1.6 Picard-Fuchs equation:

1.6.1 Prop ω and $\omega' = \frac{d\omega}{d\lambda}$ define linearly independent cohomology classes

$$\Pi'(\lambda) = \tau\, \Pi^0(\lambda)$$
$$\frac{d}{d\lambda}\Pi'(\lambda) = \frac{d\tau}{d\lambda}\Pi^0(\lambda) + \tau \frac{d\Pi^0}{d\lambda}$$

Proof: $[\omega] = \Pi^0(\lambda)\,\alpha + \Pi'(\lambda)\,\beta$

$$[\omega'] = \frac{d}{d\lambda}\Pi^0(\lambda)\,\alpha + \frac{d}{d\lambda}\Pi'(\lambda)\,\beta$$
$$= \Pi^{0\prime}(\lambda)\,\alpha + \tau'\,\Pi^0(\lambda)\,\beta + \tau\,\Pi^{0\prime}(\lambda)\,\beta$$
$$= \frac{\Pi^{0\prime}(\lambda)}{\Pi^0(\lambda)}[\omega] + \tau'\,\Pi^0(\lambda)\,\beta$$

, □ since $\tau' \neq 0$

The class of the second derivative must be expressible as a linear combination of the first two classes, consequently there is a relation

(*) $a(\lambda)\,\omega'' + b(\lambda)\,\omega' + c(\lambda)\,\omega = 0$ in cohomology.

The coefficients are meromorphic functions of λ and on the level of forms, the assertion is that the L.H.S. is exact on $\mathcal{E}_\lambda$. Let C be a one cycle, define $\Pi(\lambda) := \int_C \omega$, then (*) becomes

$$a\Pi'' + b\Pi' + c\Pi = 0.$$

1.6.2 Derivation of the Picard-Fuchs equation:

we have

$$\left.\begin{aligned} \omega &= x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-1/2}\,dx \\ \frac{d\omega}{d\lambda} &= \frac{1}{2}\,x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-3/2}\,dx \\ \frac{d^2\omega}{d\lambda^2} &= \frac{3}{4}\,x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-5/2}\,dx \end{aligned}\right\}(*)$$

Idea: look for a rational function f on E_λ such that df is a linear combination of ω, ω' and ω''
(This would give an exact form as a candidate for the R.H.S)

What about $f = x^{1/2}(x-1)^{1/2}(x-\lambda)^{-3/2}$

$$df = \frac{1}{2}x^{-1/2}(x-1)^{1/2}(x-\lambda)^{-3/2}\,dx + \frac{1}{2}x^{1/2}(x-1)^{-1/2}(x-\lambda)^{-3/2}\,dx - \frac{3}{2}x^{1/2}(x-1)^{1/2}(x-\lambda)^{-5/2}\,dx$$

use $(*)$

$$df = (x-1)\,\omega' + x\,\omega' - 2x(x-1)\,\omega''$$

but the coefficients are not functions of λ, consider instead

$$df = [(x-\lambda)+(\lambda-1)]\omega' + [(x-\lambda)+\lambda]\omega' - 2[(x-\lambda)+\lambda][(x-\lambda)+(\lambda-1)]\omega''$$

and use the relations $(x-\lambda)\omega' = \frac{1}{2}\omega$, $(x-\lambda)\omega'' = \frac{3}{2}\omega'$

to obtain $-\frac{1}{2}df = \frac{1}{4}\omega + (2\lambda-1)\,\omega' + \lambda(\lambda-1)\,\omega''$

We conclude that

$\lambda(\lambda-1)\,\Pi'' + (2\lambda-1)\,\Pi' + \frac{1}{4}\Pi = 0$ is a differential equation satisfied by the periods $\Pi(\lambda)$.

1.7 Excursion: Second order differential equations with three regular singular points

We start with a general second-order differential equation:

$$u''(z) + p(z)u'(z) + q(z) = 0 \qquad (1.7)$$

where p and q are in general meromorphic functions.

1.7.1 Definitions

A $z = a$, this equation has an:

i) ordinary (regular) point if $p(z)$ and $q(z)$ are analytic at a.

ii) regular singular point if $p(z)$ or $q(z)$ is not analytic at a but $(z-a)p(z)$ and $(z-a)^2 q(z)$ are analytic at a. (p has a simple pole at a or analytic at a.)
q has a simple or double pole at a)

iii) irregular singular point if $(z-a)p(z)$ and $(z-a)^2 q(z)$ are not analytic at a.

If a differential equation has only ordinary and regular singular points on $\mathbb{C}$, then it is called a Fuchsian differential equation.

1.7.2 Finding solutions

i) at an ordinary point u must be analytic, so we can substitute a Taylor series and compute order by order using recurrence relations.

ii) To find solutions at a regular singular point we use the method of Frobenius.

we make an ansatz with a series of the form

$$(*) \quad u(z) = \sum_{n=0} a_n (z-a)^{n+\sigma}$$

and determine σ from the indicial equation.

Plugging $(*)$ into (1.7) we get

$$\sum a_n (n+\sigma)(n+\sigma-1)(z-a)^{(n+\sigma-2)} + p(z) \sum a_n (n+\sigma)(z-a)^{(n+\sigma-1)}$$

$$+ q(z) \sum a_n (z-a)^{n+\sigma} = 0 \qquad (1.7.2)$$

if $p(z) = \frac{B}{z-a} + \mathcal{O}(1)$, $q(z) = \frac{C}{(z-a)^2} + \mathcal{O}\left(\frac{1}{z-a}\right)$ as $z \to a$

we consider the lowest order term of $(1.7.2)$ and require its vanishing

$$\sigma(\sigma-1)(z-a)^{\sigma-2} + B\sigma(z-a)^{\sigma-2} + C(z-a)^{\sigma-2}$$

$$\Rightarrow \quad \sigma^2 + (B-1)\sigma + C = 0$$

This has two roots α, α' that satisfy

$$\alpha + \alpha' = 1 - B, \quad \alpha\alpha' = C$$

we call these the exponents of the differential equation at the regular singular point

The solutions of a differential equation with three regular singular points at a, b, c and exponents α, α'; β, β'; γ, γ' is characterized by the Riemann $\mathcal{P}$-symbol (Papperitz)

$$u = \mathcal{P}\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}.$$

<u>Remark:</u> using Möbius transformation a, b, c can be mapped to $0, \infty, 1$ we further have

$$\mathcal{P}\left\{\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}\right\} = \left(\frac{z-a}{z-b}\right)^{\alpha} \left(\frac{z-c}{z-b}\right)^{\gamma} \mathcal{P}\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & w \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{matrix}\right\}$$

where $w = \dfrac{(c-b)(z-a)}{(c-a)(z-b)}$