Multiple Eisenstein series, their Fourier expansions and multiple divisor functions

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multiple zeta values

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)$$

$$\zeta(s_1,\dots,s_l) := \sum_{n_1>n_2>\dots>n_l>0} \frac{1}{n_1^{s_1}\dots n_l^{s_l}}$$

stuffle shuffle

$$\zeta(2,3)+\zeta(3,2)+\zeta(5) = \zeta(2)\cdot\zeta(3) = \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$

modular forms

$$\tau \in \mathbb{H}\,, q = \exp(2\pi i \tau)$$

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n$$
$$= \zeta(k) + (-2\pi i)^k [k]$$

$$\Delta \in S_{12}$$

Periodpolynomials

multiple zeta values

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$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)$$

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) \overset{\text{stuffle}}{=} \zeta(2) \cdot \zeta(3) \overset{\text{shuffle}}{=} \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

modular forms

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$$= \zeta(k) + (-2\pi i)^k [k]$$

$\Delta \in S_{12}$

multiple Eisenstein series

$$G_{s_1,\dots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_l \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{s_1} \dots \lambda_l^{s_l}}$$

$$G_{4,4} = \underline{\zeta(4,4)} + 20\underline{\zeta(6)}(2\pi i)^2 \underline{[2]} + 3\underline{\zeta(4)}(2\pi i)^4 \underline{[4]} + (2\pi i)^8 \underline{[4,4]}$$

Periodpolynomials

multiple zeta values

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$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2) \cdot \zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

modular forms

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$$= \zeta(k) + (-2\pi i)^k [k]$$

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multiple divisor functions

$[s_1, \ldots, s_l] \in \mathbb{Q}[[q]]$

multiple Eisenstein series

$$G_{s_1,\ldots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_l \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{s_1} \ldots \lambda_l^{s_l}}$$

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q-analouge of multiple zeta values

Periodpolynomials

multiple zeta values

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}}$$

$$\frac{5197}{691}\zeta(12) = 168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3)$$

stuffle shuffle

$$\zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2) \cdot \zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

Multiple zeta values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$ the multiple zeta value (MZV) of weight $s_1 + ... + s_l$ and length l is defined by

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \dots n_l^{s_l}} .$$

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle relation). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle relation) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

Multiple zeta-values

Example:

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2) \cdot \zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1) = \zeta(3).$$

These follow from the "extended double shuffle relations" where one use the same combinatorics as above for "$\zeta(1) \cdot \zeta(2)$" in a formal setting. The extended double shuffle relations are conjectured to give all relations between MZV.

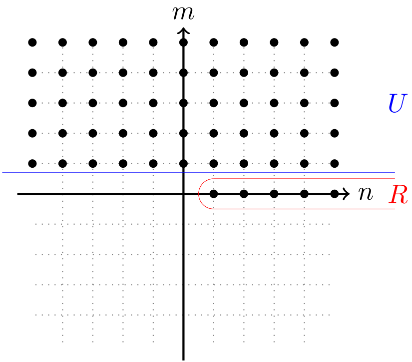
Multiple Eisenstein series

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$. We define an order $\succ$ on Λ_τ by setting

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$ and the following set which we call the set of positive lattice points

$$P := \{m\tau + n \in \Lambda_\tau \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R$$



Multiple Eisenstein series

Definition

For $s_1 \geq 3, s_2, \ldots, s_l \geq 2$ we define the *multiple Eisenstein series* of weight $k = s_1 + \cdots + s_l$ and length l by

$$G_{s_1,\ldots,s_l}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_l \succ 0 \\ \lambda_i \in \Lambda_\tau}} \frac{1}{\lambda_1^{s_1} \ldots \lambda_l^{s_l}}\,.$$

It is easy to see that these are holomorphic functions in the upper half plane and that $G_{s_1,\ldots,s_l}(\tau+1) = G_{s_1,\ldots,s_l}(\tau)$ and therefore we have a Fourier expansion of the form

$$G_{s_1,\ldots,s_l}(\tau) = \sum_{n \geq 0} a_n q^n$$

with $q = e^{2\pi i \tau}$.

Question: How to calculate the a_n ?

