Shuffle regularized multiple Eisenstein series and the Goncharov coproduct

Henrik Bachmann - University of Hamburg

joint work with Koji Tasaka (PMI, POSTECH)

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Content of this talk

In this talk we want to explain a connection between the Fourier expansion of multiple Eisenstein series and the Goncharov coproduct given on a space of formal iterated integrals. This enables us to define shuffle regularized multiple Eisenstein series $G^{\sqcup\!\sqcup}$.

- Multiple zeta values
- Multiple Eisenstein series and their Fourier expansion
- Formal iterated integrals and the Goncharov coproduct
- Shuffle regularized multiple zeta values and certain q-series
- Shuffle regularized multiple Eisenstein series
- Open questions

Multiple zeta values

Definition

For natural numbers $n_1, ..., n_{r-1} \geq 1, n_r \geq 2$, the multiple zeta value (MZV) of weight $N = n_1 + ... + n_r$ and length r is defined by

$$\zeta(n_1, ..., n_r) = \sum_{0<m_1<...<m_r} \frac{1}{m_1^{n_1} \dots m_r^{n_r}} .$$

By $\mathcal{MZ}_N$ we denote the space spanned by all MZV of weight N and by $\mathcal{MZ}$ the space spanned by all MZV.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (stuffle relation). e.g:

$$\zeta(r) \cdot \zeta(s) = \zeta(r, s) + \zeta(s, r) + \zeta(r + s) .$$

- MZV can be expressed as iterated integrals. This gives another way (shuffle relation) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.

Multiple zeta-values

Example:

$$\zeta(3,2) + 3\zeta(2,3) + 6\zeta(1,4) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(2,3) + 6\zeta(1,4) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(1,2) = \zeta(3).$$

These follow from the "extended double shuffle relations" where one use the same combinatorics as above for "$\zeta(1)\cdot\zeta(2)$" in a formal setting. The extended double shuffle relations are conjectured to give all relations between MZV.

Classical Eisenstein series

For even $k > 2$ the Eisenstein series of weight k defined by

$$G_k^{\spadesuit}(\tau) := \frac{1}{2} \sum_{\substack{(l,m)\in\mathbb{Z}^2 \\ (l,m)\neq(0,0)}} \frac{1}{(l\tau+m)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

are modular forms of weight k. These functions vanish for odd k (and there are no non trivial modular forms of odd weight) since one sums over all lattice points.

Similary if one would define the riemann zeta value as a sum over all integer, i.e.

$$\zeta^{\spadesuit}(k) := \frac{1}{2} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n^k}$$

then these series would vanish for odd k and $\zeta^{\spadesuit}(k) = \zeta(k)$ for even k.

We now want to define a multiple version of these series where we also have odd Eisenstein series in length one. For this we define an order on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ by defining what we mean by positive lattice points.

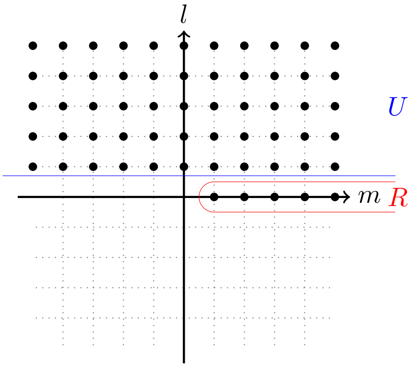
A particular order on lattices

Let $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ be a lattice with $\tau \in \mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$. We define an order $\prec$ on Λ_τ by setting

$$\lambda_1 \prec \lambda_2 :\Leftrightarrow \lambda_2 - \lambda_1 \in P$$

for $\lambda_1, \lambda_2 \in \Lambda_\tau$ and the following set which we call the set of positive lattice points

$$P := \{l\tau + m \in \Lambda_\tau \mid l > 0 \vee (l = 0 \wedge m > 0)\} = U \cup R$$



Classical Eisenstein series are ordered sums

With this order on Λ_τ one gets for even $k > 2$:

$$G_k(\tau) := \sum_{0 \prec \lambda} \frac{1}{\lambda^k} = \frac{1}{2} \sum_{\substack{(l,m)\in\mathbb{Z}^2 \\ (l,m)\neq(0,0)}} \frac{1}{(l\tau+m)^k} = G_k^{\spadesuit}(\tau).$$

Since we are not summing over all lattice points the odd Eisenstein series don't vanish anymore and we get for **all** k:

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

This order now allows us to define a multiple version of these series in the obvious way.

