Lecture 1

Variation of Hodge Structure and tt^* Geometry

Reference

Period mappings and Period Domains
Carlson, Müller-Stach, Peters

1. Intuition for VHS from elliptic curves

1.1) The Legendre family:

~~we d~~ Fix $\lambda \in \mathbb{C}$ and define

$$\mathcal{E}_\lambda := \{ y^2 = x(x-1)(x-\lambda) \subset \mathbb{C}^2 \}$$

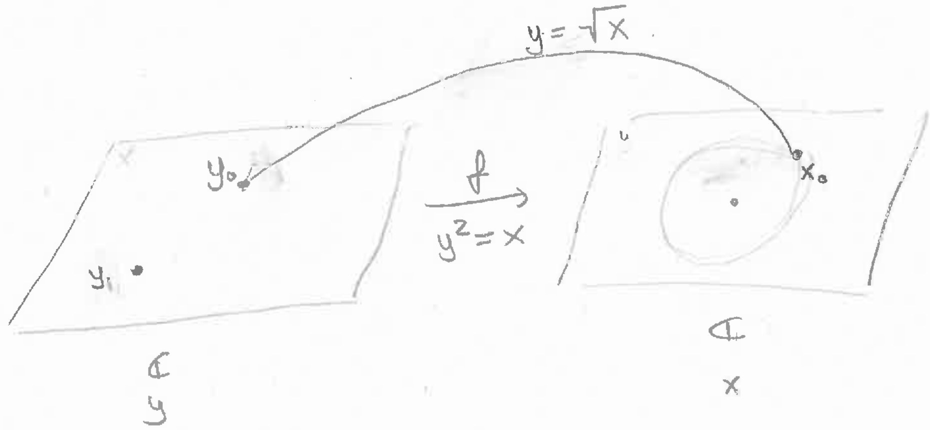
this is an elliptic curve, i.e. a complex one-dimensional Riemann surface of genus 1. For all $\lambda \neq 0, 1$ the $\mathcal{E}_\lambda$ are isomorphic as topological spaces and differentiable manifolds. (We want to show that $\mathcal{E}_\lambda, \mathcal{E}_{\lambda'}$, $\lambda \neq \lambda'$ they are not isomorphic as complex manifolds).

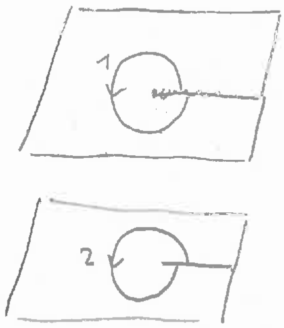
We call the family $\mathcal{E}_\lambda$ the Legendre family.

1.2) Homology of $\mathcal{E}_\lambda$

Recall the construction of R.S. associated to $\sqrt{}$

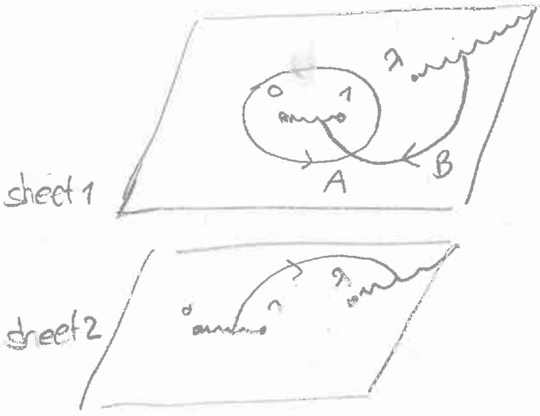
$$X = \{ y^2 = x \subset \mathbb{C}^2 \}$$



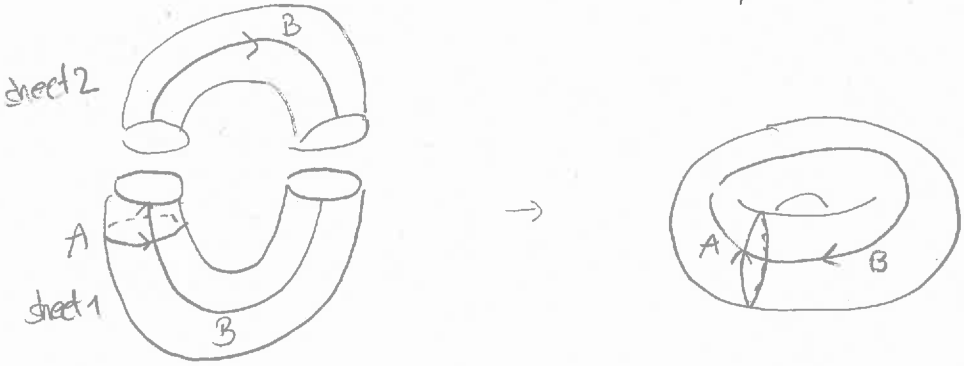


flip+glue $\Rightarrow$ R.S.

Now: $y^2 = x(x-1)(x-\lambda)$, the analytic continuation of y in the complement of the cuts defines a single valued fct, we call its graph a sheet of the R.S.



A lies on a single sheet of the R.S.



