mathbb{Z})) \subset \mathrm{Fil}^{\mathrm{W,L}}_{k,2}(\mathcal{MD})$.

Multiple divisor functions - Derivation

Theorem

The operator $\mathrm{d} = q\frac{d}{dq}$ is a derivation on $\mathcal{MD}$ and it maps $\mathrm{Fil}^{\mathrm{W,L}}_{k,l}(\mathcal{MD})$ to $\mathrm{Fil}^{\mathrm{W,L}}_{k+2,l+1}(\mathcal{MD})$.

Examples:

$$\mathrm{d}[1] = [3] + \frac{1}{2}[2] - [2,1]\,,$$
$$\mathrm{d}[2] = [4] + 2[3] - \frac{1}{6}[2] - 4[3,1]\,,$$
$$\mathrm{d}[2] = 2[4] + [3] + \frac{1}{6}[2] - 2[2,2] - 2[3,1]\,,$$
$$\mathrm{d}[1,1] = [3,1] + \frac{3}{2}[2,1] + \frac{1}{2}[1,2] + [1,3] - 2[2,1,1] - [1,2,1]\,.$$

The second and third equation lead to the first linear relation between multiple divisor functions in weight 4:

$$[4] = 2[2,2] - 2[3,1] + [3] - \frac{1}{3}[2]\,.$$

Multiple divisor functions - Derivation

To proof this theorem we need to consider the generating function of multiple divisor functions which is given by

$$
\begin{aligned}
T(X_1,\dots,X_l) &:= \sum_{s_1,\dots,s_l>0} [s_1,\dots,s_l] X_1^{s_1-1}\dots X_l^{s_l-1} \\
&= \sum_{n_1,\dots,n_l>0} \prod_{j=1}^{l} \frac{e^{n_j X_j} q^{n_1+\dots+n_j}}{1-q^{n_1+\dots+n_j}} \,.
\end{aligned}
$$

Example:

$$T(X) = \sum_{k>0} [k] X^{k-1} = \sum_{n>0} e^{nX} \frac{q^n}{1-q^n}$$

$$T(X,Y) = \sum_{s_1,s_2>0} [s_1,s_2] X^{s_1-1} Y^{s_2-1} = \sum_{n_1,n_2>0} e^{n_1 X + n_2 Y} \frac{q^{n_1}}{1-q^{n_1}} \frac{q^{n_1+n_2}}{1-q^{n_1+n_2}}$$

Multiple divisor functions - Derivation

For these functions one can prove the following identity

$$\begin{aligned} T(X)\cdot T(Y_1,\dots,Y_l) &= T(X+Y_1,\dots,X+Y_l,X) \\ &+\sum_{j=1}^{l} T(X+Y_1,\dots,X+Y_j,Y_j,\dots,Y_l) \\ &-\sum_{j=1}^{l} T(X+Y_1,\dots,X+Y_j,Y_{j+1},\dots,Y_l) \\ &+R_l(X,Y_1,\dots,Y_l)\,, \end{aligned}$$

where $R_l(X,Y_1,\dots,Y_l)$ satisfies

$$D\left(R_l(X,Y_1,\dots,Y_l)\right) = \mathrm{d}\,T(Y_1,\dots,Y_l)\,,$$

with $D(f) = \left(\frac{d}{dX} f\right)\Big|_{X=0}$. Therefore applying D to the above equation one obtains the theorem.

Multiple divisor functions - Derivation

In the lowest length we get several expressions for the derivative:

Proposition

Let $k \in \mathbb{N}$, then for any $s_1, s_2 \geq 1$ with $k = s_1 + s_2 - 2$ we have the following expression for $\mathrm{d}[k]$:

$$\binom{k}{s_1 - 1} \frac{\mathrm{d}[k]}{k} - \binom{k}{s_1 - 1} [k+1] = [s_1] \cdot [s_2] - \sum_{a+b=k+2} \left(\binom{a-1}{s_1 - 1} + \binom{a-1}{s_2 - 1} \right) [a, b] \, .$$

Observe that the right hand side is the formal shuffle product. The possible choices for s_1 and s_2 give $\lfloor \frac{k}{2} \rfloor$ linear relations in $\mathrm{Fil}_{k,2}^{\mathrm{W,L}}(\mathcal{MD})$, which are conjecturally all relations in length two.

