1.8 Solutions of the PF equation:

1.8.1 Singular points.

For the Legendre family $\mathcal{E}_\lambda$ we derived the following differential equation

$$\left.\begin{array}{l} \lambda(\lambda-1)\,\Pi'' + (2\lambda-1)\,\Pi' + \frac{1}{4}\,\Pi = 0 \\ \\ \text{i.e.} \quad \Pi'' + \dfrac{(2\lambda-1)\,\Pi'}{\lambda(\lambda-1)} + \dfrac{1}{4\,\lambda(\lambda-1)}\,\Pi = 0 \end{array}\right\} \text{(PF)}$$

which makes it obvious that $\lambda = 0, 1$ are regular singular points, what about $\lambda = \infty$

we define $\quad x = \frac{1}{\lambda} \quad , \quad \frac{\partial}{\partial\lambda} = \frac{\partial x}{\partial \lambda}\frac{\partial}{\partial x} = -x^2 \frac{\partial}{\partial x}$.

The equation (PF) becomes

$$-x^2 \frac{d}{dx}\left(-x^2 \frac{d}{dx}\right)\Pi(x) + \frac{\left(\frac{2}{x}-1\right)}{\frac{1}{x}\left(\frac{1}{x}-1\right)}\, (-x^2)\frac{d\Pi}{dx} + \frac{1}{4\,\frac{1}{x}\left(\frac{1}{x}-1\right)}\,\Pi(x) = 0$$

$$\Leftrightarrow \quad x^4\,\Pi''(x) + 2x^3\,\Pi'(x) - \frac{(2x^2-x)}{(1-x)}\,x^2\,\Pi'(x) + \frac{x^2}{4(1-x)}\,\Pi(x) = 0$$

$$\Leftrightarrow \quad \Pi''(x) + \frac{2}{x}\Pi'(x) - \frac{(2x^2-x)}{(1-x)}\frac{1}{x^2}\,\Pi'(x) + \frac{1}{4x^2(1-x)}\,\Pi(x) = 0$$

$$\Pi''(x) + \frac{2x - 2x^2 - 2x^2 - x}{(1-x)\,x^2}\,\Pi'(x) + \frac{1}{4x^2(1-x)}\,\Pi(x) = 0$$

$$\Pi''(x) + \frac{x\,(1-4x)}{x^2\,(1-x)}\,\Pi'(x) + \frac{1}{4\,(x^2)(1-x)}\,\Pi(x) = 0$$

hence $x = 0$ is a regular singular point

$\Leftrightarrow \lambda = \infty$ is.

1.8.2 Solutions at $\lambda = 0$

We rewrite (PF) as

$$\mathcal{L} = \theta^2 - \lambda\,(\theta + 1/2)^2 , \qquad \text{where} \quad \theta_1 = \lambda \frac{d}{d\lambda} \quad \text{(Exercise)}$$

we make an ansatz $\quad \Pi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{n+\sigma}$

$\mathcal{L}\,\Pi = 0$ gives $\quad \sum_{n=0}^{\infty} a_n (n+\sigma)^2 \lambda^{n+\sigma} - \lambda \sum_{n=0} a_n (n+\sigma+1/2)^2 \lambda^{n+\sigma}$

the lowest term gives the indicial equation $\sigma^2 = 0$

i.e. we have a little problem, we can only find one solution as power series.

let $\Pi(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$.

from $\mathcal{L}\Pi = 0$ we get $\quad \sum_{n=0}^{\infty} n^2 a_n \lambda^n - \lambda \sum_{n=0}^{\infty} a_n (n+1/2)^2 \lambda^n$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} n^2 a_n \lambda^n - \sum_{n=0}^{\infty} a_n (n+1/2)^2 \lambda^{n+1}$$

$$\Leftrightarrow \quad \sum_{n=1}^{\infty} \left(n^2 a_n - a_{n-1} (n-1/2)^2\right) \lambda^n = 0$$

$$\Leftrightarrow \quad a_0 \text{ is free} \quad \text{and} \quad a_n = \frac{(n-1/2)^2}{n^2}\, a_{n-1} \qquad (R)$$

Recall the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$

$$\Gamma(n) = (n-1)!$$

it has $\mathrm{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$, $n \in \mathbb{N}$, $\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$

The recurrece (R) gives (setting $a_0 = 1$)

$$\pi^0(\lambda) = \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)^2}{\Gamma(1/2)^2\, \Gamma(m+1)^2} \lambda^m .$$

What about the second solution?

Frobenius: for ~~x~~ exponents $\alpha = \alpha'$ a second linearly independent solution can be found from the first power series by

$$\pi^1(\lambda) = \frac{1}{2\pi i} \frac{d}{d\rho} \sum_{m=0}^{\infty} \frac{\Gamma(m+\rho+1/2)^2}{\Gamma(1/2)^2\, \Gamma(m+\rho+1)^2} \lambda^{m+\rho} \Big|_{\rho=0}$$

$(z^\rho = \exp(\rho \log \lambda))$

$$\Leftrightarrow \quad \pi^1(\lambda) = \frac{1}{2\pi i} \pi^0(\lambda) \log \lambda + \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)^2}{\Gamma(1/2)^2\, \Gamma(m+1)^2} \times$$

$$\left(2\psi(m+1/2) - 2\psi(m+1) \right),$$

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log z! this is multi-valued $\Rightarrow$ monodromy