Multiple Eisenstein series

Definition

For integers $n_1, \ldots, n_{r-1} \geq 2$ and $n_r \geq 3$, we define the **multiple Eisenstein series** $G_{n_1,\ldots,n_r}(\tau)$ on $\mathbb{H}$ by

$$G_{n_1,\ldots,n_r}(\tau) = \sum_{\substack{0 \prec \lambda_1 \prec \cdots \prec \lambda_r \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{n_1} \cdots \lambda_r^{n_r}}\,.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the stuffle product, i.e. it is for example

$$G_3(\tau) \cdot G_4(\tau) = G_{4,3}(\tau) + G_{3,4}(\tau) + G_7(\tau)\,.$$

Remark

The condition $n_r \geq 3$ is necessary for absolutely convergence of the sum. By choosing a specific way of summation we can also restrict this condition to get a definition of $G_{n_1,\ldots,n_r}(\tau)$ with $n_r = 2$ which also satisfies the stuffle product.

Multiple Eisenstein series - Fourier expansion

Since $G_{s_1,\ldots,s_r}(\tau+1) = G_{s_1,\ldots,s_r}(\tau)$ we have a Fourier expansion:

Theorem (B. 2012)

There exists a family of arithmetically defined q-series $g_{n_1,\ldots,n_r}(q) \in \mathbb{Q}[2\pi i][[q]]$ such that for $n_1,\ldots,n_r \geq 2$ the Fourier expansion of $G_{n_1,\ldots,n_r}$ is given by

$$\begin{aligned} G_{n_1,\ldots,n_r}(\tau) &= \zeta(n_1,\ldots,n_r) \\ &+ \sum_{\substack{k_1+k_2=N \\ k_1,k_2\geq 2}} \xi_{k_1}^{(r-1)} g_{k_2}(q) + \sum_{\substack{k_1+k_2+k_3=N \\ k_1,k_2,k_3\geq 2}} \xi_{k_1}^{(r-2)} g_{k_2,k_3}(q) \\ &+ \cdots + \sum_{\substack{k_1+\cdots+k_r=N \\ k_1,\ldots,k_r\geq 2}} \xi_{k_1}^{(1)} g_{k_2,\ldots,k_r}(q) + g_{n_1,\ldots,n_r}(q), \end{aligned}$$

where the $\xi_k^{(d)} \in \mathcal{MZ}_k$ are $\mathbb{Q}$-linear combinations of multiple zeta values of weight k and length less than or equal to d.

The length $r=2$ case was originally considered by Gangl, Kaneko and Zagier. From now on we also write $G_{n_1,\ldots,n_r}(q)$ instead of $G_{n_1,\ldots,n_r}(\tau)$.

Multiple Eisenstein series - Fourier expansion - Multitangent functions

Let

$$\Psi_{n_1,\dots,n_r}(x) := \sum_{m_1<\dots<m_r} \frac{1}{(x+m_1)^{n_1}\cdots(x+m_r)^{n_r}}.$$

This series absolutely converges for $n_i \in \mathbb{Z}_{>0}$ with $n_1, n_r \geq 2$, and is called **multitangent function**. In the case $r=1$ we also refer to these series as **monotangent function**.

Theorem (Bouillot 2011, B. 2012)

For $n_1,\dots,n_r \geq 2$ and $N = n_1+\dots+n_r$ the multitangent function can be written as

$$\Psi_{n_1,\dots,n_r}(x) = \sum_{j=2}^{N} \alpha_{N-j}\Psi_j(x)$$

with $\alpha_{N-j} \in \mathcal{MZ}_{N-j}$.

Proof idea: Use partial fraction decomposition.

Example:

$$\Psi_{2,3}(x) = \sum_{m_1<m_2} \frac{1}{(x+m_1)^2(x+m_2)^3}$$

$$= \sum_{m_1<m_2} \left(\frac{1}{(m_2-m_1)^3(x+m_1)^2} - \frac{3}{(m_2-m_1)^4(x+m_1)} \right) +$$

$$\sum_{m_1<m_2} \left(\frac{1}{(m_2-m_1)^2(x+m_2)^3} + \frac{2}{(m_2-m_1)^3(x+m_2)^2} + \frac{3}{(m_2-m_1)^4(x+m_2)} \right)$$

$$= 3\zeta(3)\Psi_2(x) + \zeta(2)\Psi_3(x)\,.$$

Since one can derive explicit formulas for the partial fraction expansions of $\frac{1}{(x+m_1)^{n_1}\ldots(x+m_r)^{n_r}}$ this reduction of multitangent into monotangent can be done explicitly.

Multiple Eisenstein series - Fourier expansion - The function g

For integers $n_1, \dots, n_r \geq 1$, define

$$g_{n_1,\dots,n_r}(q) := c \cdot \sum_{\substack{0<m_1<\dots<m_r \\ 0<v_1,\dots,v_r}} v_1^{n_1-1} \cdots v_r^{n_r-1} q^{m_1 v_1 + \dots + m_r v_r},$$

where $c = \frac{(-2\pi i)^{n_1+\dots+n_r}}{(n_1-1)!\cdots(n_r-1)!}$.