The two cycles A, B form a basis for the first homology of E_λ; their intersection matrix is given by

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

1.3 Cohomology of $\mathcal{E}_\lambda$:

Recall that a holomorphic one form ω can be written locally as $\omega = f(z)\,dz$, where z is a local coordinate and $f(z)$ is a holomorphic function (Ex. ω is closed)

We will use on $\mathcal{E}_\lambda$ the form

$$\omega = \frac{dx}{y} \quad \text{which is holomorphic (Ex)}$$

$$\omega = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

We will denote by α, β a basis of $H^1(\mathcal{E}_\lambda, \mathbb{Z})$, which is dual to A, B, i.e.

$$\int_A \alpha = 1, \quad \int_B \alpha = 0$$

$$\int_A \beta = 0, \quad \int_B \beta = 1$$

The cohomology class of $[\omega]$ can be written in terms of this basis as

$$[\omega] = \left(\int_A \omega\right)\alpha + \left(\int_B \omega\right)\beta$$

$\pi^0 = \int_A \omega$ and $\pi^1 = \int_B \omega$ are called the periods of ω

The expression $(\pi^0 \;\; \pi^1)$ is called the period vector of $\mathcal{E}_\lambda$

From the periods we want to construct an invariant which detects the changes in complex structure.

1.4 Hodge decomposition

1.4.1 Theorem Let $H^{1,0}$ be the subspace of $H^1(\mathcal{E}_\lambda, \mathbb{C})$ spanned by ω and let $H^{0,1}$ be the complex conjugate of this space, then

$$H^1(\mathcal{E}, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

this decomposition is called the Hodge decomposition.

Proof: Take the cup product

$$[\omega] \sqcup [\bar{\omega}] = (\pi^0 \overline{\pi^1} - \pi^1 \overline{\pi^0})\, \alpha \sqcup \beta$$

multiply by i and integrate the resulting form over $\mathcal{E}_\lambda$

$$i \int_{\mathcal{E}_\lambda} \omega \wedge \bar{\omega} = 2\, \mathrm{Im}(\pi^1 \overline{\pi_0}).$$

Because locally $\omega = f(z)\, dz$

$$i\,(\omega \wedge \bar{\omega}) = i |f|^2\, dz \wedge d\bar{z} = 2|f|^2\, dx \wedge dy > 0$$

hence $\mathrm{Im}(\pi^1 \overline{\pi^0}) > 0$

neither π^0 nor π^1 can be zero, therefore the cohomology class of ω cannot be zero and $H^{1,0}$ is non-zero.

We can rescale ω s.t. $\pi^0 = 1$, we then have $\mathrm{Im}\, \pi^1 > 0$

now suppose $H^{1,0}$ and $H^{0,1}$ do not give a direct sum

i.e. $H^{1,0} \cap H^{0,1} \neq \{0\}$ then $H^{1,0} = H^{0,1}$

and $[\bar{\omega}] = \lambda [\omega]$, therefore

$$\alpha + \overline{\pi^1} \beta = \lambda (\alpha + \pi^1 \beta), \qquad \lambda = 1, \ \pi^1 = \overline{\pi^1}$$

↯ since $\mathrm{Im}\, \pi^1 > 0$

1.5 τ - invariants

Suppose $f: \mathcal{E}_\mu \longrightarrow \mathcal{E}_\lambda$ is an isomorphism of complex mflds.
Let ω_μ and ω_λ be the given hol. formss, then
we claim that

$$(*) \qquad f^* \omega_\lambda = c\, \omega_\mu \quad \text{for } c \neq 0,$$

true on the level of $[\omega_\mu], [\omega_\lambda]$

On the one hand

$$\int_{[\mathcal{E}_\mu]} f^* \omega_\lambda \wedge f^* \overline{\omega_\lambda} = |c|^2 \int_{[\mathcal{E}_\mu]} \omega_\mu \wedge \overline{\omega_\mu}$$

and on the other

$$\int_{[\mathcal{E}_\mu]} f^* \omega_\lambda \wedge f^* \overline{\omega_\lambda} = \int_{f_*[\mathcal{E}_\mu]} \omega_\lambda \wedge \overline{\omega_\lambda} = \int_{[\mathcal{E}_\lambda]} \omega_\lambda \wedge \overline{\omega_\lambda}$$

$$\Rightarrow c \neq 0 \quad (**)$$

We define
$$\tau(\mathcal{E}, A, B) = \frac{\int_B \omega}{\int_A \omega}$$

from $(*), (**)$ we have shown that it is an invariant, we thus have the following theorem

1.5.1) Theorem

If $f: \mathcal{E} \longrightarrow \mathcal{E}'$ is an isomorphism of complex manifolds, then $\tau(\mathcal{E}, A, B) = \tau(\mathcal{E}', A', B')$ where
$A' = f_* A$, $B' = f_* B$.

(Exercise $\tau(\lambda)$ is not constant)

1.5.2) Theorem

Suppose $\lambda \neq 0, 1$, then there is a $\varepsilon > 0$ s.t. for all λ' within distance ε from λ, the R.S $\mathcal{E}_\lambda$ and $\mathcal{E}_{\lambda'}$ are not isomorphic as complex manifolds.