Multiple Eisenstein series - Fourier expansion

To calculate the Fourier expansion we rewrite the multiple Eisenstein series as

$$G_{s_1,\dots,s_l}(\tau) = \sum_{\lambda_1 \succ \dots \succ \lambda_l \succ 0} \frac{1}{\lambda_1^{s_1} \dots \lambda_l^{s_l}}$$
$$= \sum_{(\lambda_1,\dots,\lambda_l) \in P^l} \frac{1}{(\lambda_1 + \dots + \lambda_l)^{s_1} (\lambda_2 + \dots + \lambda_l)^{s_2} \dots (\lambda_l)^{s_l}}$$

Multiple Eisenstein series - Fourier expansion

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$$= \sum_{(\lambda_1,\dots,\lambda_l) \in P^l} \frac{1}{(\lambda_1 + \dots + \lambda_l)^{s_1} (\lambda_2 + \dots + \lambda_l)^{s_2} \dots (\lambda_l)^{s_l}}$$

We decompose the set of tuples of positive lattice points P^l into the 2^l distinct subsets $A_1 \times \dots \times A_l \subset P^l$ with $A_i \in \{R, U\}$ and write

$$G^{A_1 \dots A_l}_{s_1,\dots,s_l}(\tau) := \sum_{(\lambda_1,\dots,\lambda_l) \in A_1 \times \dots \times A_l} \frac{1}{(\lambda_1 + \dots + \lambda_l)^{s_1} (\lambda_2 + \dots + \lambda_l)^{s_2} \dots (\lambda_l)^{s_l}}$$

this gives the decomposition

$$G_{s_1,\dots,s_l} = \sum_{A_1,\dots,A_l \in \{R,U\}} G^{A_1 \dots A_l}_{s_1,\dots,s_l} .$$

In the following we identify the $A_1 \dots A_l$ with words in the alphabet $\{R, U\}$.

Multiple Eisenstein series - Fourier expansion

In length $l = 1$ we have $G_k(\tau) = G_k^R(\tau) + G_k^U(\tau)$ and

$$G_k^R(\tau) = \sum_{\substack{m_1=0 \\ n_1>0}} \frac{1}{(0\tau + n_1)^k} = \zeta(k)\,,$$

$$G_k^U(\tau) = \sum_{\substack{m_1>0 \\ n_1\in\mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^k} = \sum_{m_1>0} \Psi_k(m_1\tau)\,,$$

where Ψ_k is the so called monotangent function defined for $k > 1$ by

$$\Psi_k(x) = \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^k}\,.$$

To calculate the Fourier expansion of G_k^U one uses the Lipschitz formula.

Multiple Eisenstein series - Fourier expansion

Proposition (Lipschitz formula)

For $k > 1$ it is

$$\Psi_k(x) = \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} e^{2\pi i d x} \,.$$

With this we get

$$\begin{aligned} G_k^U(\tau) &= \sum_{m_1>0} \Psi_k(m_1\tau) = \sum_{m_1>0} \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} e^{2\pi i m_1 d\tau} \\ &= \frac{(-2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,, \end{aligned}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the classical divisor sum.

Multiple Eisenstein series - Fourier expansion

In general the $G^{U^l}_{s_1,\dots,s_l}$ can be written as

$$
\begin{aligned}
G^{U^l}_{s_1,\dots,s_l}(\tau) &= \sum_{\substack{m_1>\dots>m_l>0\\ n_1,\dots,n_l\in\mathbb{Z}}} \frac{1}{(m_1\tau+n_1)^{s_1}\dots(m_l\tau+n_l)^{s_l}}\\
&= \sum_{m_1>\dots>m_l>0} \Psi_{s_1}(m_1\tau)\dots\Psi_{s_l}(m_l\tau)\\
&= \frac{(-2\pi i)^{s_1+\dots+s_l}}{(s_1-1)!\dots(s_l-1)!} \sum_{\substack{m_1>\dots>m_l>0\\ d_1,\dots,d_l>0}} d_1^{s_1-1}\dots d_l^{s_l-1} q^{m_1d_1+\dots+m_ld_l}\\
&=: \frac{(-2\pi i)^{s_1+\dots+s_l}}{(s_1-1)!\dots(s_l-1)!} \sum_{n>0} \sigma_{s_1-1,\dots,s_l-1}(n) q^n\\
&=: (-2\pi i)^{s_1+\dots+s_l}[s_1,\dots,s_l]\,.
\end{aligned}
$$

We call the $\sigma_{r_1,\dots,r_l}$ **multiple divisor sums** and the functions $[s_1,\dots,s_l]\in\mathbb{Q}[[q]]$ **multiple divisor functions**.

Multiple Eisenstein series - Fourier expansion

The other special case $G^{R^l}_{s_1,\dots,s_l}$ can also be written down explicitly:

$$G^{R^l}_{s_1,\dots,s_l}(\tau) = \sum_{\substack{m_1=\dots=m_l=0\\ n_1>\dots>n_l>0}} \frac{1}{(0\tau+n_1)^{s_1}\dots(0\tau+n_l)^{s_l}} = \zeta(s_1,\dots,s_l)$$

What about the mixed terms in length $l>1$?

