$$\mathcal{E} \longrightarrow B = \mathbb{P}^1 \setminus \{0,1,\infty\}$$

Last time: for E_λ we had

$$\omega = \Pi^0 \alpha + \Pi^1 \beta$$

near $\lambda = 0$ $\quad \lim_{\lambda \to 0} \Pi^0(\lambda) = 1$

$$\Pi^1(\lambda) = \frac{1}{2\pi i} \Pi^0 \log \lambda + S^1(\lambda), \qquad \lim S^1(\lambda) = 0$$

$$\widetilde{\Pi} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2\pi i}\log \lambda & 1 \end{pmatrix} \Pi \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\widetilde{\Pi} = \exp\left(\frac{-1}{2\pi i} \log \lambda \, N \right) \Pi$$

at $\lambda = 0$ replace Π by $\widetilde{\Pi}$.

For a family $p : \mathcal{X} \longrightarrow B$, complete into $\bar{p} : \mathcal{X} \longrightarrow \bar{B}$

$$\bar{B} - B = \bigcup_i D_i = D$$

what is the extension of $\mathcal{F}^0 \to B$ to $\bar{\mathcal{F}} \to \bar{B}$?

$$\bar{\nabla} : \bar{\mathcal{F}}_0 \longrightarrow \bar{\mathcal{F}}^0 \otimes \Omega^1_{\bar{B}}(\log D)$$

$\gamma_j = \gamma_j(u)$ $\quad u \in [0,1]$ $\quad$ small loop going around D_j
$\gamma_j(0) = \gamma_j(1) = t \in B$

induces monodromy, for $\eta \in H^k(X_t, \mathbb{C})$

$$T_j : H^k(X_t, \mathbb{C}) \longrightarrow H^k(X_t, \mathbb{C})$$

$$T_j(\eta) = \eta(1)$$

Today (then (2.6.3) on page 35)

Let $N_j = \log(T_j)$ (where T_j is the monodromy given by going counterclockwise around the jth factor of $(\Delta^*)^r$.

The canonical extension of $\mathcal{F}^0$ can be described as follows. Let s be a flat multi-valued section of $\mathcal{H} = \mathcal{F}^0$ over $(\Delta^*)^r$, then

$$\tilde{s} = \exp\left(-\frac{1}{2\pi i} \log(z_j) N_j\right) s \quad \text{is}$$

single valued and extends to a section of the canonical extension $\overline{\mathcal{F}}^0$, this property characterizes $\overline{\mathcal{F}}^0$

Schmid's nilpotent orbit theorem implies that the bundles $\mathcal{F}^p$ extend to $\overline{\mathcal{F}}^p$, which are subbundles of the canonical extension.

At $0 \in \Delta^r$ $\overline{\mathcal{F}}^p$ gives the limiting Hodge filtration

$$\overline{\mathcal{F}}^\bullet_0 = F^\bullet_{lim},$$ it has the properties

$$N_j(F^p_{lim}) \subset F^{p-1}_{lim},$$ we get a linear map

$$N_j : F^p_{lim} / F^{p+1}_{lim} \longrightarrow F^{p-1}_{lim} / F^p_{lim}$$

If $\delta_j = z_j \frac{\partial}{\partial z_j}$, then $\overline{\nabla}_{\delta_j}(\overline{\mathcal{F}}^p) \subset \overline{\mathcal{F}}^{p-1}$ by Griffiths transversality, it follows that $\overline{\nabla}_{\delta_j}$ induces a linear map

$$\overline{\nabla}_{\delta_j} : F^p_{lim} / F^{p+1}_{lim} \longrightarrow F^{p-1}_{lim} / F^p_{lim}$$

the maps are related by

$$\overline{\nabla}\delta_j = \frac{-1}{2\pi i} N_j .$$

2.6.4 The monodromy weight filtration

The monodromy weight filtration is defined by

$$W_\bullet,\ W_0 \subset W_1 \subset \cdots \subset W_{2k-1} \subset W_{2k} = H^k(X_t, \mathbb{C})$$

in terms of the action of $N = \sum_j a_j N_j,\ a_j > 0$.

on $H^k(X_t, \mathbb{C})$

$$W_0 = \mathrm{im}(N^k)$$
$$W_1 = \mathrm{im}(N^{k-1}) \cap \ker(N)$$
$$\vdots$$

(Cox & Katz)

The main properties are:

- $N(W_m) \subset W_{m-2}$
- N^m induces an isomorphism
  $N^m\ W_{m+k}/W_{m+k-1} \cong W_{k-m}/W_{k-m-1}$
- $F^\bullet_{lim}$ induces a Hodge structure of weight m on W_m/W_{m-1}

The last property says that $(W_\bullet, F^\bullet_{lim})$ is a mixed Hodge structure

