Proposition 3.4.2.

The components of the cubic are related to the prepotential by

$$C_{ijk} = \sum_{I,J,K=0}^{n} \frac{\partial X^I}{\partial z^i} \frac{\partial X^J}{\partial z^j} \frac{\partial X^K}{\partial z^k} \frac{\partial^3 F}{\partial X^I \partial X^J \partial X^K}$$

we denote by $\quad C_{IJK} = \dfrac{\partial^3 F}{\partial X^I \partial X^J \partial X^K}$

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## 3.5 | The period matrix

Lecture 9

Let $\Omega(z_0) \in H^{3,0} = F^3$,

$F^2 = H^{3,0} \oplus H^{2,1}$ $\quad \dim F^2 = n+1$, we fix a

basis $\Omega_0, \Omega_1, \dots, \Omega_n$ of $F^2$ s.t. $\Omega_0 \in F^3$

and define

$$\Pi := \begin{pmatrix} \int_{A^I} \Omega_0 & \int_{B_J} \Omega_0 \\ \int_{A^I} \Omega_1 & \vdots \\ \vdots & \\ \int_{A^I} \Omega_n & \int_{B_J} \Omega_n \end{pmatrix}$$

as an $(n+1, 2n+2)$ matrix which we call the period matrix, the first row coincides with the period integral

$$\Omega = \sum_I X^I \alpha_I + \sum_J F_J \beta^J .$$

---

3.5.1 | Proposition.

The following are properties of the Hodge filtration and the Hodge-Riemann bilinear relations

1) $\quad \Pi \, Q \, \Pi^{tr} = 0$

2) $\quad i \, \Pi \, Q \, \overline{\Pi^{tr}} > 0$

here 2) means that we decompose the $(n+1)\times(n+1)$ hermitian matrix $i\,\Pi\, Q\, \overline{\Pi}^{tr}$ into a block form compatible with the filtration $F^3 \subset F^2$ then the first diagonal block $(1\times 1)$ is positive definite and the whole $(n+1)\times(n+1)$ hermitian matrix has 1 positive and $n$ negative eigenvalues.

---

Let $z^i$, $i=1,\dots,n=h^{2,1}=\dim \mathcal{M}$ be local coordinates on $\mathcal{M}$

we will take the basis of $F^2$ to be given by

$\partial_1 \Omega, \dots, \partial_n \Omega$ $\left(\partial_i \Omega := \frac{\partial \Omega}{\partial z^i}\right)$ together with

$\Omega$ which we also denote by $\partial_0 \Omega := \Omega$ :

We can now write the period matrix as,

$$\Pi = \left(\partial_i X^I \quad \partial_j P_J\right)$$

### 3.5.2 Proposition

We have the following.

i) $$\det\left(\partial_i X^I\right)_{0\le i, I\le n} = \left(X^0\right)^{n+1} \det\left(\partial_i\left(\frac{X^j}{X^0}\right)\right)_{1\le i,j\le n} \neq 0$$

ii) $\partial_i P_J$ can be written as $\partial_i X^J\, \tau_{IJ}$

with $$\tau_{IJ} := \frac{\partial^2 F}{\partial X^I \partial X^J}$$

Proof:

i) For the matrix $(\partial_i X^I)_{0 \le i, I \le n}$ we have

$$\partial_i X^I = \begin{pmatrix} X^0 & X^j \\ \partial_i X^0 & \partial_i X^j \end{pmatrix}_{1 \le i,j \le n}$$

we now use Prop. 3.3.1 i) ( $[X^0(z), \dots, X^n(z)] \in \mathbb{P}^n$ )

and assume $X^0 \neq 0$ to write

$$\begin{pmatrix} X^0 & X^j \\ \partial_i X^0 & \partial_i X^j \end{pmatrix} = \begin{pmatrix} X^0 & X^0 \frac{X^j}{X^0} \\ \partial_i X^0 & X^0 \partial_i\left(\frac{X^j}{X^0}\right) + (\partial_i X^0) \frac{X^j}{X^0} \end{pmatrix}$$

we thus obtain $\det \partial X^I = \det X^0 \begin{pmatrix} 1 & \frac{X^j}{X^0} \\ 0 & \partial_i\left(\frac{X^j}{X^0}\right) \end{pmatrix}$

