$M_0, M_1, M_\infty \in \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; b \equiv 0 \bmod 4 \right\}$

the group is generated by M_0, M_1.

Lecture 4

Last time: $\quad \mathcal{E}_\lambda := \{ y^2 = x(x-1)(x-\lambda) \subset \mathbb{C}^2 \}$

$\omega = \frac{dx}{y}, \qquad \pi^C = \int_C \omega, \quad C \in H_1(E_\lambda, \mathbb{Z})$

$(*) \; L\, \pi^C(\lambda) = 0, \qquad L = \theta^2 - \lambda(\theta + 1/2)^2, \quad \theta = \lambda \frac{d}{d\lambda}$

We found power series solutions of $(*)$ which converge on a disk Δ_0 around any $\lambda_0 \neq 0,1$. Analytic continuation of the resulting function produces a multivalued function defined on $\mathbb{P}^1 \setminus \{0,1,\infty\}$

Let π^0, π^1 be two linearly independent solutions defined on Δ_0 and let γ be a loop in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ based at λ_0. Let $\pi^{0\prime}, \pi^{1\prime}$ be the fcts on Δ_0 obtained by analytic continuation of π^0, π^1 along γ. $\pi^{0\prime}, \pi^{1\prime}$ are also solutions of $L\pi' = 0$, they can be expressed as linear combinations of π^0, π^1

$$\begin{pmatrix} \pi^{0\prime} \\ \pi^{1\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}$$

The indicated matrix, which we denote by $\rho(\gamma)$ depends only on the homotopy class of γ is called the monodromy matrix.

The map that send γ to $\rho(\gamma)$ defines a homomorphism

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, \lambda_0) \longrightarrow GL(2,\mathbb{C})$$

it is called the <u>monodromy representation</u> and its image is called the monodromy group.

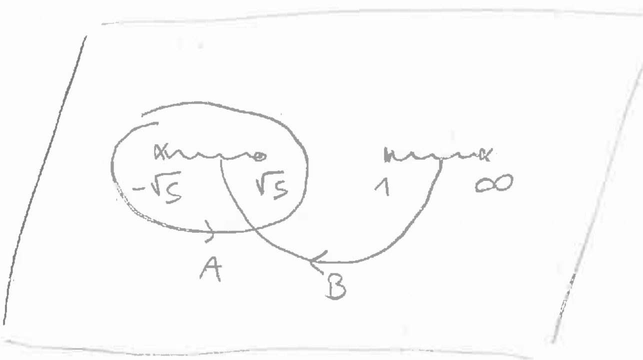
Last time $\quad \operatorname{im} \rho = \Gamma_0(4)$

<u>Today:</u> what "causes" monodromy?

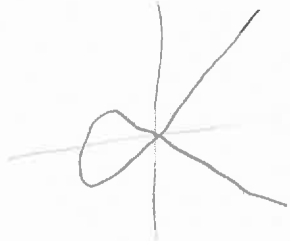
1.7 | Local monodromy representation (Topology)

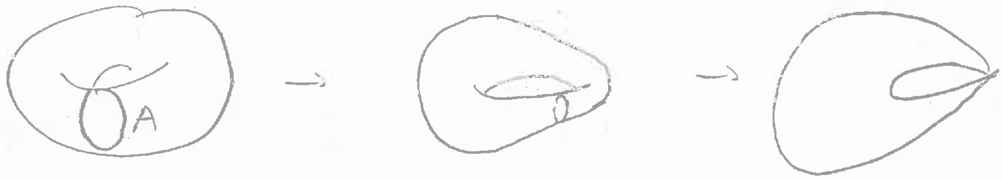
Consider $y^2 = (x^2 - s)(x-1)$

The fiber $\mathcal{E}_0$ given by $y^2 = x^2(x-1)$ has a node at $p = (0,0)$



node
(easier to see in $\mathbb{R}$ for $y^2 = x^2(x+1)$)

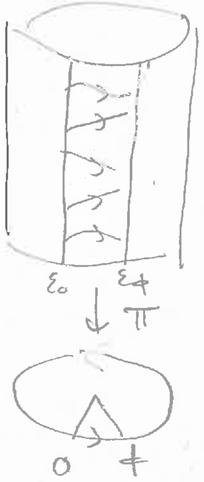




A copy of the loop A is slowly contracted to a point, producing the double point at p. In the limit of $s=0$, the cycle A is homologous to 0.