Multiple Eisenstein series - Fourier expansion

In length 2 we have $G_{s_1,s_2} = G^{RR}_{s_1,s_2} + G^{UR}_{s_1,s_2} + G^{RU}_{s_1,s_2} + G^{UU}_{s_1,s_2}$ and

$$G^{UR}_{s_1,s_2} = \sum_{\substack{m_1>0, m_2=0 \\ n_1\in\mathbb{Z}, n_2>0}} \frac{1}{(m_1\tau+n_1)^{s_1}(0\tau+n_2)^{s_1}}$$
$$= \sum_{m_1>0} \Psi_{s_1}(m_1\tau) \sum_{n_2>0} \frac{1}{n_2^{s_2}} = (-2\pi i)^{s_1}[s_1]\zeta(s_2)\,,$$

$$G^{RU}_{s_1,s_2}(\tau) = \sum_{\substack{m_1=0, m_2>0 \\ n_1>n_2 \\ n_i\in\mathbb{Z}}} \frac{1}{(m_1\tau+n_1)^{s_1}(m_1\tau+n_2)^{s_2}} = \sum_{m>0} \Psi_{s_1,s_2}(m\tau).$$

where we call $\Psi_{s_1,s_2}(x) = \sum_{n_1>n_2} \frac{1}{(x+n_1)^{s_1}(x+n_2)^{s_2}}$ the multitangent function of length 2.

Multiple Eisenstein series - Fourier expansion

Using partial fraction expansion one can show that

$$\Psi_{s_1,s_2}(x) = \sum_{k_1+k_2=s_1+s_2} \left((-1)^{s_2} \binom{k_2-1}{s_2-1} + (-1)^{k_1-s_1} \binom{k_2-1}{s_1-1} \right) \zeta(k_2) \Psi_{k_1}(x).$$

and therefore

$$G^{RU}_{s_1,s_2}(\tau) = \sum_{m>0} \Psi_{s_1,s_2}(m\tau)$$

$$= \sum_{m>0} \sum_{k_1+k_2=s_1+s_2} \left((-1)^{s_2} \binom{k_2-1}{s_2-1} + (-1)^{k_1-s_1} \binom{k_1-1}{s_1-1} \right) \zeta(k_2) \Psi_{k_1}(m\tau)$$

$$= \sum_{k_1+k_2=s_1+s_2} \left((-1)^{s_2} \binom{k_2-1}{s_2-1} + (-1)^{k_2-s_1} \binom{k_1-1}{s_1-1} \right) \zeta(k_2) (-2\pi i)^{k_1} [k_1].$$

Therefore we obtain for the Fourier expansion of the double Eisenstein series

$$\begin{aligned} G_{s_1,s_2}(\tau) &= G^{RR}_{s_1,s_2} + G^{UR}_{s_1,s_2} + G^{RU}_{s_1,s_2} + G^{U}_{s_1,s_2} \\ &= \zeta(s_1,s_2) + (-2\pi i)^{s_1}[s_1]\zeta(s_2) \\ &+ \sum_{k_1+k_2=s_1+s_2} C^{k_2}_{s_1,s_2}\zeta(k_2)(-2\pi i)^{k_1}[k_1] + (-2\pi i)^{s_1+s_2}[s_1,s_2]\,. \end{aligned}$$

where

$$C^{k_2}_{s_1,s_2} := (-1)^{s_2}\binom{k_2-1}{s_2-1} + (-1)^{k_2-s_1}\binom{k_2-1}{s_1-1}\,.$$

Multiple Eisenstein series - Fourier expansion

In the case G^{UR} we saw that we could write it as G^{U} multiplied with a zeta value.

In general having a word w of length l ending in the letter R, i.e. there is a word w' ending in U with $w = w'R^r$ and $1 \leq r \leq l$ we can write

$$G^{w}_{s_1,\dots,s_l}(\tau) = G^{w'}_{s_1,\dots,s_{l-r}}(\tau) \cdot \zeta(s_{l-r+1},\dots,s_l)\,.$$

Example: $G^{RUURR}_{3,4,5,6,7} = G^{RUU}_{3,4,5} \cdot \zeta(6,7)$

Hence one can concentrate on the words ending in U when calculating the Fourier expansion of a multiple Eisenstein series.

Multiple Eisenstein series - Fourier expansion

Definition

For $s_1, \dots, s_l \geq 2$ we define the multitangent function of length l by

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{\substack{n_1 > \dots > n_l \\ n_i \in \mathbb{Z}}} \frac{1}{(x+n_1)^{s_1} \dots (x+n_l)^{s_l}} .$$

In the case $l = 1$ we also refer to these as monotangent function.