Multiple divisor functions - $\mathrm{q}\mathcal{MZ}$

We define the space of all admissible multiple divisor functions $\mathrm{q}\mathcal{MZ}$ as

$$\mathrm{q}\mathcal{MZ} := \big\langle [s_1, \ldots, s_l] \in \mathcal{MD} \;\big|\; s_1 > 1 \big\rangle_{\mathbb{Q}}.$$

Theorem

- The vector space $\mathrm{q}\mathcal{MZ}$ is a subalgebra of $\mathcal{MD}$.
- We have $\mathcal{MD} = \mathrm{q}\mathcal{MZ}\big[\,[1]\,\big]$.
- The algebra $\mathcal{MD}$ is a polynomial ring over $\mathrm{q}\mathcal{MZ}$ with indeterminate $[1]$, i.e. $\mathcal{MD}$ is isomorphic to $\mathrm{q}\mathcal{MZ}[\,T\,]$ by sending $[1]$ to T.

Multiple divisor functions - $\mathrm{q}\mathcal{MZ}$

We define the space of all admissible multiple divisor functions $\mathrm{q}\mathcal{MZ}$ as

$$\mathrm{q}\mathcal{MZ} := \left\langle [s_1, \dots, s_l] \in \mathcal{MD} \,\middle|\, s_1 > 1 \right\rangle_{\mathbb{Q}}.$$

Theorem

- The vector space $\mathrm{q}\mathcal{MZ}$ is a subalgebra of $\mathcal{MD}$.
- We have $\mathcal{MD} = \mathrm{q}\mathcal{MZ}[\,[1]\,]$.
- The algebra $\mathcal{MD}$ is a polynomial ring over $\mathrm{q}\mathcal{MZ}$ with indeterminate $[1]$, i.e. $\mathcal{MD}$ is isomorphic to $\mathrm{q}\mathcal{MZ}[\,T\,]$ by sending $[1]$ to T.

Sketch of proof:

- The first two statements can be proven by induction using the quasi-shuffle product.
- The algebraic independency of $[1]$ for the third statements follows from the fact that near $q = 1$ one has $[1] \approx \frac{-\log(1-q)}{1-q}$ but $[s_1, \dots, s_l] \approx \frac{1}{(1-q)^{s_1+\dots+s_l}}$ for $[s_1, \dots, s_l] \in \mathrm{q}\mathcal{MZ}$.

Multiple divisor functions - Connections to MZV

For the natural map $Z : \mathrm{q}\mathcal{MZ} \to \mathcal{MZ}$ of vector spaces given by

$$Z([s_1, \dots, s_l]) = \zeta(s_1, \dots, s_l)$$

we have a factorization

$$\begin{tikzcd}
\mathrm{q}\mathcal{MZ} \arrow[r] \arrow[dr, "Z"'] & \bigoplus_k \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}) \arrow[d, "\bigoplus_k Z_k"] \\
& \mathcal{MZ},
\end{tikzcd}$$

Remark

If we endow $\overline{\mathrm{q}\mathcal{MZ}} = \bigoplus_k \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ with the induced multiplication given on equivalence classes, then the products in $\overline{\mathrm{q}\mathcal{MZ}}$ satisfy the stuffle relations. Moreover the right hand arrow is a homomorphism of algebras.

Extending Z by setting

$$Z([s_1,\dots,s_l][1]^r) = \zeta(s_1,\dots,s_l)T^r$$

we obtain a homomorphisms of vector spaces

$$\begin{tikzcd}
\mathcal{MD} = \mathrm{q}\mathcal{MZ}[[1]] \arrow[r] \arrow[dr, "Z"'] & \overline{\mathrm{q}\mathcal{MZ}}[[1]] \arrow[d, "\oplus_k Z_k"] \\
 & \mathcal{MZ}[T],
\end{tikzcd}$$

such that the image of a multiple divisor function $[s_1,\dots,s_l]$ with $s_1 = 1$ equals the stuffle regularised $\zeta(s_1,\dots,s_l)$ as given by Ihara, Kaneko and Zagier (2006).

Remark

Again the right hand arrow is a homomorphism of algebras.

All relations between stuffle regularised multiple zeta values of weight k correspond to elements in the kernel of Z_k. We approach $\ker(Z_k) \subseteq \mathcal{MD}$ by using an analytical description of Z_k.

Proposition

For $[s_1, \dots, s_l] \in \mathrm{Fil}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ we have the alternative description

$$Z_k\left([s_1, \dots, s_l]\right) = \lim_{q \to 1} (1-q)^k [s_1, \dots, s_l].$$

In particular we have

$$Z_k\left([s_1, \dots, s_l]\right) = \begin{cases} \zeta(s_1, \dots, s_l)\,, & s_1 + \dots + s_l = k, \\ 0\,, & s_1 + \dots + s_l < k\,. \end{cases}$$