$\lambda \to e^{2\pi i}\lambda$, $\log \lambda \to \log e^{2\pi i}\lambda = \log \lambda + 2\pi i$

$$\begin{pmatrix} \pi^{0\,\prime} \\ \pi^{1\,\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{=: M_0} \begin{pmatrix} \pi^0 \\ \pi^1 \end{pmatrix}$$

We need to find M_1 and M_∞, what we should do:

i) Solve $\mathcal{L}\Pi = 0$ around $\lambda = 1$ and $\lambda = \infty$

ii) find the monodromy in the basis of solutions

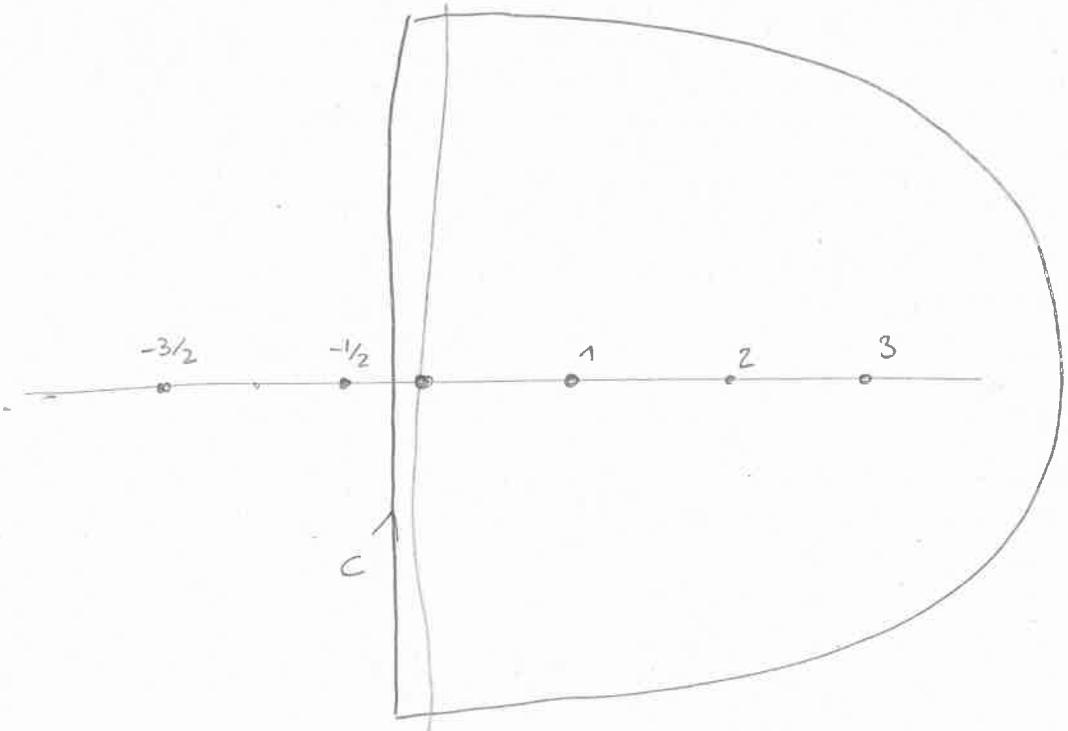
iii) Relate the different bases of solutions (analytic continuation)

For iii) it is useful to use analytic properties of the Γ function.

for example

$$\Pi^0(\lambda) = \frac{1}{2\pi i\, \Gamma(1/2)} \int_C \frac{\Gamma(-s)\, \Gamma(s+1/2)^2}{\Gamma(s+1)} (-\lambda)^s \, ds$$

$|\arg(-\lambda)| < \pi$



1.8.3 Monodromy of ${}_2F_1$

We use a shortcut, namely the expressions for $\pi^0(\lambda)$ and $\pi^1(\lambda)$ in terms of hypergeometric functions

recall: $${}_2F_1(a,b;c;z) := P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{Bmatrix}$$

we get $$\pi^0(\lambda) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \lambda)$$

$$\pi^1(\lambda) = \frac{-1}{2i}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-\lambda)$$

with monodromy $M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

to find the monodromy around $\lambda = 1$ we write

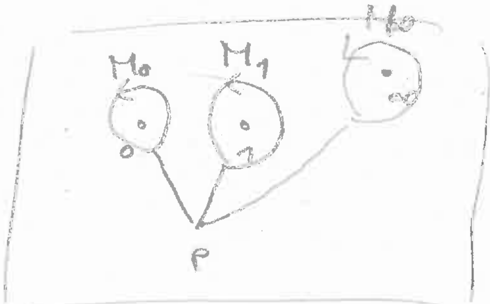
$u = 1-\lambda$ and get

$$\pi^0(u) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-u)$$

$$\pi^1(u) = \frac{-1}{2i}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; u)$$

and immediately obtain $M_1 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$

we know that



$$M_\infty^{-1} \overset{!}{=} M_1 \cdot M_0$$

$$\Rightarrow \quad M_\infty = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$$

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$$M_0, M_1, M_\infty \in \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \;\middle|\; b \equiv 0 \bmod 4 \right\}$$

the group is generated by M_0, M_1.
