Based on Dubrovin: hep-th\9206037

4 tt^* geometry

4.1 Real structure

Let M be a complex manifold of dim. n with a non-degenerate holomorphic complex quadratic form

$$\eta = \eta_{ab}(z)\, dz^a dz^b \quad , \quad \det(\eta_{ab}) \neq 0$$

and a Hermitian positive definite form

$$g = g_{\bar a b}(z, \bar z)\, d\bar z^a dz^b \quad , \quad \overline{g_{\bar b a}} = g_{\bar a b}$$

$$\det(g_{\bar a b}) \neq 0$$

<u>Definition 4.1.1</u>.

The pair η, g is called compatible if there exists a complex connection $D = (\Gamma^c_{ab} , \Gamma^{\bar c}_{\bar a \bar b} = \overline{\Gamma^c_{ab}})$ where for any complex vector field $X = X^a \partial_a$

$$D_c X^a = \partial_c X^a + \Gamma^a_{cb} X^b \, ,$$

$$D_{\bar c} X^a = \bar\partial_c X^a$$

$$\overline{D_c} = D_{\bar c}$$

such that

$$D_c \eta_{ab} \equiv \partial_c \eta_{ab} - \Gamma^d_{ca} \eta_{db} - \Gamma^d_{cb} \eta_{db} = 0$$

$$D_c g_{\bar a b} \equiv \partial_c g_{\bar a b} - \Gamma^d_{cb} g_{\bar a d} = 0 \quad (*)$$

(likewise $D_{\bar c} \overline{\eta_{ab}} = 0, \; D_{\bar c}\, g_{\bar a b} = 0$)

The connection ∇ for a compatible pair η, g is determined uniquely

from ⊛ we have $\Gamma_c = g^{-1}\partial_c g$, $g = (g_{\bar{a}b})$

Definition/prop 4.1.2:

The tensor $M = (M^{b}_{\bar{a}})$ is called the real structure by Cecotti & Vafa. It is defined by

$$M^{b}_{\bar{a}} = g_{\bar{a}c}\,\eta^{cb}, \qquad (\eta^{cb}) = (\eta_{cb})^{-1}$$

it obeys the equation $M\overline{M} = \text{const.}\ id$

(tedious but straightforward)

The compatible pair η, g is called normalized if

$$M\overline{M} = id$$

4.1.3 Anti-complex involution τ

On the complexified tangent space $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ we define an anti-complex involution τ as follows

$$\tau(X^a\partial_a + X^{\bar{a}}\bar{\partial}_a) = M^{b}_{\bar{a}}\,\overline{X^a}\,\partial_b + \overline{M^{b}_{\bar{a}}}\,\overline{X^{\bar{a}}}\,\bar{\partial}_b$$

$$\tau(T^{1,0}M) = T^{1,0}M, \quad \tau(T^{0,1}M) = T^{0,1}M$$

$$\tau^2 = 1, \quad \tau(\lambda x) = \bar{\lambda}\,\tau(x) \quad \text{for } \lambda \in \mathbb{C}$$

The operator τ commutes with complex conjugation

$$T^{1,0} \longrightarrow T^{0,1}, \quad X \mapsto \overline{X}$$

i.e. $\quad \tau(\overline{X}) = \overline{\tau(X)}$

The complex inner product $\quad \langle X, Y \rangle = \eta_{ab} X^a Y^b$

and the Hermitian scalar product $\quad (X, Y) = g_{\bar{a}b} \overline{X^a} Y^b$

are related by the equation

$$(X, Y) = \langle \tau(X), Y \rangle$$

The operator τ is anti-orthogonal w.r.t. $\langle , \rangle$:

$$\langle \tau(X), \tau(Y) \rangle = \overline{\langle X, Y \rangle}.$$

It is covariantly constant w.r.t. D

$$D_c M^{b}_{\bar{a}} = \partial_c M^{b}_{\bar{a}} + \Gamma^{b}_{cd} M^{d}_{\bar{a}} = 0$$

All compatible normalized pairs η, g with fixed η are in 1-1 correspondence with anti-complex involutions of the form above

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The group of holomorphic automorphisms

$A = (A_a^b(z))$ of $T^{1,0}H$ acts on normalized

compatible pairs as follows

$$\eta \longmapsto A^T \eta A, \quad g \longmapsto A^+ g A, \quad M \longmapsto A^{-1} M \overline{A}$$

$$\Gamma_a \longmapsto A^{-1} \Gamma_a A + A^{-1} \partial_a A$$

The connection D for the compatible pair η, g is

not symmetric. If $\hat{\Gamma} = (\hat{\Gamma}^c_{ab})$ is the Levi-Civita

connection for the metric η (i.e. $\hat{\Gamma}^c_{ab} = \hat{\Gamma}^c_{ba}$

$$\hat{D}_c \eta_{ab} \equiv \partial_c \eta_{ab} - \hat{\Gamma}^d_{ca} \eta_{db} - \hat{\Gamma}^d_{cb} \eta_{ad} = 0$$

then the difference $T^c_{ab} = \Gamma^c_{ab} - \hat{\Gamma}^c_{ab}$

is a $\binom{1}{2}$ tensor. It obeys the symmetry

$$T^c_{ab} \eta_{cd} + T^c_{ad} \eta_{cb} = 0$$

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4.1.4 Proposition:

For any anticomplex involution τ in an n-dim complex space T there exists an n-dimensional τ-invariant real subspace $V \subset T$ s.t. T is isomorphic to the complexification of V

$$\tau|_V = id, \quad T = V \oplus iV$$

Proof let $V_\pm = \left(\frac{1 \pm \tau}{2}\right) T$

$$T = V_+ \oplus V_- \quad , \quad \tau|_{V_\pm} = \pm 1$$

$iV_+ \subset V_-$, let $x \in V_+$, $ix \in V_x$

$$\tau(x) = x \;, \quad \tau(ix) = -ix \Rightarrow ix \in V_-$$

let $x \in V_-$ $\quad \tau(x) = -x \;, \quad x = iy$

$$\tau(iy) = -iy \;, \quad y \in V_+ \;, \quad x \in iV_+$$

hence $V_- = iV_+$, putting $V = V_+$ we get $T = V \oplus iV$

Outlook: If a basis of T is chosen in V, then the operator τ is represented by the unity matrix. ie. $M = (M^b_a)$ can be represented $M = \Phi \bar{\Phi}^{-1}$

$G = \Phi^T g \Phi = \Phi^\dagger g \Phi$ is real symmetric

4.2 Frobenius manifolds.

4.2.1 Definition: i) A commutative associative algebra A with a unity is called Frobenius if there is a non-degenerate invariant inner product $\langle , \rangle$ on A

$$\langle a.b , c \rangle = \langle a, b.c \rangle .$$

ii) M is called a complex quasi-Frobenius manifold if a a structure of a Frobenius algebra over the ring $F(M)$ of holomorphic functions on M is fixed on the space $Vect(M)$ of holomorphic vector fields.

It is assumed that the invariant inner product on $Vect(M)$ is specified by a non-degenerate holomorphic quadratic form η.

In local complex coordinates the multiplication law and inner product read

$$(X.Y)^c(z) = X^a(z) Y^b(z) c^c_{ab}(z)$$

$$\langle X, Y \rangle = \eta_{ab}(z) X^a(z) Y^b(z)$$

where c^c_{ab}, η_{ab} are hol tensors on M

These satisfy

$$c^c_{ba} = c^c_{ab}$$

$$c^s_{ab} c^d_{sc} = c^d_{as} c^s_{bc}$$

$$c_{abc} \equiv c^s_{ab} \eta_{sc} = c_{acb}$$

(ii) A quasi-Frobenius M is called a Frobenius manifold (Dubrovin) if the curvature of the connection

$$\tilde{\nabla}^{(\lambda)}_X Y = \nabla_X Y + \lambda\, X \cdot Y$$

vanish identically in the spectral parameter λ. here ∇ is the Levi-Civita connection for η.

The complex metric η on a Frobenius manifold M is flat. That means that in an appropriate local coordinate system t^α, $\alpha = 1, \ldots, n$ η has constant form

$$\eta = \eta_{\alpha\beta}\, dt^\alpha dt^\beta, \qquad \eta_{\alpha\beta} = \text{const}.$$

The structure constants $c_{\alpha\beta\gamma}(t)$ can be represented as

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t) \quad , \quad \partial_\alpha := \frac{\partial}{\partial t^\alpha}$$

for some function $F(t)$ (Recall the prepotential)

4.2.2 Remarks

i) In a topological field theory with n primary fields $\phi_1, \dots, \phi_n$ the tensors $\eta_{\alpha\beta}$ and $c_{\alpha\beta\gamma}$ are the two-point and three-point functions.

The coordinates $t^1, \dots, t^n$ are the coupling constants of the perturbed TFT $\quad \mathcal{L} \to \mathcal{L} - \sum t^\alpha \int \phi_\alpha$

$F(t)$ is the tree level free energy

ii) The associativity conditions

$$c^s_{ab}\, c^d_{sc} = c^d_{as}\, c^s_{bc}$$

$$c_{abm}\, \eta^{ms}\, c_{scn}\, \eta^{nd} = \eta^{nd}\, c_{asn}\, \eta^{sm}\, c_{mbc}$$

is a system of PDE's called the WDVV equations

(Witten - Dijkgraaf - Verlinde - Verlinde)

The flatness of the connection (**) gives the zero-curvature representation depending on a spectral parameter of WDVV.