Proposition

For $n_1, \dots, n_r \geq 2$ these series can be written as an ordered sum of monotangent functions:

$$g_{n_1,\dots,n_r}(q) = \sum_{0<m_1<\dots<m_r} \Psi_{n_1}(m_1\tau) \dots \Psi_{n_r}(m_r\tau).$$

The q-series $g_{n_1,\dots,n_r}(q)$ divided by $(-2\pi i)^{n_1+\dots+n_r}$ are the generating series of multiple divisor sums $[n_r, \dots, n_1] \in \mathbb{Q}[[q]]$ which were studied in a joint paper with U. Kühn.

Multiple Eisenstein series - Fourier expansion

Summing over $0 \prec \lambda_1 \prec \cdots \prec \lambda_r$ is by definition equivalent to summing over all $\lambda_1, \dots, \lambda_r$ with

$$\lambda_i - \lambda_{i-1} \in P = U \cup R \qquad (\lambda_0 := 0)\,.$$

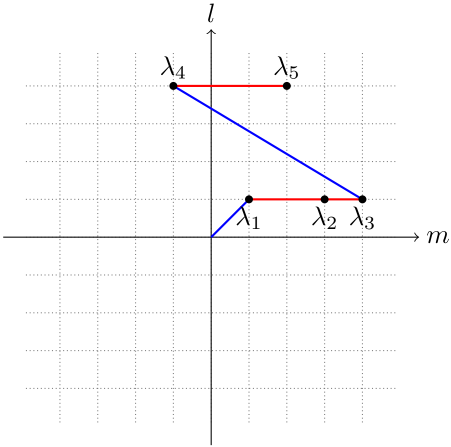
Since $\lambda_i - \lambda_{i-1}$ can be either in U or in R we can split up the sum in the definition of the MES into 2^r terms. For $w_1, \dots, w_r \in \{U, R\}$ we define

$$G^{w_1 \dots w_r}_{n_1,\dots,n_r}(\tau) = \sum_{\substack{\lambda_1,\dots,\lambda_r \in \Lambda_\tau \\ \lambda_i - \lambda_{i-1} \in w_i}} \frac{1}{\lambda_1^{n_1} \cdots \lambda_r^{n_r}}\,.$$

With this we get

$$G_{n_1,\dots,n_r}(\tau) = \sum_{w_1,\dots,w_r \in \{U,R\}} G^{w_1 \dots w_r}_{n_1,\dots,n_r}(\tau)\,.$$

Example: $URRUR$



A summand of $G^{URRUR}_{n_1,n_2,n_3,n_4,n_5}$.

By definition of the multitangent functions we can write

$$G^{URRUR}_{n_1,n_2,n_3,n_4,n_5}(\tau) = \sum_{0<l_1<l_2} \Psi_{n_1,n_2,n_3}(l_1\tau)\Psi_{n_4,n_5}(l_2\tau)\,.$$

Multiple Eisenstein series - Fourier expansion

In length $r = 2$ the $2^2 = 4$ terms are given by

$$G^{RR}_{n_1,n_2}(\tau) = \sum_{\substack{0=l_1=l_2\\0<m_1<m_2}} \frac{1}{(l_1\tau+m_1)^{n_1}(l_2\tau+m_2)^{n_2}} = \zeta(n_1,n_2),$$

$$G^{UR}_{n_1,n_2}(\tau) = \sum_{\substack{0<l_1=l_2\\m_1<m_2}} \frac{1}{(l_1\tau+m_1)^{n_1}(l_2\tau+m_2)^{n_2}} = \sum_{0<l} \Psi_{n_1,n_2}(l\tau),$$

$$G^{RU}_{n_1,n_2}(\tau) = \sum_{\substack{0=l_1<l_2\\0<m_1,m_2\in\mathbb{Z}}} \frac{1}{(l_1\tau+m_1)^{n_1}(l_2\tau+m_2)^{n_2}} = \zeta(n_1)\sum_{0<l} \Psi_{n_2}(l\tau),$$

$$G^{UU}_{n_1,n_2}(\tau) = \sum_{\substack{0<l_1<l_2\\m_1,m_2\in\mathbb{Z}}} \frac{1}{(l_1\tau+m_1)^{n_1}(l_2\tau+m_2)^{n_2}}$$

$$= \sum_{0<l_1<l_2} \Psi_{n_1}(l_1\tau)\Psi_{n_2}(l_2\tau).$$

Fourier expansion - example

$$G_{2,3}(\tau) = \zeta(2,3) + \sum_{0<l} \Psi_{2,3}(l\tau) + \zeta(2) \sum_{0<l} \Psi_3(l\tau) + \sum_{0<l_1<l_2} \Psi_2(l_1\tau)\Psi_3(l_2\tau)\,.$$