26.9 Monodromy and Picard-Fuchs equations

assume $\dim B = 1$, let z be a coordinate in a disk $\Delta \subset \overline{B}$ centered at $p \in \overline{B} - B$. Choosing a basis $\omega_1 \ldots \omega_r$ of $\overline{F}^0$ over Δ, the Gauss-Manin connection ∇ is completely determined by its connection matrix

$$(\Gamma_{ij}) \text{ defined by } \nabla_{d/dz}\, \omega_i = \sum_j \Gamma_{ij}\omega_j$$

we get the residue matrix (has at most a simple pole at $z=0$)

$$\mathrm{Res}(\nabla) = \mathrm{Res}_{z=0}(\Gamma_{ij})$$

which has the properties

- The eigenvalues λ of $\mathrm{Res}(\nabla)$ are rational $0 \le \lambda < 1$
- $\exp(-2\pi i\, \mathrm{Res}(\nabla))$ is conjugate to the monodromy T
- T is unipotent if and only if $\mathrm{Res}(\nabla)$ is nilpotent

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2.7) Period domains:

Let $(H_{\mathbb{Z}}, H^{p,q}, b)$ be a polarized Hodge structure of weight k, recall (2.1.1)

i) $b(x,y) = 0$ if x is in $H^{p,q}$, $y \in H^{r,s}$ $(r+p, q+s) \neq (k,k)$

ii) $\pm i^{p-q}(b(x,\bar{x})) > 0$ if $x \neq 0$ in $H^{p,q}$ $\quad \pm$ is $(-1)^{k(k-1)/2}$

(Hodge-Riemann bilinear relations)

2.7.1) Interlude, Hodge diamond

Def. Let X be a compact Kähler manifold, then

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

the integers $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ are called the Hodge numbers of X. Note that the Hodge decomposition implies $h^{p,q} = h^{q,p}$.

The n-th Betti number $b_n(X) = \sum_{p+q=n} h^{p,q}$

it follows that the odd Betti numbers $b_{2k-1}(X)$ are even.

the Hodge numbers of a compact Kähler manifold are often displayed in the Hodge diamond

$$\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{m,0} & h^{m-1,1} & & & h^{0,m} \\
 & & & & \\
 & h^{m,m-1} & & & \\
 & & h^{m,m} & &
\end{array}$$

examples

curve of genus g

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

K3 surface
compact complex surface with trivial canonical bundle and $h^1(S, \mathcal{O}_S) = 0$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

CY threefold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2.7.2| The period domain.

Def. Let $\mathcal{D}$ denote the set of all collections of subspaces $\{H^{p,q}\}$ of $H_{\mathbb{C}}$ such that $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ and $\dim(H^{p,q}) = h^{p,q}$ on which b satisfies the Hodge-Riemann bilinear relations i) and ii).

$\mathcal{D}$ is called the local period domain. It can be realized as a homogeneous domain G/K where G is the real Lie group of linear automorphisms of $H_{\mathbb{R}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ which preserve b and K is the subgroup of elements fixing a reference structure in $\mathcal{D}$.

2.7.1.2) Example

Consider the example of an elliptic curve E

The polarized Hodge structure on the first cohomology

$H^1(E,\mathbb{Z})$ has Hodge numbers $h^{1,0} = h^{0,1} = 1$

The Hodge filtration is

$$\{0\} \subset F^1 \subset F^0 \subset H \quad , \; F^1 = H^{1,0}$$

$\mathcal{D}$ is the set of all filtrations $\{0\} \subset F^1 \subset \mathbb{C}^2$

with $\dim(F^1) = 1$.

To specify a point in $\mathcal{D}$, it suffices to give $\lambda \in H_{\mathbb{C}}$

that spans F^1, the relations i) ii) become

i) $b(\lambda,\lambda) = 0$, ii) $i\, b(\lambda, \bar{\lambda}) > 0$

we write λ in the canonical basis as $\lambda = z_1 \alpha + z_2 \beta$

for $z_1, z_2 \in \mathbb{C}$, the relations become

i) $$\int (z_1 \alpha + z_2 \beta) \wedge (z_1 \alpha + z_2 \beta) = 0$$

$$\Rightarrow \quad z_1 z_2 - z_2 z_1 = 0$$

ii) $$i \int (z_1 \alpha + z_2 \beta) \wedge (\bar{z}_1 \alpha + \bar{z}_2 \beta) > 0$$

$$i\,(z_1 \bar{z}_2 - z_2 \bar{z}_1) > 0$$

(in particular $z_1 \neq 0$, so we rescale

$$\lambda = \alpha + z_2 \beta \qquad ii) \Rightarrow \quad \mathrm{Im}\, z_2 > 0$$

specifying z_2 is equivalent to specifying λ

$$\Rightarrow \quad \mathcal{D} \cong \mathbb{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$$