$$= (X^0)^{n+1} \det \begin{pmatrix} 1 & \frac{X^j}{X^0} \\ 0 & \partial_i\left(\frac{X^j}{X^0}\right) \end{pmatrix} = (X^0)^{n+1} \det \partial_i\left(\frac{X^j}{X^0}\right)_{1 \le i,j \le n} \quad \square$$

by prop. 3.3.1 $t^j := \frac{X^j}{X^0}$ are inhomogeneous coordinates on $\mathbb{P}^n$

and hence $(X^0)^{n+1} \det \partial_i\left(\frac{X^j}{X^0}\right) \neq 0$

---

ii) The homogeneity of $\mathcal{F}$ gives $X^I \frac{\partial \mathcal{F}}{\partial X^I} = 2\mathcal{F}$

but $P_J = \frac{\partial \mathcal{F}}{\partial X^J} \Rightarrow \mathcal{F} = \frac{1}{2} X^I P_I$

now consider $\partial_j P_J$

first $j=0$ $\quad \partial_0 P_J = P_J = \frac{\partial \mathcal{F}}{\partial X^J} = \frac{\partial}{\partial X^J}\left(\frac{1}{2} X^I \frac{\partial \mathcal{F}}{\partial X^I}\right)$

$$= \frac{1}{2} \delta^I{}_J \frac{\partial \mathcal{F}}{\partial X^I} + \frac{1}{2} X^I \frac{\partial^2 \mathcal{F}}{\partial X^I \partial X^J}$$

$$P_J = \frac{1}{2} \underbrace{\frac{\partial \mathcal{F}}{\partial X^J}}_{P_J} + \frac{1}{2} X^I \frac{\partial^2 \mathcal{F}}{\partial X^I \partial X^J}$$

$$\Rightarrow P_J = X^I \tau_{IJ} .$$

we further have $\partial_K P_J = \delta^I{}_K \tau_{IJ} + X^I C_{IJK}$

$$\tau_{KJ} = \tau_{KJ} + X^I C_{IJK}$$

$$\Rightarrow X^I C_{IJK} = 0$$

$1 \leq j \leq n$

hence $\partial_j P_J = (\partial_i X^I) \tau_{IJ} + \underline{X^I \partial_j \tau_{IJ}}$

$$= \underset{\overset{\shortparallel}{0}}{X^I \partial_j X^K C_{IJK}} \qquad \square$$

---

## 3.5.3) Normalization of the period matrix.

using the prop 3.5.2 we can normalize the period matrix as

$$(\partial_i X^I)^{-1}\, \Pi = \begin{pmatrix} E & \tau_{IJ} \end{pmatrix}$$

prop. 3.5.1 1) is trivial $\tau_{IJ} = \tau_{JI}$

and 2) $i(\bar\tau - \tau) > 0$

which means that $\operatorname{Im} \tau_{IJ}$ has one positive and $n$ negative eigenvalues.

For $\begin{pmatrix} D & B \\ C & A \end{pmatrix} \in \Gamma < Sp(2n+2, \mathbb{Z})$

$$\tau \longmapsto (C\tau + D)^{-1} (A\tau + B)$$

since $\tau_{IJ} = \dfrac{\partial^2 F}{\partial X^I \partial X^J}$ this describes the transformation property of $F$

## 3.6 | The Kähler metric on $\mathcal{M}$.

Def. On $\mathcal{M}$ except at degeneration loci there exists a Kähler metric whose components are given by

$$G_{i\bar{j}} := \partial_i \partial_{\bar{j}} K$$

where the Kähler potential is given by

$$K = -\log i \int \Omega \wedge \bar{\Omega}$$

this Kähler metric is called the Weil-Petersson metric on $\mathcal{M}$.

We modify the basis $\partial_0 \Omega = \Omega$, $\partial_i \Omega$ $i = 1, \ldots, n$ which was compatible with the filtration i.e. $\Omega \in F^3$, $\partial_i \Omega \in F^2$ $i = 0, \ldots, n$

to $D_0 \Omega = \Omega$, $D_i \Omega = (\partial_i + K_i) \Omega$ which is compatible with the Hodge decomposition i.e. $\Omega \in H^{3,0}$, $D_i \Omega \in H^{2,1}$ $i = 1, \ldots, n$

