2. Variation of Hodge Structure

Reference

Carlson et al. Chapter 1.2

2.1 Polarized Hodge Structure

2.1.1 Definition

a) A real Hodge structure of weight k is a real vector space H on the complexification of which there is a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

and where $\overline{H^{p,q}} = H^{q,p}$

The conjugation is relative to the real structure on $H_{\mathbb{C}} = H \otimes \mathbb{C}$. We call such an object a lattice Hodge structure if in addition there is a $H_{\mathbb{Z}} \subset H$, i.e. a free abelian group such that $H_{\mathbb{Z}} \otimes \mathbb{R} \cong H$.

b) Suppose that a Hodge structure of any kind carries a bilinear form b satisfying the relations

i) $b(x,y) = 0$ if x is in $H^{p,q}$ and y is in $H^{r,s}$ for $(r+p, s+q) \neq (k,k)$

ii) $\pm i^{p-q} b(x,\bar{x}) > 0$ if x is a nonzero vector in $H^{p,q}$, $\pm$ is $(-1)^{k(k-1)/2}$

We then say that the Hodge structure is polarized. When an integral lattice is part of the structure, one requires that b take integer values on it.

2.1.21 Example: i) fix $\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}$, $E_\lambda := \{ y^2 = x(x-1)(x-\lambda) \subset \mathbb{C}^2 \}$

$H^1(E_\lambda, \mathbb{C})$ carries a Hodge structure of weight 1

$H^1(E_\lambda, \mathbb{Z}) = \mathrm{span}\{\alpha, \beta\}$, $\quad H^1(E_\lambda, \mathbb{Z}) \otimes \mathbb{R} = H^1(E_\lambda, \mathbb{R})$

$H^1(E_\lambda, \mathbb{R}) \otimes \mathbb{C} = H^1(E_\lambda, \mathbb{C})$

$H^1(E_\lambda, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, $\quad H^{1,0} = \mathrm{span}\left\{ \frac{dx}{y} \right\}$

Polarized

- $b(\omega_1, \omega_2) = \int \omega_1 \wedge \omega_2$, $\quad b\Big(\underset{\in H^{1,0}}{\frac{dx}{y}}, \underset{\in H^{1,0}}{\frac{dx}{y}}\Big) = 0$

- $(-1)^0\, i \int \frac{dx}{y} \wedge \overline{\frac{dx}{y}} = i \int (\pi^0 \alpha + \pi^1 \beta) \wedge (\overline{\pi}^0 \alpha + \overline{\pi}^1 \beta)$

$\frac{dx}{y} = \pi^0 \alpha + \pi^1 \beta \qquad = i\,(\pi^0 \overline{\pi}^1 - \overline{\pi}^0 \pi^1) \qquad \tau = \frac{\pi^1}{\pi^0}$

$= i\, |\pi^0|^2 (\overline{\tau} - \tau)$

$= i\, |\pi^0|^2 \, (-2i\, \mathrm{Im}\,\tau)$

$= 2 |\pi^0|^2\, \mathrm{Im}\,\tau > 0 \quad \square$

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ii) The k-th cohomology of a projective algebraic manifold carries a Hodge structure of weight k, where $H^{p,q}$ is represented by closed k-forms whose local expressions contain p "dz's" and q '$d\bar{z}$'s'.

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iii) Suppose the algebraic manifold is denoted by X and is n-dimensional and that Y is a smooth hyperplane section. The primitive cohomology is defined by

$$H^n_{prim}(X) = \mathrm{Ker}\left[H^n(X) \longrightarrow H^n(Y) \right].$$

This is a polarized structure of weight n.

2.2 | The Hodge Filtration

2.2.1 Definition to a given Hodge decomposition of weight k we can associate the Hodge Filtration

$$\underset{F^{k+1}}{\{0\}} \subset F^k \subset \cdots \subset F^0 = H_{\mathbb{C}}$$

defined in terms of the decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ by

$$F^p = \bigoplus_{a \geq p} H^{a,k-a}$$

(e.g. for $H^1(E_\lambda, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$, $F^2 = \{0\} \subsetneq F^1 = H^{1,0} \subset F^0 = H^1(E_\lambda, \mathbb{C})$)

A given Filtration defines a Hodge decomposition by

$$H^{p,q} = F^p \cap \overline{F^q}$$

it satisfies $\quad H_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}} \qquad (*)$

Any filtration of $H_{\mathbb{C}}$ satisfying $(*)$ is called a Hodge filtration

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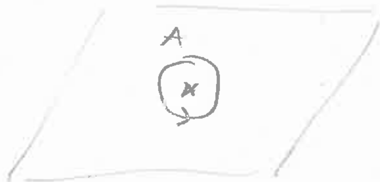
2.3 | Tate structures.