To evaluate the second term we use $\Psi_{2,3}(x) = 3\zeta(3)\Psi_2(x) + \zeta(2)\Psi_3(x)$ and obtain

$$G_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3) \sum_{0<l} \Psi_2(l\tau) + 2\zeta(2) \sum_{0<l} \Psi_3(l\tau) + \sum_{0<l_1<l_2} \Psi_2(l_1\tau)\Psi_3(l_2\tau)\,.$$

With this we get the Fourier expansion of $G_{2,3}$:

$$G_{2,3}(q) = \zeta(2,3) + 3\zeta(3)g_2(q) + 2\zeta(2)g_3(q) + g_{2,3}(q)\,.$$

Fourier expansion - general idea

The general idea to compute the Fourier expansion of $G_{n_1,\dots,n_r}(\tau)$:

- For each of the 2^r words of length r in the alphabet $\{U, R\}$, i.e a word of the form

$$w_1 \dots w_r = \underset{1}{R}\underset{2}{R} \dots R\underset{t_1}{U} R \dots R\underset{t_2}{U} \dots \underset{t_k}{U} R \dots \underset{r}{R},$$

  where $1 \leq t_1 < \cdots < t_k \leq r$ are the positions of the U, we get a term of the form

$$G^{w_1 \dots w_r}_{n_1,\dots,n_r} =$$
$$\zeta(n_1, \dots, n_{t_1-1}) \sum_{0<l_1<\dots<l_k} \Psi_{n_{t_1},\dots,n_{t_2-1}}(l_1\tau) \dots \Psi_{n_{t_k}\dots,n_r}(l_k\tau) \,.$$

- Reduce the multitangent functions $\Psi_{n_{t_j},\dots,n_{t_i-1}}(x)$ to a linear combination of MZV and monotangent functions $\Psi_n(x)$ and then write the remaining sums of monotangent functions in terms of the q-series g.

Summary: Multiple Eisenstein series

For $n_1, \ldots, n_r \geq 2$ the multiple Eisenstein series $G_{n_1,\ldots,n_r}(\tau)$ are holomorphic functions having a Fourier expansion with the multiple zeta value $\zeta(n_1, \ldots, n_r)$ as the constant term. By construction they fulfill the stuffle product.

This leads to the following questions:

- Is there a "good" definition of multiple Eisenstein series for $n_1, \ldots, n_r \geq 1$?
- Does then these multiple Eisenstein series fulfill the shuffle and stuffle product?
- Basis, dimension, modularity defect ?

The space of formal iterated integrals

Following Goncharov we consider the algebra $\mathcal{I}$ generated by the elements

$$\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1}), \quad a_i \in \{0,1\}, N \geq 0.$$

together with the following relations

(i) For any $a, b \in \{0,1\}$ the unit is given by $\mathbb{I}(a;b) := \mathbb{I}(a;\emptyset;b) = 1$.

(ii) The product is given by the shuffle product $\sqcup\!\sqcup$

$$\mathbb{I}(a_0; a_1, \ldots, a_M; a_{M+N+1})\mathbb{I}(a_0; a_{M+1}, \ldots, a_{M+N}; a_{M+N+1})$$
$$= \sum_{\sigma \in sh_{M,N}} \mathbb{I}(a_0; a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(M+N)}; a_{M+N+1}),$$

(iii) The path composition formula holds: for any $N \geq 0$ and $a_i, x \in \{0,1\}$, one has

$$\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1}) = \sum_{k=0}^{N} \mathbb{I}(a_0; a_1, \ldots, a_k; x)\mathbb{I}(x; a_{k+1}, \ldots, a_N; a_{N+1}).$$

(iv) For $N \geq 1$ and $a_i, a \in \{0,1\}$, $\mathbb{I}(a; a_1, \ldots, a_N; a) = 0$.

(v)*

$$\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1}) = (-1)^N \mathbb{I}(a_{N+1}; a_N, \ldots, a_1; a_0)$$

Formal iterated integrals - coproduct

Goncharov desfines a coproduct on $\mathcal{I}$ by

$$\Delta\left(\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1})\right) :=$$

$$\sum \left(\prod_{p=0}^{k} \mathbb{I}(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes \mathbb{I}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{N+1}),$$

where the sum runs over all $i_0 = 0 < i_1 < \cdots < i_k < i_{k+1} = N+1$ with $0 \leq k \leq N$.

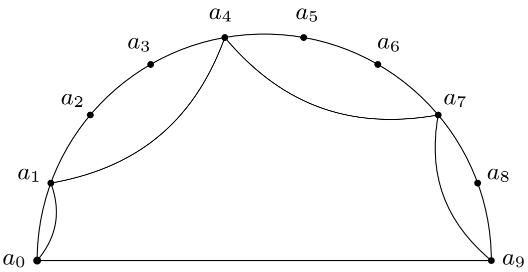
Proposition (Goncharov)

The triple $(\mathcal{I}, \sqcup\!\sqcup, \Delta)$ becomes a commutative graded Hopf algebra over $\mathbb{Q}$.