(Exercise)

we have $\det (D_i X^I)_{0 \le i, I \le n} = \det (\partial_i X^I)_{0 \le i, I \le n} \neq 0$

the normalized period matrix becomes

$$(D_i X^I)^{-1} \Pi = (E \ \tau)$$

---

## 3.7 | Special Kähler geometry on $\mathcal{M}$.

Prop: we have

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = -e^K \, i \int D_i \Omega \wedge D_{\bar{j}} \bar{\Omega}$$

Proof. ex.

with respect to this metric we introduce the connections

$$\Gamma^h_{ij} = G^{h\bar{h}} \partial_i G_{j\bar{h}} \quad , \quad \Gamma^{\bar{h}}_{\bar{i}\bar{j}} = G^{h\bar{h}} \partial_{\bar{i}} G_{\bar{j}h}$$

The curvature tensor for these connections is given by

$$R_{i\bar{j}}{}^{k}{}_{l} = -\partial_{\bar{j}} \Gamma^{k}_{il} \quad , \quad R_{i\bar{j}}{}^{\bar{k}}{}_{\bar{l}} = \partial_i \Gamma^{\bar{k}}_{\bar{j}\bar{l}}$$

---

Let us introduce the following $(2n+2) \times (2n+2)$ matrix

$$\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} := \begin{pmatrix} X^I & P_J \\ D_i X^I & D_j P_J \\ \overline{D_i X^I} & \overline{D_j P_J} \\ \overline{X^I} & \overline{P_J} \end{pmatrix} \quad i,j = 0, \dots, n$$

whose row vector represent the Hodge decomposition

$$\begin{matrix} H^{3,0} & \oplus & H^{2,1} & \oplus & H^{1,2} & \oplus & H^{0,3} \\ \psi & & \psi & & \psi & & \psi \\ \Omega & & D_i\Omega & & \overline{D_i\Omega} & & \overline{\Omega} \end{matrix}$$

we define

$$D_i = \begin{cases} \partial_i + K_i - \Gamma^{*}_{i*} & \text{on } \Pi \\ \partial_i & \text{on } \overline{\Pi} \end{cases}$$

$$\overline{D}_{\bar{i}} = \begin{cases} \partial_{\bar{i}} & \text{on } \Pi \\ \partial_{\bar{i}} + K_{\bar{i}} - \Gamma^{*}_{\bar{i}*} & \text{on } \overline{\Pi} \end{cases}$$

---

<u>Theorem 3.7.1</u>

The period matrix satisfies

$$(D_i + A_i) \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = 0 \quad , \quad (\overline{D}_{\bar{i}} + \overline{A}_{\bar{i}}) \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = 0$$

with

$$A_i = \begin{pmatrix} 0 & -\delta_i^n & 0 & 0 \\ 0 & 0 & C_{im}^{\bar{n}} & 0 \\ 0 & 0 & 0 & -G_{i\bar{m}} \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \overline{A}_{\bar{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -G_{\bar{i}m} & 0 & 0 & 0 \\ 0 & \overline{C}_{\bar{i}\bar{m}}^{n} & 0 & 0 \\ 0 & 0 & -\delta_{\bar{i}}^{\bar{n}} & 0 \end{pmatrix}$$

where $C_{im}^{\bar{n}} := i e^K C_{imn} G^{n\bar{n}}$, $\quad \overline{C}_{\bar{i}\bar{m}}^{n} = -i e^K \overline{C}_{\bar{i}\bar{m}\bar{n}} G^{n\bar{n}}$

Theorem 3.7.2:

The compatibility condition

$$[D_i + A_i \,,\, \bar{D}_{\bar{j}} + \bar{A}_{\bar{j}}] = 0$$

is equivalent to

$$-R_{i\bar{j}}{}^{k}{}_{\ell} = G_{i\bar{j}}\,\delta^{k}_{\ell} + G_{\bar{j}\ell}\,\delta^{k}_{i} - e^{2K} C_{i\ell m}\,\bar{C}_{\bar{j}\bar{k}\bar{n}}\,G^{m\bar{m}}\,G^{k\bar{k}}$$

this property defines special Kähler geometry

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We further have $[D_i + A_i \,,\, D_j + A_j] = 0 = [\bar{D}_{\bar{i}} + \bar{A}_{\bar{i}} \,,\, \bar{D}_{\bar{j}} + \bar{A}_{\bar{j}}]$

which is ensured by the existence of the prepotential.

This leads to a flat connection on $M$

$$\nabla = D + A$$ this is an example

of a $tt^*$ flat connection.

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