Multiple Eisenstein series - Fourier expansion

Definition

For $s_1, \dots, s_l \geq 2$ we define the multitangent function of length l by

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{\substack{n_1 > \dots > n_l \\ n_i \in \mathbb{Z}}} \frac{1}{(x+n_1)^{s_1} \dots (x+n_l)^{s_l}} .$$

In the case $l = 1$ we also refer to these as monotangent function.

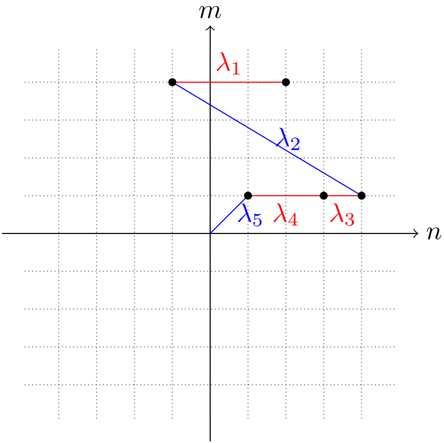
Let w a word ending in U then there are integers $r_1, \dots, r_j \geq 0$ with $w = R^{r_1} U R^{r_2} U \dots R^{r_j} U$. With this one can write

$$G^w_{s_1,\dots,s_l}(\tau) = \sum_{m_1 > \dots > m_j > 0} \Psi_{s_1,\dots,s_{r_1+1}}(m_1\tau) \cdot \Psi_{s_{r_1+2},\dots}(m_2\tau) \dots \Psi_{s_{l-r_j},\dots,s_l}(m_j\tau) .$$

This will become clear in an example...

Example: $w = RURRU$

$$G^{RURRU}_{s_1,\dots,s_l} = \sum_{m_1 > m_2 > 0} \Psi_{s_1,s_2}(m_1\tau)\Psi_{s_3,s_4,s_5}(m_2\tau)$$



A summand of $G^{RURRU}_{s_1,\dots,s_l}$.

To calculate the Fourier expansion of such terms we need the following theorem which reduces the multitangent functions into monotangent functions.

Theorem (Bouillot 2011, B. 2012)

Let $\mathcal{MZ}_k$ be the $\mathbb{Q}$-vector space spanned by all MZVs of weight k. Then for $s_1, \dots, s_l \geq 2$ and $k = s_1 + \cdots + s_l$ the multitangent function can be written as

$$\Psi_{s_1,\dots,s_l}(x) = \sum_{h=2}^{k} c_{k-h} \Psi_h(x)$$

with $c_{k,h} \in \mathcal{MZ}_{k-h}$.

Proof idea: Use partial fraction decomposition.

Multiple Eisenstein series - Fourier expansion

To summarize one can compute the Fourier expansion of the multiple Eisenstein series $G_{s_1,\dots,s_l}$ in the following way

- Split up the summation into 2^l distinct parts $G^w_{s_1,\dots,s_l}$ where w are a words in $\{R,U\}$.
- For w being a word ending in R one can write $G^w_{s_1,\dots,s_l}$ as $G^{w'}_{s_1,\dots} \cdot \zeta(\dots,s_l)$ with a word w' ending in U.
- For w being a word ending in U one can write $G^w_{s_1,\dots,s_l}$ as

$$G^w_{s_1,\dots,s_l}(\tau) = \sum_{m_1 > \dots > m_j > 0} \Psi_{s_1,\dots}(m_1\tau)\dots\Psi_{\dots,s_l}(m_l\tau)\,.$$

- Using the reduction theorem for multitangent functions this can be written as a MZV-linear combination of sums of the form

$$\sum_{m_1 > \dots > m_j > 0} \Psi_{k_1}(m_1\tau)\dots\Psi_{k_j}(m_j\tau) = (2\pi i)^{k_1+\dots+k_l}[k_1,\dots,k_l]$$

for which the Fourier expansions are known.