The calculation of Δ can be visualized by marking k of $N+2$ points on a half circle.

Coproduct - diagrams

To calculate $\Delta\left(\mathbb{I}(a_0; a_1, \ldots, a_8; a_9)\right)$ one sums over all possible diagrams of the following form.



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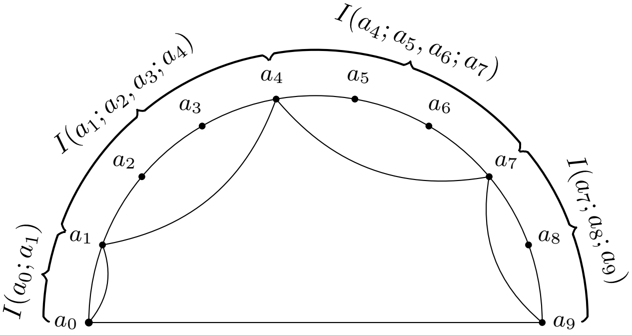


This diagram corresponds to the summand

$$I(a_0; a_1) I(a_1; a_2, a_3; a_4) I(a_4; a_5, a_6; a_7) I(a_7; a_8; a_9) \otimes I(a_0; a_1, a_4, a_7; a_9)\,.$$

The space $\mathcal{I}^1$

We will consider the quotient space

$$\mathcal{I}^1 = \mathcal{I}/\mathbb{I}(0;0;1)\mathcal{I}.$$

Let us denote by

$$I(a_0; a_1, \ldots, a_N; a_{N+1})$$

an image of $\mathbb{I}(a_0; a_1, \ldots, a_N; a_{N+1})$ in $\mathcal{I}^1$. The quotient map $\mathcal{I} \to \mathcal{I}^1$ seems to induce a Hopf algebra structure on $\mathcal{I}^1$, but for our application we just need the following statement.

Proposition

For any $w_1, w_2 \in \mathcal{I}^1$, one has $\Delta(w_1 ⧢ w_2) = \Delta(w_1) ⧢ \Delta(w_2)$.

The coproduct on $\mathcal{I}^1$ is given by the same formula as before by replacing $\mathbb{I}$ with I.

The space $\mathcal{I}^1$

For integers $n \geq 0, n_1, \dots, n_r \geq 1$, we set

$$I_{n}(n_1, \dots, n_r) := I(0; \underbrace{0, \dots, 0}_{n}, \underbrace{1, 0, \dots, 0}_{n_1}, \dots, \underbrace{1, 0, \dots, 0}_{n_r}; 1).$$

In particular, we write $I(n_1, \dots, n_r)$ to denote $I_0(n_1, \dots, n_r)$.

Proposition

- We have $I_n(\emptyset) = 0$ if $n \geq 1$ or 1 if $n = 0$.
- For integers $n \geq 0, n_1, \dots, n_r \geq 1$,

$$I_{n}(n_1, \dots, n_r) = (-1)^n \sum^{*} \left(\prod_{j=1}^{r} \binom{k_j - 1}{n_j - 1} \right) I(k_1, \dots, k_r).$$

  where the sum runs over all $k_1 + \cdots + k_r = n_1 + \cdots + n_r + n$ with $k_1, \dots, k_r \geq 1$.
- The set $\{I(n_1, \dots, n_r) \mid r \geq 0, n_i \geq 1\}$ forms a basis of the space $\mathcal{I}^1$.

Example : Write I_n as a linear combination in I's

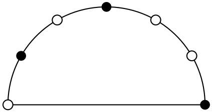
In $\mathcal{I}^1$ it is $I(0;0;1) = 0$ and therefore

$$\begin{aligned} 0 &= I(0;0;1)I(0;1,0;1) \\ &= I(0;0,1,0;1) + I(0;1,0,0;1) + I(0;1,0,0;1) \\ &= I_1(2) + 2I(3) \end{aligned}$$

which gives $I_1(2) = -2I(3) = (-1)^1 \binom{2}{1} I(3)$.

Coproduct - example

In the following we are going to calculate $\Delta(I(2,3)) = \Delta(I(0;1,0,1,0,0;1))$ and therefore we have to determine all possible markings of the diagram



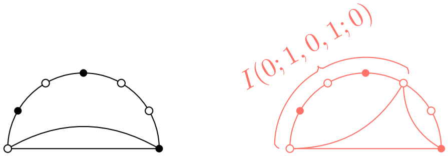
,

where the corresponding summand in the coproduct does not vanish. For simplicity we draw $\circ$ to denote a 0 and $\bullet$ to denote a 1.

We will consider the $4 = 2^2$ ways of marking the two $\bullet$ in the top part of the circle separately .