Restrict the family to the circle $|s| = \varepsilon$ and consider the vector field $\frac{\partial}{\partial \theta}$ in the s-plane. It lifts to a vector field ξ on the manifold

$M = \{ (x,y,s) \mid y^2 = (x^2-s)(x-1) \}$ of which fibers over the circle via $\pi: (x,y,s) \longmapsto s$



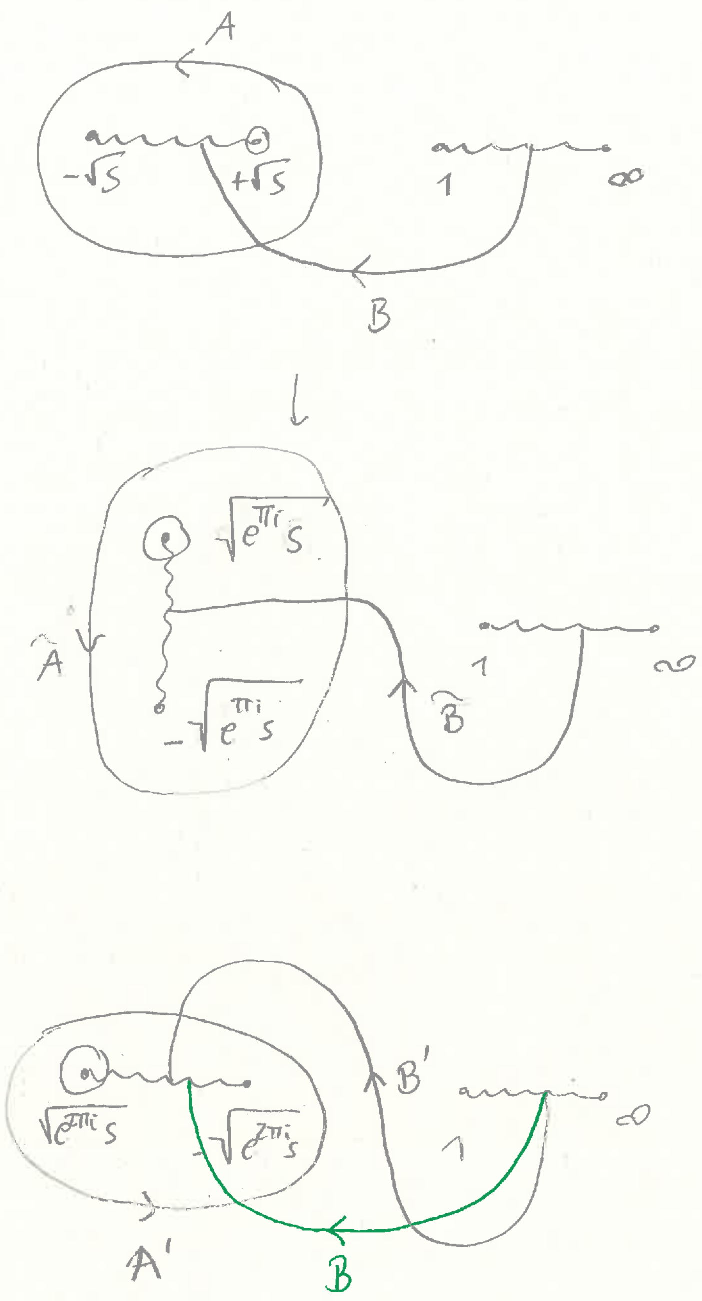
By letting the flow which is tangent to ξ act for time ϕ one obtains a diffeomorphism g_ϕ of the fiber at $\theta = 0$ onto the fiber at $\theta = \phi$.

