3. Special Geometry

Lecture 8

3.1 Middle dim cohomology of threefolds.

We will consider a family of Calabi-Yau threefolds $\pi : \mathcal{X} \longrightarrow B$ over a small complex domain B with fibers $\pi^{-1}(z) = X_z$.

- We will assume that the local family extends to a toric variety $\mathcal{M}$.
- we will further assume that $\dim \mathcal{M} = h^{2,1}(X_{z_0}) = n$ for a smooth X_{z_0}

We fix a smooth X_{z_0}

3.1.1 Proposition: $H^3(X_{z_0}, \mathbb{C})$ carries a polarized Hodge structure of weight 3.

we have $H^3(X_{z_0}, \mathbb{C}) = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$

the lattice is given by $H^3(X_{z_0}, \mathbb{Z})$, which has a symplectic form given by

$$Q(u,v) := \int_{X_{z_0}} u \wedge v$$

We choose a symplectic basis $\{\alpha_I, \beta^J\}_{0 \leq I,J \leq n}$ satisfying $Q(\alpha_I, \beta^J) = \delta_I^J$, $Q(\alpha_I, \alpha_J) = 0 = Q(\beta^I, \beta^J)$ with respect to this basis the symplectic form is

$$Q = \begin{pmatrix} 0 & \mathbb{1}_{n+1} \\ -\mathbb{1}_{n+1} & 0 \end{pmatrix}$$

denote the dual homology basis by $\{A^I, B_J\}_{0 \leq I, B \leq n}$

3.2 | The period map:

We define the period domain

$$\mathcal{D} = \{ [\omega] \in \mathbb{P}(H^3(X_{z_0}, \mathbb{C})) \mid Q(\omega,\omega)=0, \; iQ(\omega,\overline{\omega})>0 \}$$

We choose a holomorphic three form of the fiber X_{z_0} and denote it by $\Omega(z_0)$, then the period map

$\mathcal{P}_0 : B \longrightarrow \mathcal{D}$ is defined by

$$\mathcal{P}_0(z) = \left[\sum_I \left(\int_{A^I} \Omega(x) \right) \alpha_I + \sum_J \left(\int_{B_J} \Omega(x) \right) \beta^J \right]$$

Using path independent identification $H_3(X_z, \mathbb{Z}) \cong H_3(X_{z_0}, \mathbb{Z})$ we may globalize this period map on B to M by introducing a covering space $\widetilde{M}$. We denote the resulting period map $\mathcal{P} : \widetilde{M} \longrightarrow \mathcal{D}$ and assume $\Gamma \subset Sp(2r+2, \mathbb{Z})$ as the covering group.

We denote by $\mathcal{U}$ the restriction to $\mathcal{D}$ of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}(H^3(X_{z_0}, \mathbb{C}))$. We set $\mathcal{L} = \mathcal{P}^* \mathcal{U}$, i.e. the pullback to $\widetilde{M}$. The Complex conjugate of $\mathcal{L}$ will be denoted by $\overline{\mathcal{L}}$.

The period integral may be considered as a section of $\mathcal{L}$.

3.3 The prepotential.

recall that $\dim H^3(X_{z_0}, \mathbb{C}) = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3}$

$= 2 + 2h^{2,1} = 2 + 2n$

$\mathcal{D}$ is a projectivization, hence $\dim \mathcal{D} = 2n+1$.

The symplectic form Q on $H^3(X_{z_0}, \mathbb{Z})$ induces

a one form on $\mathcal{D}$ by $\theta := \langle d\omega, \omega \rangle$.

With this one form $(\mathcal{D}, \theta)$ becomes a $2n+1$ dim

contact manifold.

Def. A contact manifold P is a smooth manifold P of odd dimension $2n+1$ which is equipped with a differential 1-form A that is non-degenerate in the sense that the wedge product $A \wedge (dA)^{\wedge n}$ does not vanish.