Calculation of $\Delta(I(2,3))$

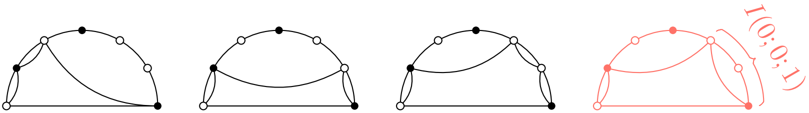
Diagrams with no marked ●:



Corresponding sum in the coproduct:

$$I(0;1,0,1,0,0;1)\otimes I(0;\emptyset;1)=I(2,3)\otimes 1\,.$$

Diagrams with the first ● marked:



Corresponding sum in the coproduct:

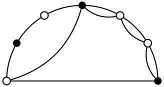
$$\begin{aligned} & I(0;1)\cdot I(1;0)\cdot I(0;1,0,0;1)\otimes I(0;1,0;1) \\ +& I(0;1)\cdot I(1;0,1,0;0)\cdot I(0;1)\otimes I(0;1,0;1) \\ +& I(0;1)\cdot I(1;0,1;0)\cdot I(0;0)\cdot I(0;1)\otimes I(0;1,0,0;1) \\ =& \, I(3)\otimes I(2) - I_1(2)\otimes I(2) + I(2)\otimes I(3)\,. \end{aligned}$$

Together with $I_1(2) = -2I(3)$ this gives

$$3I(3)\otimes I(2) + I(2)\otimes I(3)\,.$$

Calculation of $\Delta(I(2,3))$

Diagrams with the second ● marked:

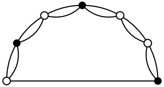


Corresponding sum in the coproduct:

$$I(0;1,0;1) \cdot I(1;0) \cdot I(0;0) \cdot I(0;1) \otimes I(0;1,0,0;1) = I(2) \otimes I(3)\,.$$

Calculation of $\Delta(I(2,3))$

Diagrams with both ● marked:



Corresponding sum in the coproduct:

$$1 \otimes I(2,3)\,.$$

Comparison of $\Delta(I(2,3))$ and $G_{2,3}(\tau)$

Summing all 4 parts together we obtain

$$\Delta(I(2,3)) = I(2,3) \otimes 1 + 3I(3) \otimes I(2) + 2I(2) \otimes I(3) + 1 \otimes I(2,3)\,.$$

Compare this to the Fourier expansion of $G_{2,3}(\tau)$:

$$G_{2,3}(\tau) = \zeta(2,3) + 3\zeta(3)g_2(q) + 2\zeta(2)g_3(q) + g_{2,3}(q)\,.$$

Since $\Delta(I(n_1,\dots,n_r)) \in \mathcal{I}^1 \otimes \mathcal{I}^1$ exists for all $n_1,\dots,n_r \geq 1$ this comparison suggests, that there might be a extended definition of $G_{n_1,\dots,n_r}$ by defining a map

$$\mathcal{I}^1 \otimes \mathcal{I}^1 \to \mathbb{C}[[q]]$$

which sends the first component to the corresponding zeta values and the second component to the functions g.

Shuffle regularized zeta values and $g^{ш}$

Theorem (Ihara, Kaneko, Zagier)

There exist an algebra homomorphism $Z^{ш} : \mathcal{I}^1 \to \mathcal{MZ}$ with $\zeta^{ш}(n_1, \dots, n_r) = Z^{ш}(I(n_1, \dots, n_r))$ such that

$$\zeta^{ш}(n_1, \dots, n_r) = \zeta(n_1, \dots, n_r)$$

for $n_1, \dots, n_{r-1} \geq 1$ and $n_r \geq 2$. It is uniquely determined by $Z^{ш}(I(1)) = 0$.

Theorem (B., K. Tasaka)

There exist an algebra homomorphism $\mathfrak{g}^{ш} : \mathcal{I}^1 \to \mathbb{C}[[q]]$ with $g^{ш}_{n_1,\dots,n_r}(q) := \mathfrak{g}^{ш}(I(n_1, \dots, n_r))$ such that

$$g^{ш}_{n_1,\dots,n_r}(q) = g_{n_1,\dots,n_r}(q)$$

for $n_1, \dots, n_r \geq 2$.

Proof sketch: We use generating functions and give an explicit form of $\mathfrak{g}^{ш}$.

Shuffle regularized MES

Definition

For integers $n_1, \dots, n_r \geq 1$, we define the q-series $G^{⧢}_{n_1,\dots,n_r}(q) \in \mathbb{C}[[q]]$, which we call **shuffle regularized multiple Eisenstein series**, as the image of the generator $I(n_1, \dots, n_r)$ in $\mathcal{I}^1$ under the algebra homomorphism $(Z^{⧢} \otimes \mathfrak{g}^{⧢}) \circ \Delta$:

$$G^{⧢}_{n_1,\dots,n_r}(q) := (Z^{⧢} \otimes \mathfrak{g}^{⧢}) \circ \Delta\big(I(n_1, \dots, n_r)\big).$$

We denote the space spanned by all shuffle regularized multiple Eisenstein series where the corresponding MZV exists by

$$\mathcal{E}_N = \langle G^{⧢}_{n_1,\dots,n_r}(q) \mid N = n_1 + \dots + n_r,\ r \geq 0, n_i \geq 1, n_r \geq 2 \rangle_{\mathbb{Q}}.$$

In the definition we identify $\mathcal{MZ} \otimes \mathbb{C}[[q]]$ with $\mathbb{C}[[q]]$ in the obvious way.