Multiple divisor functions - Connections to MZV

Sketch of proof: The multiple divisor functions can be written as

$$[s_1, \dots, s_l] = \frac{1}{(s_1-1)! \dots (s_l-1)!} \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^{l} \frac{q^{n_j} P_{s_j-1}(q^{n_j})}{(1-q^{n_j})^{s_j}} ,$$

where $P_k(t)$ is the k-th Eulerian polynomial. These polynomials have the property that $P_k(1) = k!$ and therefore

$$\lim_{q \to 1} \frac{q^n P_{k-1}(q^n)}{(k-1)!} \frac{(1-q)^k}{(1-q^n)^k} = \frac{1}{n^k} .$$

Now doing some careful argumentation to justify the interchange of limit and summation one obtains

$$Z_{s_1+\dots+s_l}([s_1, \dots, s_l]) = \lim_{q \to 1} (1-q)^{s_1+\dots+s_l} [s_1, \dots, s_l] = \zeta(s_1 + \dots + s_l) .$$

$\square$

Multiple divisor functions - Connections to other q-analogues

Another example of a **q-analogue of multiple zeta values** are the modified multiple q-zeta values. They are defined for $s_1 > 1$,$s_2, \dots, s_l \geq 1$ as

$$\overline{\zeta}_q(s_1, \dots, s_l) = \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^{n} \frac{q^{n_j(s_j - 1)}}{(1 - q^{n_j})^{s_j}} \, .$$

Similar to the multiple divisor functions one derives

$$\lim_{q \to 1} (1-q)^{s_1 + \dots + s_l} \overline{\zeta}_q(s_1, \dots, s_l) = \zeta(s_1, \dots, s_l) \, .$$

They seem to have a close connection to multiple divisor functions an if all entries $s_i > 1$ they are indeed elements of $\mathcal{MD}$, e.g.

$$\overline{\zeta}_q(4) = [4] - [3] + \frac{1}{3}[2] \, .$$

But if at least one entry is 1 this connection is conjectural.

Multiple divisor functions - Connections to MZV

Theorem

For the kernel of $Z_k \in \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ we have

- If $s_1 + \cdots + s_l < k$ then $Z_k([s_1, \ldots, s_l]) = 0$.
- For any $f \in \mathrm{Fil}_{k-2}^{\mathrm{W}}(\mathcal{MD})$ we have $Z_k(\mathrm{d}(f)) = 0$.
- If $f \in \mathrm{Fil}_k^{\mathrm{W}}(\mathcal{MD})$ is a cusp form for $\mathrm{SL}_2(\mathbb{Z})$, then $Z_k(f) = 0$.

Multiple divisor functions - Connections to MZV

Sketch of proof: First we extend the Z_k to a larger space. We define for $\rho \in \mathbb{R}$

$$\mathcal{Q}_\rho = \left\{ \sum_{n>0} a_n q^n \in \mathbb{R}[[q]] \mid a_n = O(n^{\rho-1}) \right\}$$

and

$$\mathcal{Q}_{<\rho} = \left\{ \sum_{n>0} a_n q^n \in \mathbb{R}[[q]] \mid \exists\, \varepsilon > 0 \text{ with } a_n = O(n^{\rho-1-\varepsilon}) \right\}.$$

For $\rho > 1$ define the map Z_ρ for a $f = \sum_{n>0} a_n q^n \in \mathbb{R}[[q]]$ by

$$Z_\rho(f) = \lim_{q\to 1} (1-q)^\rho \sum_{n>0} a_n q^n .$$

Lemma

- Z_ρ is a linear map from $\mathcal{Q}_\rho$ to $\mathbb{R}$
- $\mathcal{Q}_{<\rho} \subset \ker Z_\rho$.
- $\mathrm{d}\, \mathcal{Q}_{<\rho-1} \subset \ker(Z_\rho)$, where as before $\mathrm{d} = q\frac{d}{dq}$.
- For any $s_1, \dots, s_l \geq 1$ we have $[s_1, \dots, s_l] \in \mathcal{Q}_{<s_1+\dots+s_l+1}$.

Example I: We have seen earlier that the derivative of $[1]$ is given by

$$\mathrm{d}[1] = [3] + \frac{1}{2}[2] - [2,1]$$

and because of the theorem it is $\mathrm{d}[1], [2] \in \ker Z_3$ from which $\zeta(2,1) = \zeta(3)$ follows.

