Last time

M complex manifold of dim n

η (non-degenerate hol. complex quadratic form)

g Hermitian positive definite form

compatible $\exists$ D s.t. $D\eta = 0$, $Dg = 0$, $\Gamma_a = g^{-1}\partial_a g$

M real structure $M_a^{\,b} = g_{\bar{a}c}\eta^{cb}$, $M\overline{M} = id$

$M \to$ anti-complex involution τ

$$\underset{\substack{\uparrow \\ g}}{(X,Y)} = \underset{\substack{\uparrow \\ \eta}}{\langle \tau(X), Y \rangle}, \quad DM = 0$$

M Quasi-Frobenius $(X \cdot Y)^c(z) = X^a(z)\, Y^b(z)\, c_{ab}^{\,c}(z)$

$\exists$ commutative associative algebra on $\mathrm{Vect}(M)$, the space of hol. vector fields.

Frobenius if the curvature of

$\tilde{\nabla}_X^{(\lambda)} Y = \nabla_X Y + \lambda X \cdot Y$ vanishes identically in λ.

4.3 tt^* structure

Denote by C_a the operators $(C_{ab}^c(z))$

4.3.1 Definition: A compatible pair η, g on a quasi-Frobenius manifold M determines a tt^* structure on it if

$$D_a C_b = D_b C_a$$

$$[D_a, D_{\bar b}] = -[C_a, C_{\bar b}]$$

where $C_{\bar b} = M \overline{C_b} \overline{M}$

the normalized Hermitian metric

$$\tilde g_{\bar a b}\, d\bar z^a dz^b = \frac{g_{\bar a b}\, d\bar z^a dz^b}{g_{\bar a b}\, \bar e^a e^b}$$

is called the Zamolodchikov metric on M.

Rem. 4.3.2:

The tt^* equations can be written in the form

$$\bar\partial_b (\partial_a M \cdot \overline{M}) = C_a M \overline{C_b} \overline{M} - M \overline{C_b} \overline{M} C_a,$$

Proposition 4.3.3 The equations

$$\partial_a \xi = \lambda C_a \xi - \Gamma_a \xi$$
$$\bar\partial_a \xi = \lambda^{-1} M \overline{C_a} \overline{M} \xi \qquad (*)$$

where the matrix coefficients

$$C_a = (C_{ab}^c(z)), \quad \Gamma_a = (\Gamma_{ab}^c), \quad M = M_a^b$$

obey the conditions

$$M\overline{M} = 1$$
$$M^T \eta M = \overline{\eta}$$
$$\partial_k \eta = \eta \Gamma_k + \Gamma_k^T \eta$$
$$\eta C_a = C_a^T \eta$$

for a hol. symmetric nondegenerate $\eta = (\eta_{ab})$ are compatible identically in the spectral parameter λ iff the pair η, M is compatible with Γ^c_{ab} ~~as~~ determining a tt^* structure on the quasi-Frobenius mfd.

Remark 4.3.4

The tt^* equations can be interpreted as vanishing of the curvature of the λ-dependent connection

$$\tilde{D}^{(\lambda)}_X Y = D_X Y - \lambda X \cdot Y$$
$$\tilde{D}^{(\lambda)}_{\overline{X}} Y = D_{\overline{X}} Y - \lambda^{-1} \tau(X \cdot \tau(Y))$$

4.4 2d physical origin

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