Shuffle regularized multiple Eisenstein series

Theorem (B., K. Tasaka 2014)

For all $n_1, \dots, n_r \geq 1$ the shuffle regularized multiple Eisenstein series $G^{ш}_{n_1,\dots,n_r}$ have the following properties:

- They are holomorphic functions on the upper half plane having a Fourier expansion with the shuffle regularized multiple zeta values as the constant term.
- They fulfill the shuffle product, i.e. we have an algebra homormorphism $\mathcal{I}^1 \to \mathbb{C}[[q]]$ by sending the generators $I(n_1, \dots, n_r)$ to $G^{ш}_{n_1,\dots,n_r}(q)$.
- For integers $n_1, \dots, n_r \geq 2$ they equal the multiple Eisenstein series

$$G^{ш}_{n_1,\dots,n_r}(q) = G_{n_1,\dots,n_r}(q)$$

  and therefore they fulfill the stuffle product in these cases.

Proof sketch: The first statement follows directly by definition. The second statement follows from the fact that Δ, $Z^{ш}$ and $\mathfrak{g}^{ш}$ are algebra homomorphism and hence $(Z^{ш} \otimes \mathfrak{g}^{ш}) \circ \Delta$ is also an algebra homomorphism.

Coproduct $\leftrightarrow$ multiple Eisenstein series

Proof sketch continued:

- To show that the $G^{\sqcup\!\sqcup}$ coincide with the G in the case $n_1, \dots, n_r \geq 2$ we give an explicit construction of the coproduct.
- For this we also split up the possible diagrams into $2^r = \sum_{k=0}^{r} \binom{r}{k}$ groups, where $\binom{r}{k}$ gives the number of ways marking k of the r $\bullet$.
- We show that the term

$$G^{w_1 \dots w_r}_{n_1, \dots, n_r}$$

  in the calculation of the Fourier expansion corresponds to the diagrams where the positions of the U in the word $w_1 \dots w_r$ coincide with the positions of the marked $\bullet$ by giving explicit formulas for both terms.
- The reduction of multitangent to monotangent functions (i.e. partical fraction expansion) in some sense then corresponds to the reduction of the I_n into linear combinations of the I's.

$\square$

Example: Shuffle regularized g in length 2 and 3

Before we give an explicit example in small length for $g^{\sqcup\!\sqcup}$ we define the following generalization of $g_{n_1,\dots,n_r}$

$$g_{\substack{n_1,\dots,n_r\\d_1,\dots,d_r}}(q) := c \cdot \sum_{\substack{0<u_1<\dots<u_r\\0<v_1,\dots,v_r}} u_1^{d_1} v_1^{n_1-1} \cdots u_r^{d_r} v_r^{n_r-1} q^{u_1 v_1 + \dots + u_r v_r} \in \mathbb{Q}[[q]]$$

where

$$c = \frac{(-2\pi i)^{n_1+\dots+n_r+d_1+\dots+d_r}}{d_1!(n_1-1)!\cdots d_r!(n_r-1)!} .$$

These q-series are up to the power $-2\pi i$ the recently introduced bi-brackets $\begin{bmatrix} n_r,\dots,n_1 \\ d_r,\dots,d_1 \end{bmatrix}$ which are currently studied independently by the speaker.

In the case $d_1 = \dots = d_r = 0$ these are the functions g, i.e.

$$g_{\substack{n_1,\dots,n_r\\0,\dots,0}} = g_{n_1,\dots,n_r} .$$

Example: Shuffle regularized g in length 2 and 3

For simplicity we write $g_{\dots}$ instead of $g_{\dots}(q)$.

Proposition (B. , K. Tasaka)

Assume $n_1, n_2, n_3 \geq 1$.

- In length two it is

$$g^{ш}_{n_1,n_2} = g_{n_1,n_2} + \delta_{n_1,1} \cdot \frac{1}{2}\left(g_{\substack{n_2\\1}} - (-2\pi i) g_{n_2}\right)$$

- And in length three it is

$$\begin{aligned} g^{ш}_{n_1,n_2,n_3} &= g_{n_1,n_2,n_3} + \delta_{n_1,1} \cdot \frac{1}{2}\left(g_{\substack{n_2,n_3\\1,0}} - (-2\pi i) g_{n_2,n_3}\right) \\ &+ \delta_{n_2,1} \cdot \frac{1}{2}\left(g_{\substack{n_1,n_3\\0,1}} - g_{\substack{n_1,n_3\\1,0}} - (-2\pi i) g_{n_1,n_3}\right) \\ &+ \delta_{n_1 \cdot n_2,1} \cdot \left(\frac{1}{6} g_{\substack{n_3\\2}} - \frac{1}{4}(-2\pi i) g_{\substack{n_3\\1}} + \frac{1}{6}(-2\pi i)^2 g_{n_3}\right). \end{aligned}$$

A variant of this was originally considered by Gangl, Kaneko and Zagier in length two.