Example II: (Shuffle product) We saw that for $s_1 + s_2 = k+2$

$$\binom{k}{s_1-1}\frac{d[k]}{k} = [s_1]\cdot[s_2] + \binom{k}{s_1-1}[k+1] - \sum_{a+b=k+2}\left(\binom{a-1}{s_1-1}+\binom{a-1}{s_2-1}\right)[a,b]$$

Applying Z_{k+2} on both sides we obtain the shuffle product for single zeta values

$$\zeta(s_1)\cdot\zeta(s_2) = \sum_{a+b=k+2}\left(\binom{a-1}{s_1-1}+\binom{a-1}{s_2-1}\right)\zeta(a,b)\,.$$

Example III: For the cusp form $\Delta \in S_{12} \subset \ker(Z_{12})$ one can derive the representation

$$\frac{1}{2^6 \cdot 5 \cdot 691}\Delta = 168[5,7] + 150[7,5] + 28[9,3] + \frac{1}{1408}[2] - \frac{83}{14400}[4] + \frac{187}{6048}[6] - \frac{7}{120}[8] - \frac{5197}{691}[12]\,.$$

Letting Z_{12} act on both sides one obtains the relation

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)\,.$$

Multiple divisor functions - Connections to MZV

To summarize we have the following objects in the kernel of Z_k, i.e. ways of getting relations between multiple zeta values using multiple divisor functions.

- Elements of lower weights, i.e. elements in $\mathrm{Fil}^{\mathrm{W}}_{k-1}(\mathcal{MD})$.
- Derivatives
- Modular forms, which are cusp forms
- Since $0 \in \ker Z_k$, any linear relation between multiple divisor functions in $\mathrm{Fil}^{\mathrm{W}}_{k}(\mathcal{MD})$ gives an element in the kernel.

Remark

The number of admissible multiple divisor functions $[s_1, \dots, s_l]$ of weight k and length l equals $\binom{k-2}{l-1}$. Since we have $\mathcal{MD} = \mathrm{q}\mathcal{MZ}\left[[1]\right]$, it follows that knowing the dimension of $\mathrm{gr}^{\mathrm{W,L}}_{k,l}(\mathrm{q}\mathcal{MZ})$ is sufficient to know the number of independent relations.

Multiple divisor functions - Dimensions

$k \setminus l$	0	1	2	3	4	5	6	7	8	9	10	11
0	**1**	**0**										
1	**0**	**0**										
2	**0**	**1**	**0**									
3	**0**	**1**	**1**	**0**								
4	**0**	**1**	**1**	**1**	**0**							
5	**0**	**1**	**2**	**2**	**1**	**0**						
6	**0**	**1**	**2**	**3**	3	1	**0**					
7	**0**	**1**	**3**	4	5	4	1	**0**				
8	**0**	**1**	**3**	6	8	8	5	1	**0**			
9	**0**	**1**	**4**	7	11	14	12	6	1	**0**		
10	**0**	**1**	**4**	10	16	21	23	17	7	1	**0**	
11	**0**	**1**	**5**	11	21	32	38	36	23	8	1	**0**
12	**0**	**1**	**5**	14	28	44	60	?	?	30	9	1
13	**0**	**1**	**6**	16	35	?	?	?	?	?	38	10
14	**0**	**1**	**6**	20	43	?	?	?	?	?	?	47
15	**0**	**1**	**7**	21	?	?	?	?	?	?	?	?

$\dim_{\mathbb{Q}} \mathrm{gr}^{\mathrm{W,L}}_{k,l}(\mathrm{q}\mathcal{MZ})$: **proven**, conjectured.

Multiple divisor functions - Dimensions

We observe that $d'_k := \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ})$ satisfies: $d'_0 = 1$, $d'_1 = 0$, $d'_2 = 1$ and

$$d'_k = 2d'_{k-2} + 2d'_{k-3}, \quad \text{for } \; 5 \leq k \leq 11.$$

We see no reason why this shouldn't hold for all $k > 11$ also, i.e. we ask whether

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathrm{q}\mathcal{MZ}) X^k = \sum_{k \geq 0} d'_k x^k \stackrel{?}{=} \frac{1 - X^2 + X^4}{1 - 2X^2 - 2X^3}. \tag{1}$$

Multiple divisor functions - Dimensions

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Compare this to the Zagier conjecture for the dimension d_k of $\mathcal{MZ}_k$

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathrm{gr}^{\mathrm{W}}_k(\mathcal{MZ}) X^k = \sum_{k\geq 0} d_k X^k \stackrel{?}{=} \frac{1}{1 - X^2 - X^3}$$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21
d'_k	1	0	1	2	3	6	10	18	32	56	100	176	312	552	976

Multiple divisor functions - Summary

- Multiple divisor functions are formal power series in q with coefficient in $\mathbb{Q}$ coming from the calculation of the Fourier expansion of multiple Eisenstein series.
- The space spanned by all multiple divisor functions form an differential algebra which contains the algebra of (quasi-) modular forms.
- A connection to multiple zeta values is given by the map Z_k whose kernel contains all relations between multiple zeta values of weight k.
- Questions:
    - What is the kernel of Z_k ?
    - Is there an analogue of the Broadhurst-Kreimer conjecture?
    - Is there a geometric/motivic interpretation of the multiple divisor functions?
    - L-Series?