Examples

A few examples:

$$G_{4,4}(\tau) = \zeta(4,4) + 20\zeta(6)(2\pi i)^2[2] + 3\zeta(4)(2\pi i)^4[4] + (2\pi i)^8[4,4]\,,$$

$$\begin{aligned} G_{3,2,2}(\tau) =& \zeta(3,2,2) + \left(\frac{54}{5}\zeta(2,3) + \frac{51}{5}\zeta(3,2)\right)(2\pi i)^2[2] \\ &+ \frac{16}{3}\zeta(2,2)(2\pi i)^3[3] + 3\zeta(3)(2\pi i)^4[2,2] + 4\zeta(2)(2\pi i)^5[3,2] \\ &+ (2\pi i)^7[3,2,2]\,. \end{aligned}$$

In the following we also use the notation

$$\tilde{\zeta}(s_1,\dots,s_l) = (-2\pi i)^{-(s_1+\dots+s_l)}\zeta(s_1,\dots,s_l)$$

and similarly $\tilde{G}(s_1,\dots,s_l)$.

Examples

$$
\begin{aligned}
\tilde{G}_{4,5,6}(\tau) &= (-2\pi i)^{-15} G_{4,5,6}(\tau) = \tilde{\zeta}(4,5,6) \\
&+ \frac{(-1)^{4+5+6}}{c} \sum_{n>0} \left(\sigma_{3,4,5}(n) - \frac{281}{2882880}\sigma_0(n) + \frac{130399}{605404800}\sigma_2(n) - \frac{37}{1330560}\sigma_4(n) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(3600\sigma_4(n)\tilde{\zeta}(6,4) + 293760\sigma_2(n)\tilde{\zeta}(7,5) + 302400\sigma_2(n)\tilde{\zeta}(8,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(1814400\sigma_0(n)\tilde{\zeta}(8,6) + 2903040\sigma_0(n)\tilde{\zeta}(9,5) + 2177280\sigma_0(n)\tilde{\zeta}(10,4) \right) q^n \\
&- \frac{1}{c} \sum_{n>0} \left(-\frac{1}{168}\sigma_{2,5}(n) - \frac{1}{120}\sigma_{3,2}(n) + \frac{1}{168}\sigma_{3,4}(n) + \frac{1}{240}\sigma_{4,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(-\frac{\zeta(5)}{20\pi^5}\sigma_1(n) + \frac{\zeta(5)}{14\pi^5}\sigma_3(n) - \frac{\zeta(5)}{80\pi^5}\sigma_5(n) - \frac{3\zeta(5)}{\pi^5}\sigma_{3,5}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{45\zeta(5)^2}{32\pi^{10}}\sigma_4(n) + \frac{25\zeta(7)}{64\pi^7}\sigma_1(n) + \frac{21\zeta(7)}{32\pi^7}\sigma_3(n) - \frac{105\zeta(7)}{64\pi^7}\sigma_5(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{315\zeta(7)}{8\pi^7}\sigma_{1,5}(n) - \frac{315\zeta(7)}{4\pi^7}\sigma_{3,3}(n) - \frac{2835\zeta(5)\zeta(7)}{16\pi^{12}}\sigma_2(n) + \frac{42525\zeta(7)^2}{128\pi^{14}}\sigma_0(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{189\zeta(9)}{16\pi^9}\sigma_1(n) - \frac{945\zeta(9)}{16\pi^9}\sigma_3(n) + \frac{1125\zeta(9)}{64\pi^9}\sigma_5(n) + \frac{2835\zeta(9)}{4\pi^9}\sigma_{3,1}(n) \right) q^n \\
&- \frac{i}{c} \sum_{n>0} \left(\frac{8505\zeta(5)\zeta(9)}{16\pi^{14}}\sigma_0(n) + \frac{28755\zeta(11)}{64\pi^{11}}\sigma_3(n) - \frac{135135\zeta(13)}{128\pi^{13}}\sigma_1(n) \right) q^n, \quad (c = 3! \cdot 4! \cdot 5!)
\end{aligned}
$$

Multiple Eisenstein series

The sum in the definition of $G_{s_1,\ldots,s_l}$ is absolutely convergent for $s_1 \geq 3$ and $s_2, \ldots, s_l \geq 2$. Therefore using the same combinatorial argument as in the MZV case these functions fulfill the **stuffle product**, for example it is

$$G_4(\tau) \cdot G_6(\tau) = G_{4,6}(\tau) + G_{6,4}(\tau) + G_{10}(\tau)\,.$$

But the **shuffle product** can't be fulfilled because for example it is

$$\zeta(4)\zeta(6) = \zeta(4,6) + 4\zeta(4,6) + 11\zeta(6,4) + 26\zeta(7,3) + 56\zeta(8,2) + 112\zeta(9,1)$$

and this equation does not make sense in terms of multiple Eisenstein series because we didn't define $G_{9,1}$.

What about the (extended) double shuffle relations when all corresponding multiple Eisenstein series are defined? For this we need to recall the definition of modular forms and cusp forms.