Example: $G^{ш}_{1,2}(q) = \zeta(1,2) + g^{ш}_{1,2} = \zeta(1,2) + g_{1,2} + \frac{1}{2} g_{\substack{2\\1}} - \frac{1}{2}(-2\pi i) g_2$.

Shuffle regularized MES - double shuffle relations

Since the shuffle regularized Eisenstein series fulfill the shuffle product we have

$$G_2^{\sqcup\!\sqcup}(q) \cdot G_3^{\sqcup\!\sqcup}(q) \overset{\text{shuffle}}{=} G_{3,2}^{\sqcup\!\sqcup}(q) + 3G_{2,3}^{\sqcup\!\sqcup}(q) + 6G_{1,4}^{\sqcup\!\sqcup}(q)$$

We also have the stuffle product whenever the indices are greater equal to 2:

$$G_2^{\sqcup\!\sqcup}(q) \cdot G_3^{\sqcup\!\sqcup}(q) \overset{\text{stuffle}}{=} G_{3,2}^{\sqcup\!\sqcup}(q) + G_{2,3}^{\sqcup\!\sqcup}(q) + G_5^{\sqcup\!\sqcup}(q)\,.$$

This gives the same relation between MES as we had before for MZV:

$$2G_{2,3}^{\sqcup\!\sqcup}(q) + 6G_{1,4}^{\sqcup\!\sqcup}(q) \overset{\text{double shuffle}}{=} G_5^{\sqcup\!\sqcup}(q)\,.$$

But we don't have all relations of MZV since the stuffle product of MES fails when at least one $n_j = 1$. While Euler has shown that $\zeta(3) - \zeta(1,2) = 0$ we get

$$G_3^{\sqcup\!\sqcup}(q) - G_{1,2}^{\sqcup\!\sqcup}(q) = \frac{1}{2} g_{\substack{2\\1}} = \frac{1}{2} q \frac{d}{dq} G_1^{\sqcup\!\sqcup}(q) \neq 0\,.$$

Shuffle regularized MES - double shuffle relations

Euler also showed that

$$\zeta(6)^2 = \frac{715}{691}\zeta(12)$$

and this relation can also be proven by using the extended double shuffle relations of multiple zeta values.

For multiple Eisenstein series this relation does not hold since there are cusp forms in weight 12 and it is

$$G_6(\tau)^2 = \frac{715}{691}G_{12}(\tau) + \alpha\Delta(q)$$

with some $\alpha \in \mathbb{C}\backslash\{0\}$ and $\Delta(q) = q\prod_{n>0}(1-q^n)^{24}$.

Shuffle regularized MES - Open questions

There are a lot of open questions for shuffle regularized multiple Eisenstein series.

- What is exactly the failure of the stuffle product of shuffle regularized multiple Eisenstein series?
- What is the dimension of the space $\mathcal{E}_N$?
- Define the differential operator $\mathrm{d} = q\frac{d}{dq}$. Is the space spanned by all shuffle regularized multiple Eisenstein series closed under d ?
- Consider the projection $\pi : \mathcal{E}_N \longrightarrow \mathcal{MZ}_N$ to the constant term, i.e

$$\pi(G^{⧢}_{n_1,\dots,n_r}(q)) = \zeta(n_1,\dots,n_r)\,.$$

  What is the kernel of π and are there elements in the kernel which are not derivatives of MES or cusp forms?

Summary

- Multiple Eisenstein series $G_{n_1,\dots,n_r}$ which are defined for $n_1,\dots,n_r \geq 2$ are multiple versions of the classical Eisenstein series and they fulfill the stuffle product.
- Their Fourier expansions are similar to the coproduct Δ on the space $\mathcal{I}^1$ of formal iterated integrals.
- This connections enables one to define shuffle regularized multiple Eisenstein series $G^{\sqcup\!\sqcup}_{n_1,\dots,n_r}$ for all $n_1,\dots,n_r \geq 1$.
- They fulfill the shuffle product and for $n_1,\dots,n_r \geq 2$ the stuffle product since in these cases they are equal to the multiple Eisenstein series.
- Since the algebra of shuffle regularized Eisenstein series contains all modular forms this setup gives a framework to study the connection of multiple zeta values and modular forms. Yet there are a lot of open and interesting problems to be solved.