We have the following:

i) Locally the period map $\mathcal{P}_0 : B \to \mathcal{D}$ is an embedding (Bogomolov, Tian, Todorov)

ii) $\theta \big|_{\mathcal{P}_0(B)} = 0$ (due to Griffiths' transversality)

remember

$$\begin{aligned} \Omega &\in F^3 = H^{3,0} \\ & \quad\ \cap \\ \nabla\Omega &\in F^2 = H^{3,0} \oplus H^{2,1} \end{aligned}$$

A result of Bryant and Griffiths states that the image of the period map $\mathcal{P}_0$ can be recovered by half of the period integrals

$$X^I(z) = \int_{A^I} \Omega(z).$$

3.3.1 Proposition/Definition

i) The map $z \longmapsto [X^0(z), \dots, X^n(z)] \in \mathbb{P}^n$
is a local isomorphism $B \to \mathbb{P}^n$

ii) Integrating $\Theta|_{P_0(B)} = 0$ on B we can write the other half of the period integrals

$$P_J(z) = \frac{\partial F(X^I)}{\partial X^J} \qquad (J = 0, 1, \dots, r)$$

in terms of a holomorphic function $F(X)$ called prepotential

The function $F(X)$ is a holomorphic function of $X^0(z), \dots X^n(z)$ and has the following homogeneity property

$$\sum_{I=0}^{n} X^I \frac{\partial F}{\partial X^I} = 2F(X)$$

This potential function exists locally for the small domain B. When globalizing to $\widehat{M}$ the monodromy group plays a role (The group action is referred to as duality transformation in physics)

3.4 The Griffiths-Yukawa cubic:

3.4.1 Definition:
We define the Griffiths-Yukawa cubic to be a section of $\mathcal{L}^2 \otimes \mathrm{Sym}^3 T^*\widehat{M}$, whose components are given by (fixing a section Ω of $\mathcal{L}$)

$$C_{ijk}(z) = -\int_{X_z} \Omega(z) \wedge \partial_i \partial_j \partial_k \Omega(z), \qquad \partial_i = \frac{\partial}{\partial z^i}$$

$i = 1, \dots, \dim \widehat{M} = n$

Proposition 3.4.2:

The components of the cubic are related to the prepotential by

$$C_{ijk} = \sum_{I,J,K=0}^{n} \frac{\partial X^I}{\partial z^i} \frac{\partial X^J}{\partial z^j} \frac{\partial X^K}{\partial z^k} \frac{\partial^3 F}{\partial X^I \partial X^J \partial X^K}$$

we denote by $\quad C_{IJK} = \dfrac{\partial^3 F}{\partial X^I \partial X^J \partial X^K}$

3.5| The period matrix

Let $\Omega(z_0) \in H^{3,0} = F^3$,

$F^2 = H^{3,0} \oplus H^{2,1}$ $\quad \dim F^2 = n+1$, we fix a basis $\Omega_0, \Omega_1, \dots, \Omega_n$ of F^2 s.t. $\Omega_0 \in F^3$

and define

$$\Pi := \begin{pmatrix} \int_{A^I} \Omega_0 & \int_{B_J} \Omega_0 \\ \int_{A^I} \Omega_1 & \vdots \\ \vdots & \vdots \\ \int_{A^I} \Omega_n & \int_{B_J} \Omega_n \end{pmatrix}$$

as an $(n+1, 2n+2)$ matrix which we call the period matrix, the first row coincides with the period integral

$$\Omega = \sum_I X^I \alpha_I + \sum_J \mathcal{F}_J \beta^J .$$

3.5.1| Proposition:

The following are properties of the Hodge filtration and the Hodge-Riemann bilinear relations

1) $\Pi \, Q \, \Pi^{tr} = 0$

2) $i \, \Pi \, Q \, \overline{\Pi^{tr}} > 0$

here 2) means that we decompose the

$(n+1)\times(n+1)$ hermitian matrix $i\,\Pi\, Q\, \overline{\Pi}^{tr}$

into a block form compatible with the filtration $F^3 \subset F^2$

then the first diagonal block (1×1) is positive definite

and the whole $(n+1)\times(n+1)$ hermitian matrix

has 1 positive and n negative eigenvalues.