Similarly we get a diffeomorphism $g_{2\pi}$ which carries the fiber to itself and therefore defines a map T on the homology of the fiber which depends only on the homotopy class of $g_{2\pi}$.

This is the Picard-Lefschetz transformation of the degeneration.

We now want to determine the matrix

$$T : \begin{pmatrix} A \\ B \end{pmatrix} \longmapsto T \begin{pmatrix} A \\ B \end{pmatrix}$$



we get $A' = A$

$$B' = a A + b B$$

we compute $B' \cdot A = -1 = a\,\underbrace{A\cdot A}_{=0} + b\,\underbrace{B\cdot A}_{=-1} \Rightarrow b = 1$

$$B' \cdot B = 1 = a\,\underbrace{A\cdot B}_{1} + b\,\underbrace{B\cdot B}_{=0} \Rightarrow a = 1$$

$$\Rightarrow B' = A + B$$

The matrix T relative to the basis $\{A, B\}$ is $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

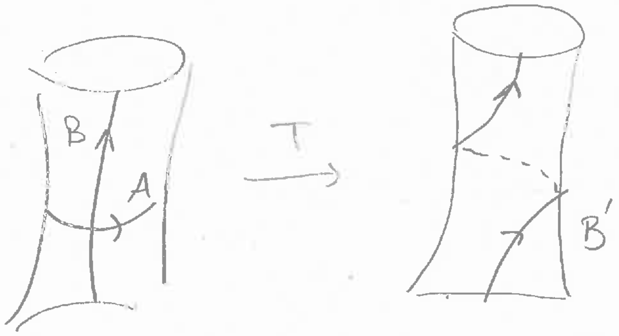
The Picard-Lefschetz formula for an arbitrary homology cycle x (for vanishing A) is:

$$T(x) = x - (x \cdot A_i) A.$$

The Picard-Lefschetz formula is valid in great generality; it holds for any degeneration of Riemann surfaces acquiring a node where the local analytic equation of the degeneration is $y^2 = x^2 - s$.

For such a degeneration the cycle A is the one that is pinched to obtain the singular fiber

In a neighbourhood of the vanishing cycle the Picard-Lefschetz diffeomorphism $g_{2\pi}$ acts as



It is called <u>Dehn twist</u>.

1.8 The global monodromy representation.

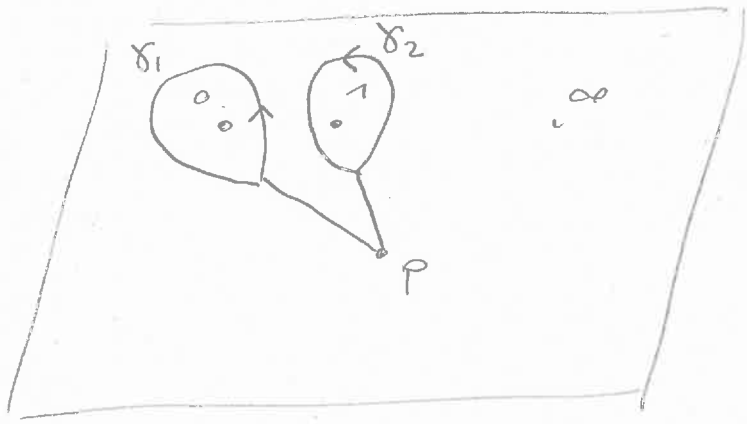
The Picard-Lefschetz transformation determines the local monodromy representation

$$\rho : \pi_1(\Delta^*, p) \longrightarrow GL(2, \mathbb{C})$$ for a family of Riemann surfaces defined on the punctured disk $0 < |s| < \varepsilon$ where the fiber at $s=0$ has a node