- The trivial Hodge structure is defined by the lattice $\mathbb{Z} \subset \mathbb{R}$ and the Hodge decomposition $\mathbb{C} = \mathbb{C}^{0,0}$
- The $(-k)$-th Tate twist $\mathbb{Z}(-k)$ is the Hodge structure of weight $2k$ given by the lattice $\frac{1}{(2\pi i)^k}\mathbb{Z}$ on the real line $\frac{1}{i^k}\mathbb{R}$ by the decomposition $\mathbb{C} = \mathbb{C}^{k,k}$.

e.g. Let's consider $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$



let $A \in H_1(\mathbb{C}^*, \mathbb{Z})$

and α its dual, $\alpha \in H^1(\mathbb{C}^*, \mathbb{Z})$

$$\alpha = \frac{1}{2\pi i} \frac{dz}{z} \quad , \quad \text{since } \alpha \in H_{\mathbb{Z}} \,, \; \bar{\alpha} = \alpha$$

also $\alpha \in F^1$, and in $\overline{F^1}$

$$\alpha \in F^1 \cap \overline{F^1} = H^{1,1} .$$

This is $\mathbb{Z}(-1)$.

2.4 Variation of Hodge Structure

Ref.

- Morrison: Mirror symmetry and rational curves on quintic threefolds
- Cox & Katz

We now consider a family of algebraic varietie given by $\pi : \mathcal{X} \longrightarrow B$

(B quasi-projective)

$\pi^{-1}(b) = X_b$

The cohomology of the fibers of the map π with coefficients in $\mathbb{Z}$ and $\mathbb{C}$ fit together into local systems $R^n \pi_* \mathbb{Z}$ and $R^n \pi_* \mathbb{C}$ on B.

The Hodge filtration becomes a filtration of the vector bundle $\mathcal{F}^0 := R^n \pi_* \mathbb{C} \otimes \mathcal{O}_{\mathcal{C}}$ by holomorphic subbundles

$$0 \subset \mathcal{F}^n \subset \quad \cdots \quad \subset \mathcal{F}^0 := R^n \pi_* \mathbb{C} \otimes \mathcal{O}_{\mathcal{C}}$$

The vector bundle $\mathcal{F}^0$ has a natural flat connection

$$\nabla : \mathcal{F}^0 \longrightarrow \mathcal{F}^0 \otimes \Omega_B$$ called the

<u>Gauß-Manin</u> connection, whose horizontal sections determine the local system $R^n \pi_* \mathbb{C}$.

The Griffiths transversality property says that

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_B .$$

If we set $\mathcal{H} = \mathcal{F}^0$, then $\mathcal{H}$ has the locally constant subsheaf $\mathcal{H}_{\mathbb{C}} = R^k \pi_* \mathbb{C}$ and this in turn has the subsheaf ~~of~~ $\mathcal{H}_{\mathbb{Z}}$ of integer sections (the image of $R^k \pi_* \mathbb{Z} \to R^k \pi_* \mathbb{C}$)

we call $(\mathcal{H}, \nabla, \mathcal{H}_{\mathbb{Z}}; \mathcal{F}^\bullet)$ a <u>variation of Hodge structure</u>.

<u>Example</u>: Let $\pi : \mathcal{E} \longrightarrow B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the Legendre family of elliptic curves whose fibers for $\lambda \in B$ are given by $\pi^{-1}(\lambda) = E_\lambda = \{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{C}^2$

$\varphi : \mathcal{F}^0 \longrightarrow B$ is the bundle whose fibers are $\varphi^{-1}(\lambda) = H^1(E_\lambda, \mathbb{C})$

let s be a section of $\mathcal{F}^0$ given by $s = \begin{pmatrix} \omega \\ \partial_\lambda \omega \end{pmatrix}$

we have

$$\nabla_{\frac{\partial}{\partial \lambda}} \begin{pmatrix} \omega \\ \partial_\lambda \omega \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \frac{1}{4\lambda(1-\lambda)} & \frac{2\lambda - 1}{\lambda(1-\lambda)} \end{pmatrix}}_{:= C_\lambda} \begin{pmatrix} \omega \\ \partial_\lambda \omega \end{pmatrix} \underline{d\lambda}$$

$C_\lambda \, d\lambda$ is the Gauss-Manin connection one-form.