Modular forms

Definition

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a *modular form* of weight k if it satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and if it has a Fourier expansion of the form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ with $a_n \in \mathbb{C}$ and $q = e^{2\pi i \tau}$. If $a_0 = 0$ then f is called cusp form.

The Eisenstein series G_k for even k are the building blocks of all modular forms and it is well known that the space of all modular forms is a graded algebra given by $\mathbb{C}[G_4, G_6]$.

Modularity

Because of of the stuffle product we have for example

$$G_4^2 = 2G_{4,4} + G_8$$

so $G_{4,4}$ is a modular form of weight 8. The space of weight 8 modular forms has dimension one and therefore $G_{4,4}$ is a multiple of G_8 from which one can deduce the following relations between multiple divisor functions

$$[8] = 12[4,4] + \frac{1}{40}[4] - \frac{1}{252}[2]\,,$$

which can be proven without using the theory of modular forms (see tomorrow). In general we have

Theorem

If all $s_1, ..., s_l$ are even and all $s_j > 2$, then we have

$$\sum_{\sigma \in \Sigma_l} G_{s_{\sigma(1)},\dots,s_{\sigma(l)}} \in M_k(\mathrm{SL}_2(\mathbb{Z}))\,,$$

where the weight k is given by $k = s_1 + ... + s_l$.

Proof: Easy induction using stuffle relation.

Modularity

By the double shuffle relations and Eulers formula $\zeta(2k) = \lambda \cdot \pi^{2k}$ for $\lambda \in \mathbb{Q}$ one can show:

$$
\begin{aligned}
&-\frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17}\zeta(12) - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17}\zeta(6,3,3) \\
&\quad - \frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17}\zeta(4,5,3) + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17}\zeta(7,5) \\
&= -40\zeta(4)^3 + 49\zeta(6)^2 \\
&= 0 \text{ because of Euler's formula} .
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But in the context of multiple Eisenstein series we get:

Theorem

$$
\begin{aligned}
&-\frac{2^5 \cdot 3 \cdot 5 \cdot 757}{17}\tilde{G}_{12} - \frac{2^9 \cdot 3 \cdot 5^2 \cdot 7 \cdot 691}{17}\tilde{G}_{6,3,3} \\
&-\frac{2^9 \cdot 3^2 \cdot 5 \cdot 7 \cdot 691}{17}\tilde{G}_{4,5,3} + \frac{2^8 \cdot 3^2 \cdot 5^2 \cdot 691}{17}\tilde{G}_{7,5} = \Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z})) .
\end{aligned}
$$

Modularity & cusp forms

Proof: The first identity of the above MZV Relation hold also for multiple Eisensten series. It follows from the Stuffle relation and partial fraction decompositions which replaces Shuffle.
But in the second identity, i.e., the place when Eulers formula is needed, one gets the "error term" , because in general whenever $s_1 + s_2 \geq 12$ the following function doesn't vanish

$$G_{s_1} \cdot G_{s_2} - \frac{\zeta(s_1)\zeta(s_2)}{\zeta(s_1 + s_2)} G_{s_1+s_2} \in S_{s_1+s_2}(\mathrm{SL}_2(\mathbb{Z})).$$

So the failure of Euler's relation give us the cusp forms $\square$

Remark

- There are many more such linear relations which give cusp forms

Modularity, cusp forms

From such identities we get new relations between Fourier coefficients of modular forms and multiple divisor sums, e.g.:

Corollary - Formula for the Ramanujan τ-function

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\tau(n) = {} & \frac{2\cdot 7\cdot 691}{3^2\cdot 11\cdot 17}\sigma_1(n) - \frac{43\cdot 691}{2^3\cdot 3^2\cdot 5\cdot 17}\sigma_3(n) + \frac{691}{2\cdot 3^3\cdot 7\cdot 17}\sigma_5(n) \\ & - \frac{757}{2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 17}\sigma_{11}(n) - \frac{2^3\cdot 5\cdot 691}{3\cdot 17}\sigma_{2,2}(n) + \frac{2^2\cdot 5\cdot 691}{3\cdot 17}\sigma_{3,1}(n) \\ & - \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{3,3}(n) + \frac{2\cdot 7\cdot 691}{17}\sigma_{4,2}(n) + \frac{2^2\cdot 7\cdot 691}{3\cdot 17}\sigma_{5,1}(n) \\ & + \frac{2\cdot 5\cdot 691}{3\cdot 17}\sigma_{6,4}(n) + \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{3,4,2}(n) - \frac{2^4\cdot 5\cdot 7\cdot 691}{17}\sigma_{5,2,2}(n)\,.\end{aligned}$$

$$\Delta(\tau) = q\prod_{i=1}^{\infty}(1-q^n)^{24} = \sum_{n>0}\tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + \ldots$$

"Non convergent" multiple Eisenstein series

The multiple Eisenstein series are defined for $s_1 \geq 3$ and $s_2, \ldots, s_l \geq 2$ but the MZVs are defined for $s_1 \geq 2$ and $s_2, \ldots, s_l \geq 1$.