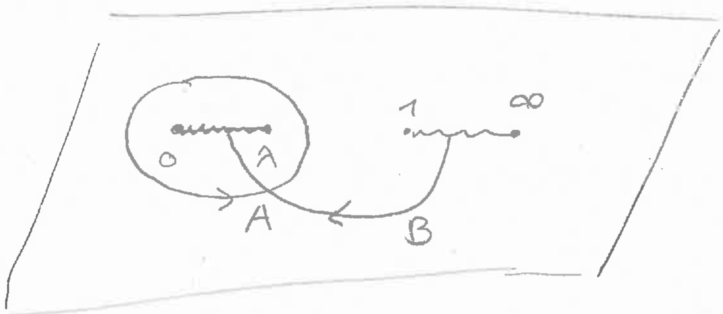
Let's determine the global monodromy representation for the Legendre family $y^2 = x(x-1)(x-\lambda)$

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, p) \longrightarrow GL(2, \mathbb{C})$$

Fix a base point $p \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ and define γ_1, γ_2 as follows



i) Consider the degeneration as $\lambda \to 0$



(note here branch cuts are different from before, adapted to λ close to 0) as λ moves counter-clockwise around 0 we get twice the monodromy considered in the nodal degeneration.

hence $\rho(\gamma_1) = T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

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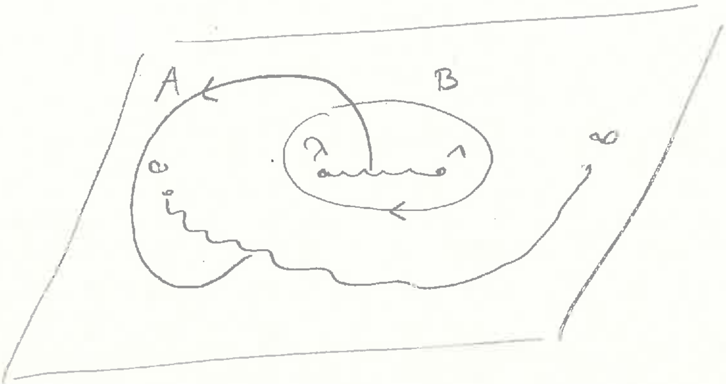
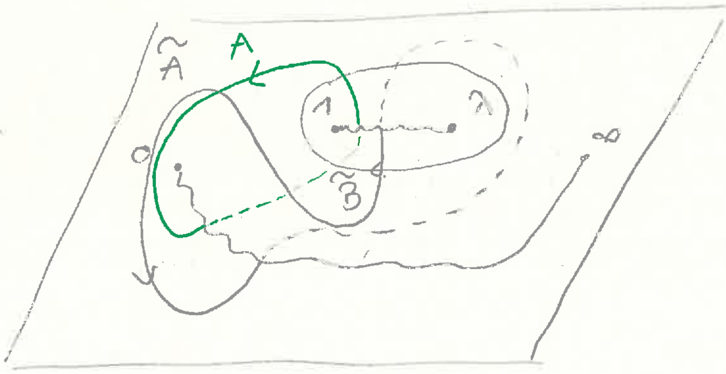
ii) Consider deg. $\lambda \to 1$



$\lambda \to \tilde{\lambda} = e^{\pi i} \lambda$



$\tilde{B} = B$

$\tilde{A} = aA + bB$

$\tilde{A} \cdot A = 1 = -b$

$\tilde{A} \cdot B = 1 = a$

$\tilde{A} = A - B$

$\tilde{T} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$\rho(\gamma_2) = \tilde{T}^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

Now let $\Gamma = \rho(\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}))$ be the monodromy group. It is the group generated by $\rho(\gamma_1)$ and $\rho(\gamma_2)$

$$\Gamma = \langle \rho(\gamma_1), \rho(\gamma_2) \rangle$$

$\rho(\gamma_1)$, $\rho(\gamma_2)$ are congruent to $\mathbb{1}$ modulo 2 so every matrix in Γ has this property. We have the following

Theorem 1.8.1:

a) $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$ is a free group on two generators.

b) the monodromy representation is injective

c) The image of the monodromy representation is $\Gamma(2)$

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \right\}$$

d) $\Gamma(2)$ has index six in $SL(2,\mathbb{Z})$

Proof: (Book of Carlson et al / exercise)

exercise $\quad \Gamma(2) \cong \Gamma_0(4)$
