Lecture 6

Reminder:

* A Hodge structure of weight k, real v.s H with $H_{\mathbb{Z}} \subset H$

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}$$

* Filtration $\{0\} \subset F^k \subset \cdots \subset F^0 = H_{\mathbb{C}}$

$$F^p = \bigoplus_{a \geq p} H^{a,k-a}$$

$$H^{p,q} = F^p \cap \overline{F^q}, \qquad H_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$$

Example: $E_\lambda := \{ y^2 = x(x-1)(x-\lambda) \subset \mathbb{C}^2 \}$

$H^1(E_\lambda, \mathbb{C}) = H^{1,0} + H^{0,1}$, $\quad \{0\} \subset \underbrace{H^{1,0}}_{=: F^1} \subset \underbrace{H^{1,0} \oplus H^{0,1}}_{=: F^0}$

Variation of Hodge structure

Now consider a family $\pi : \mathcal{X} \to B$, $\pi^{-1}(b) = X_b$

The Hodge filtration becomes a filtration of

$\mathcal{F}^0 \to B$ with hol. subbundles

$$0 \subset \mathcal{F}^k \subset \cdots \subset \mathcal{F}^0$$

$\mathcal{F}^0$ has a natural flat connection, ∇ the Gauss manin connection

$$\nabla \mathcal{F}^0 \subset \mathcal{F}^0 \otimes \Omega^1_B,$$

it has the Griffith's transversality property

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_B.$$

If we set $\mathcal{H} = \mathcal{F}^0$, then $\mathcal{H}$ has the locally constant subsheaf $\mathcal{H}_{\mathbb{C}} = R^k \pi_* \mathbb{C}$ and this in turn has the subsheaf $\mathcal{H}_{\mathbb{Z}}$ of integer sections (the image of $R^k\pi_*\mathbb{Z} \to R^k\pi_*\mathbb{C}$)

we call $(\mathcal{H}, \nabla, \mathcal{H}_{\mathbb{Z}}, \mathcal{F}^\cdot)$ a variation of Hodge structure

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e.g. $\varphi : \mathcal{E} \longrightarrow B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$\tilde{\omega}(\lambda) = \dfrac{\omega(\lambda)}{\pi^0(\lambda)}$ is a section of $\varphi_* \mathcal{F}^1 \longrightarrow B$

$$\varphi^{1\,-1}(\lambda) = H^{1,0}(E_\lambda)$$

$$\tilde{\omega}(\lambda) = \alpha + \tau\beta \,,$$

τ hol. in λ on B

$$\partial_\lambda \tilde{\omega}(\lambda) = \frac{\partial \tau}{\partial \lambda} \beta$$

$\frac{\partial \tau}{\partial \lambda}$ hol.

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however,
$$\overline{\tilde{\omega}(\lambda)} = \alpha + \overline{\tau}\beta \in H^{0,1}(E_\lambda)$$

$\overline{\tau}$ non-hol.

2.5| Limiting mixed Hodge structure (Intuition)

Question: does the VHS extend to the singular points $\lambda = 0, 1, \infty$?

For $\mathcal{E} \longrightarrow B = \mathbb{P}^1 \setminus \{0,1,\infty\}$

we had $\Pi^0 = \int_A \frac{dx}{y}$, $\Pi^1 = \int_B \frac{dx}{y}$

around $\lambda = 0$ we found the solutions

$$\Pi^0(\lambda) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1, \lambda) \qquad \text{with } \lim_{\lambda \to 0} \Pi^0 = 1 \quad \text{fine}$$

and $$\Pi^1(\lambda) = \frac{-1}{2i} \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1-\lambda)$$

$$= \frac{1}{2\pi i} \Pi^0 \log \lambda + S^1(\lambda) \qquad \lim_{\lambda \to 0} \Pi^1 = ☹$$

but $\lim_{\lambda \to 0} \Pi^1 = $ fine.

Idea: at $\lambda = 0$ replace $\Pi = \begin{pmatrix} \Pi^0 \\ \Pi^1 \end{pmatrix}$ by $\tilde{\Pi} = \begin{pmatrix} \Pi^0 \\ S^1 \end{pmatrix}$

$$\tilde{\Pi} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2\pi i} \log z & 1 \end{pmatrix} \Pi$$

(Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, verify that $T = \exp N$, $N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$)

$$\tilde{\Pi} = \exp\left(-\frac{1}{2\pi i} \log \lambda \, N \right) \Pi$$

$N = \log T$, T is the monodromy of Π around $\lambda = 0$.

2.6 Degeneration of Hodge structure

2.6.1 Extension to singular pts.

Suppose that the smooth family $\pi : \mathcal{X} \to B$ can be completed to a family $\overline{\pi} : \mathcal{X} \to \overline{B}$ where $\overline{B}$ is a compactification of B with normal crossing boundary divisor $D = \cup_i D_i = \overline{B} - B$

Deligne showed that the bundle $\mathcal{F}^0$ on B has an canonical extension on $\overline{\mathcal{F}^0}$ on $\overline{B}$.

The Gauss-Manin connection does not necessarily extend to a connection on $\overline{\mathcal{F}^0}$ (because of singularities)

The singularities are very mild (regular singular pts. nodes) which means that ∇ extends to a map

$$\overline{\nabla} : \overline{\mathcal{F}^0} \longrightarrow \overline{\mathcal{F}^0} \otimes \Omega^1_{\overline{B}}(\log D)$$

The sheaf $\Omega^1_{\overline{B}}(\log D)$ is a subsheaf of rational 1-forms on $\overline{B}$. in local coordinates $z_1, -, z_r$ for $\overline{B}$ such that D is defined by $z_1 - z_\ell = 0$ the sheaf $\Omega^1_{\overline{B}}(\log D)$ is generated by

$$\frac{dz_1}{z_1}, - , \frac{dz_\ell}{z_\ell}, dz_{\ell+1}, -, dz_r$$

2.6.2 Monodromy

Let $\gamma_j = \gamma_j(u)$ be a small loop going around the boundary divisor D_j based at $\gamma_j(0) = \gamma_j(1) = t \in B$.

A cohomology class $\eta \in H^k(X_t, \mathbb{C})$ can be uniquely lifted to a ∇-flat section $\eta(u) \in H^k(X_{\gamma_j(u)}, \mathbb{C})$ over $[0,1]$ s.t. $\eta(0) = \eta$.

The monodromy transformation

$$T_j : H^k(X_t, \mathbb{C}) \longrightarrow H^k(X_t, \mathbb{C})$$

is defined by $\quad T_j(\eta) = \eta(1)$

A theorem by Landman says that for some integer $m \geq 0$

$$(T_j^m - I)^{k+1} = 0$$

T_j is quasi-unipotent, with index of unipotency at most $k+1$. Eigenvalues of T_j are roots of unity.

2.6.3 The limiting mixed Hodge structure

Assume that the monodromy is unipotent and that we are at a point $p \in \overline{B}$ such that $D = \overline{B} - B$ is defined by $z_1 \ldots z_r = 0$ where $z_1, \ldots z_r$ are local coordinates at p.

We may assume $B = (\Delta^*)^r$, $\overline{B} = \Delta^r$ and $p = 0$