Question: What should $G_{s_1,\ldots,s_l}$ be for $s_1 \geq 2$ and $s_2, \ldots, s_l \geq 1$?

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Question: What should $G_{s_1,\ldots,s_l}$ be for $s_1 \geq 2$ and $s_2, \ldots, s_l \geq 1$?

The answer depends on what properties you want these functions to fulfill. If you want them to fulfill the double shuffle relations (up to cusp forms) then this questions is answered for length $l = 2$ by Gangl, Kaneko and Zagier. If you want them to have the algebraic structure of a stuffle algebra then this is currently work in progress (Bouillot, B.). In the following we will focus on the "double shuffle relations"-version.

"Non convergent" multiple Eisenstein series

Theorem (GKZ)

Define for $s_1 \geq 2, s_2 \geq 1$ the double Eisenstein series by their Fourier expansion

$$\tilde{G}_{s_1,s_2}(\tau) = \tilde{\zeta}(s_1, s_2) + \frac{(-1)^{s_1+s_2}}{(s_1-1)!(s_2-1)!} \sum_{n>0} a_n q^n \,,$$

where

$$\begin{aligned} a_n = {} & \sigma_{s_1-1,s_2-1}(n) + (-1)^{s_2}(s_2-1)!\tilde{\zeta}(s_2)\sigma_{s_1-1}(n) \\ & + (s_1-1)!(s_2-1)! \sum_{k_1+k_2=s_1+s_2} C^{k_2}_{s_1,s_2} \tilde{\zeta}(k_2)\sigma_{k_1-1}(n) \\ & + \delta_{s_2,1}\left(\frac{n}{2}\sigma_{s_1-2}(n) - \frac{1}{2}\sigma_{s_1-1}(n)\right) \\ & + \delta_{s_1,2}\left(\frac{n}{2s_2}\sigma_{s_2-1}(n)\right) \,, \end{aligned}$$

Then these functions fulfill the (extended) double shuffle relations modulo cusp forms.

"Non convergent" multiple Eisenstein series

For example these extended version of double Eisenstein series fulfill $G_3 = G_{2,1}$. This is equivalent to $[3] = [2,1] - \frac{1}{2}[2] + \mathrm{d}[1]$ which follows also from the theory of multiple divisor functions (tomorrow).

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What about $G_{2,\dots,2}$? We want our multiple Eisenstein series to fulfill the same linear relations as the corresponding MZV (modulo cusp forms), therefore we have to imitate the following well known result in the context of multiple Eisenstein series:

Theorem

For $\lambda_n := (-1)^{n-1} \cdot 2^{2n-1} \cdot (2n+1) \cdot B_{2n}$ we have

$$\zeta(2n) - \lambda_n \cdot \zeta(\underbrace{2, ..., 2}_{n}) = 0\,.$$

"Non convergent" multiple Eisenstein series

Ansatz: Define $G_{2,\dots,2}$ to be the function obtained by setting all s_i to 2 in the formula of the Fourier Expansion. e.g.

$$G_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n>0} \sigma_1(n) q^n \,,$$

$$G_{2,2}(\tau) = \zeta(2,2) + (-2\pi i)^4 \sum_{n>0} \left(\sigma_{1,1}(n) - \frac{1}{8} \sigma_1(n) \right) q^n \,.$$

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Will this give the "right" definition of $G_{2,\dots,2}$?
No, because with this definition the function

$$G_{2n} - \lambda_n \cdot G_{\underbrace{2,\dots,2}_{n}} \notin S_{2n}(\mathrm{SL}_2(\mathbb{Z}))$$

is not a cusp form. It is not modular but quasi-modular, this property is used for the following modified definition:

Theorem

Let $X_n(\tau) := (2\pi i)^{-2l} G_{\underbrace{2,\,\ldots,\,2}_{n}}(\tau)$, $\mathrm{d} := q\frac{d}{dq}$ and

$$\tilde{G}^*_{\underbrace{2,\,\ldots,\,2}_{n}}(\tau) := X_n(\tau) + \sum_{j=1}^{n-1} \frac{(2n-2-j)!}{2^j \cdot j! \cdot (2n-2)!} \, \mathrm{d}^j \, X_{n-j}(\tau)\,.$$

then we have with $\lambda_n \in \mathbb{Q}$ as above

$$\tilde{G}_{2n} - \lambda_n \cdot \tilde{G}^*_{2,\ldots,2} \in S_{2n}(\mathrm{SL}_2\,\mathbb{Z})\,.$$

