Lecture (3) - Presheaves to Sheaves

Recall: • A PRESHEAF on a category $\mathcal{T}$ is a contravariant functor to sets

$$P : \mathcal{T}^{op} \longrightarrow Sets$$

• A presheaf VALUED IN another category $\mathcal{C}$ is

$$P : \mathcal{T} \longrightarrow \mathcal{C}$$

(often interested when $\mathcal{C}$ = [Abelian Groups]
= [Rings]
etc...

Note: These are cases where $\mathcal{C}$ is an Abelian Category. We'll see that $\mathcal{C}$-valued presheaves then also form an Abelian category

eg: $\exists$ biproduct: $P_1 \oplus P_2 (x) = P_1(x) \oplus P_2(x)$

(structures inherited pointwise).)

- We'll return to special properties of $\mathcal{C}$ later...

Recall: A SHEAF is a presheaf on a category $\mathcal{T}$ which has some special properties "gluing" & "locality".

Susama defined these when $\mathcal{T} = TOP(X)$

i.e. $\mathcal{T}$ has: objects: open sets of $X \in TOP$

Morphisms: Inclusions $U \hookrightarrow W$

Note: Top(X) is a PARTIAL ORDER. Therefore it is a category w/ "$\subseteq$" as the morphisms.

28 Q: What special property of $Top(X)$ allows the defining properties of a sheaf to be stated?

i.e.: 1) <u>LOCALITY</u>

If $\{U_i\}_I$ is an open cover of U

then $\forall s,t \in \mathcal{F}(U)$ such that

$s|_{U_i} = t|_{U_i} \ \forall i$

we also have $s = t$.

2) <u>GLUING</u>

If $\{U_i\}_I$ is an open cover of U,

and $s_i \in \mathcal{F}(U_i)$, $s_j \in \mathcal{F}(U_j)$, ... (for all $i \in I$)

agree on $U_{ij} = U_i \cap U_j$ $(\forall i,j)$

(i.e. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$)

then there is a unique $s \in \mathcal{F}(U)$

s.t. $s_i = s|_{U_i}$

Notation: $s|_{U_i} = \mathcal{F}(\leq)(s)$

since "$\leq$" is a morphism $U_i \leq U$

and $\mathcal{F}$ is contravariant

$$\mathcal{F}(\leq) : \mathcal{F}(U) \longrightarrow \mathcal{F}(U_i)$$
$$\underset{s}{\wedge} \longmapsto s|_{U_i}$$

<u>Model</u>: These are supposed to capture the fact that "CONTINUITY IS A LOCAL PHENOMENON"

eg// $\mathcal{F} = \mathcal{C} : Top(X) \longrightarrow Sets$

$U \longmapsto \mathcal{C}(U, \mathbb{R})$ cts f^{n}s on U

check Sheaf Conditions

1) Two cts f^{n}s that agree on all open subsets are equal

2) Continuous functions can be extended...

~~[illegible] (From Presheaves to Sheaves I)~~

<u>Idea</u>: · The sheaf condition uses the notion of "OPEN COVER" of an object.

· An open cover $\{\mathcal{U}_i\}$ of $\mathcal{U}$ is a SUBSET of $SUB(\mathcal{U}) = \{$sub~~sets~~ objects of $\mathcal{U}\}$

· In $TOP(X)$, there is a natural notion of cover, because we implicitly use:

<u>Fact</u>: $(TOP(X), \leq)$ is a COMPLETE LATTICE:

· It has operations $\bigvee \overset{(\text{join})/\sup}{=} \bigcup$, $\wedge \overset{(\text{meet})/\inf}{=} \bigcap$

(existence of limits) – closed under joins: $\bigvee_{i \in I} \mathcal{U}_i \in Top(X)$

i.e. unions of open sets are open complete

– Infinite distributive law:

$$V \wedge (\bigvee \mathcal{U}_i) = \bigvee (V \wedge \mathcal{U}_i)$$

<u>Defn</u>: · A FRAME is a complete lattice w/ infinite distributive law.