The multiple Eisenstein series $G^*_{2,\dots,2}$

Examples:

$$\begin{aligned}
G^*_{2,2} &= \tilde{G}_{2,2} + \frac{1}{4}\,\mathrm{d}\,\tilde{G}_2\,,\\
G^*_{2,2,2} &= \tilde{G}_{2,2,2} + \frac{1}{8}\,\mathrm{d}\,\tilde{G}_{2,2} + \frac{1}{96}\,\mathrm{d}^2\,\tilde{G}_2\,,\\
G^*_{2,2,2,2} &= \tilde{G}_{2,2,2,2} + \frac{1}{12}\,\mathrm{d}\,\tilde{G}_{2,2,2} + \frac{1}{240}\,\mathrm{d}^2\,\tilde{G}_{2,2} + \frac{1}{5760}\,\mathrm{d}^3\,\tilde{G}_2\,.\\
&\dots
\end{aligned}$$

In weight 12 we get

$$\tilde{G}_{12} - \lambda_6 \cdot \tilde{G}^*_{2,2,2,2,2,2} = \frac{17}{3^6 \cdot 5^4 \cdot 7^2}\Delta$$

which gives another expression for $\tau(n)$ in $\sigma_{11}(n)$ and $\sigma_{1,\dots,1}(n)$ (see last slide).

Sketch of the proof

- First show following identity for the generating function of X_n

$$\Phi(\tau,T) := \sum_{n\geq 0} X_n(\tau)(-4\pi i T)^n = \exp\left(-2\sum_{l\geq 1}\frac{(-1)^l}{(2l)!}E_{2l}(\tau)(-4\pi i T)^l\right)$$

where $E_k(\tau) = -\frac{B_k}{2k} + \sum_{n>0}\sigma_{k-1}(n)q^n$.

- Using the modularity of the E_k for $k>2$ one sees easily that Φ is a Jacobi-like form of weight 0, i.e.:

$$\Phi\left(\frac{a\tau+b}{c\tau+d},\frac{T}{(c\tau+d)^2}\right) = \exp\left(\frac{cT}{(c\tau+d)}\right)\Phi(\tau,T) \quad ,\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

- One can show that coefficients of such functions always give rise to modular forms as above.

Application

New formular for fourier coefficients of cusp forms, e.g.:

$$
\begin{aligned}
-\frac{17}{9794925}\tau(n) = &\left(\frac{n^5}{7560} - \frac{n^4}{1008} + \frac{13n^3}{4320} - \frac{41n^2}{9072} + \frac{671n}{201600} - \frac{73}{76032}\right)\sigma_1(n) \\
&+ \left(\frac{1}{126}n^4 - \frac{5}{54}n^3 + \frac{23}{54}n^2 - \frac{227}{252}n + \frac{631}{864}\right)\sigma_{1,1}(n) \\
&+ \left(\frac{4}{9}n^3 - \frac{56}{9}n^2 + \frac{154}{5}n - \frac{479}{9}\right)\sigma_{1,1,1}(n) \\
&+ \left(\frac{64}{3}n^2 - 288n + 1032\right)\sigma_{1,1,1,1}(n) \\
&+ (768n - 7040)\,\sigma_{1,1,1,1,1}(n) \\
&+ 15360\sigma_{1,1,1,1,1,1}(n) - \frac{1}{17512704}\sigma_{11}(n)
\end{aligned}
$$

Reminder:

$$
\sigma_{\underbrace{1,\ldots,1}_{l}}(n) = \sum_{\substack{u_1 v_1 + \ldots + u_l v_l = n \\ u_1 > \ldots > u_l > 0}} v_1 \cdot \ldots \cdot v_l \, .
$$

Summary

- Multiple Eisenstein series are holomorphic functions on the upper half plane which are defined as a sum over ordered lattice points.
- They have a Fourier expansion where the constant term is given by the corresponding multiple zeta value and the remaining terms are rational linear combinations of products of multiple zeta values and multiple divisor functions.
- A general (i.e. $s_1 \geq 2$ and $s_2, \dots, s_l \geq 1$) definition of $G_{s_1,\dots,s_l}$ such that these functions fulfill similar properties as multiple zeta values (for example double shuffle relations or the algebraic structure) is still an open question.
- We believe that this question can be understand better when studying the multiple divisor functions on its own.