· A FRAME MORPHISM is a lattice map (preserves all meets & joins, including $0, 1$ i.e. min & max elt, namely $\emptyset, X$ in $Top(X)$)

<u>Note</u>: $TOP(-)$ is contravariant, since any cts map $f : X \to Y$ induces a frame map $TOP(Y) \xrightarrow{f^{-1}} TOP(X)$

(which gives $f^{-1}(U) \subseteq_{op} X \ \forall\, U \subseteq_{op} Y$. by definition of continuity!)

<u>Defn</u>: The CATEGORY OF LOCALES is

$$Loc = Frm^{op}$$

<u>Note</u>: In any locale, we can recognize an open cover of U **as**: a collection of subobjects $U_i \leq U$ s.t. $\bigvee_{i \in I} U_i = U$ ("$U = \sup_i U_i$")

(In $\mathrm{TOP}(X)$, we might have U_i not contained in U
$\Rightarrow$ replace U_i by $U_i' = U \wedge U_i$
Then this definition works).

<u>Lemma</u>: If $\mathrm{Sob} \subset \mathrm{Top}$ is the category of SOBER SPACES ~~…~~.

[i.e. T_0 space (. ◯)
s.t. the only $U \subset X$ s.t. ($W \wedge V \subset U \Rightarrow$ either $W \subset U$ or $V \subset U$)
are $U = X \setminus \{x\}$).

e.g. (loc. compact Hausdorff) ✓

Then the map

$$\begin{aligned} \mathrm{TOP}(-) : \mathrm{Top} &\longrightarrow \mathrm{Loc} \\ X &\longmapsto \mathrm{TOP}(X) \\ f &\longmapsto f^{-1} \end{aligned}$$

when restricted to Sob, is a FULL EMBEDDING

(In particular: all locale maps come from actual continuous maps).

<u>Idea</u>: · Sheaves can be defined on Locales
· Want to define them on other categories...

Let's see some properties...

(i) Notice that, given $f : X \to Y$, f^{-1} has a right adjoint. Namely: $f^{-1} : \mathrm{TOP}(Y) \to \mathrm{T}$ $\dashv$

$$\begin{aligned} f_* : \mathrm{TOP}(X) &\longrightarrow \mathrm{TOP}(Y) \\ U &\longmapsto \bigvee \{ V_\alpha \mid f^{-1}(V) \subseteq U \} \end{aligned}$$

~~(…)~~ (union of all open sets whose preimage lands in U

<u>check rt. adjoint</u>: Amounts to

$$\mathrm{Hom}(f^{-1}(U), V) = \mathrm{Hom}(U, f_* V)$$

nontrivial when $f^{-1}(U) \subset V \iff U \subset f_*(V)$

PROPERTY TO REMEMBER:

"good maps" $\Longleftrightarrow$ "adjoint pair of functors"

$[f]$ $\qquad\qquad$ $[f^{-1} \dashv f_*]$

(ii) <u>POINTS</u>

A point of a space $X \in$ Top "is" a map $p : 1 \to X$ ($1 = \{*\}$)

<u>Defⁿ</u>: A point of a locale L is a locale morphism $p : \mathbf{1} \to L$

(where $\mathbf{1}$ is the lattice $\left(\begin{matrix} 1 \\ \uparrow \\ 0 \end{matrix}\right) \cong \left(\begin{matrix} \{*\} \\ \uparrow \\ \{\} \end{matrix}\right)$)

i.e. lattice map $L \to 2$

<u>Defⁿ</u>: Let $Pt(L)$ be the topological space with

<u>points</u>: $\{p \mid p : L \to 2 \text{ in Frm}\}$

<u>open sets</u>: $\{\Sigma_l \mid l \in L\}$, where $\Sigma_l = \{p : L \to 2 \text{ s.t. } p(l) = 1\}$

ex: (check!)

<u>Thm</u>: Then $Pt : Loc \to Sob \subset Top$ is a functor

(Also, it's rt. adjoint to TOP(-) i.e. hom-spaces agree!)

<u>SHEAVES ON LOCALES</u>

Locales have enough information to define a sheaf.

<u>Motivation</u>: Often sheaves describe "functions on a space / sections of a bundle" - but SHEAF means thing of a sort which can be recognized locally.

eg// - Continuous functions can be recognized from behaviour on (<u>ENOUGH</u>) arbitrarily small open sets

- Bounded functions can't.

(32) **Def^n**: A sheaf on a locale $\mathcal{L}$ is

- a presheaf $\mathcal{F}$ in $Sets^{\mathcal{L}^{op}}$, such that
- for $U = \bigvee_{i\in I} U_i$, and $x_i \in \mathcal{F}(U_i)$ $\forall i \in I$

  $\exists!\, x \in \mathcal{F}(U)$ s.t. $x|_{U_i} = x_i$ $\forall i$

(The example of continuous functions illustrates the point)

Note: The definition above uses the lattice join operation $\bigvee$ to define a class of collections of subobjects $\{U_i\}$ s.t. $U_i < U$

Namely, $\{U_i\}_{i\in I}$ is a cover if $\bigvee_{i\in I} U_i = U$.

Goal: A "Grothendieck topology" will be the abstract structure which mimics this, for more complicated categories than locales
(e.g.) the category Top, in which ... representable presheaves ~ topological spaces

Recall: A SIEVE S on an object $x \in \mathcal{J}$ is a collection of morphisms into x closed under precomposition.

Recall: The Yoneda embedding

$$y : \mathcal{J} \longrightarrow Sets^{\mathcal{J}^{op}}$$
$$x \longmapsto Hom(-, x) \longleftarrow \text{ALL morphisms into } x$$

Any particular seive is a sub-functor of $y(x)$:

Prop: A seive on x is determined by a subobject $S \subseteq y(x)$ in $Sets^{\mathcal{J}^{op}}$.

(i.e. all $(y \xrightarrow{f} x)$ in S are given by $S(y)$

and given $y \xrightarrow{\phi} y'$, get $\phi^* : S(y') \longrightarrow S(y)$, $f \longmapsto f\circ\phi$)

Intuitions Seives: • are right ideals in a category
• are subobjects of presheaves
• pick collections of "subobjects" of X

Q: When are the subobjects in a sieve a "cover"?

Def^n: If $\mathcal{L}$ is a locale, [or any poset) a subset $S \subseteq \mathcal{L}$ is HEREDITARY when

$\forall u \in S, \forall v \in \mathcal{L}, \quad v \leq u \Rightarrow v \in S$

A HEREDITARY COVERING of $u \in \mathcal{L}$ is a hereditary subset S with $u = \bigvee S$

eg// the "down segment" $\downarrow u = \{v \in \mathcal{L} \mid v \leq u\}$ always...

Lemma: If $u \in \mathcal{L}$, the hereditary subsets of $\downarrow u$ are in bijection with

Subfunctors of the representable $\mathcal{L}(-,u)$

Pf: $\mathcal{L}(-,u)(v) = \begin{cases} \{T\} & \text{if } v \leq u \\ \{\} & \text{if } v \not\leq u \end{cases}$

A subfunctor $S \subseteq \mathcal{L}(-,u)$ is equivalent to specifying all $v \leq u$ with $S(v) = \{T\}$

Functoriality means if $S(v) = \{T\}$, $w \leq v \Rightarrow S(w) = \{T\}$
(since it determines a set map $S(v) \to S(w)$!)

$\boxtimes$

Lemma: If $\mathcal{L}$ a locale, $\mathcal{F}$ a presheaf in $\mathrm{Sets}^{\mathcal{L}^{op}}$ then $\mathcal{F}$ is a sheaf if:

· $\forall u \in \mathcal{L}$, $S \subseteq \mathcal{L}(-,u)$ (hereditary covering)
every natural transformation $S \Rightarrow \mathcal{F}$
extends uniquely to a natural trans.
$\mathcal{L}(-,u) \Rightarrow F$

This is a good form to generalize!

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SITES

Def^n: A GROTHENDIECK TOPOLOGY on a category $\mathcal{T}$ is a function J which assigns to $x \in \mathcal{T}$, a family $J(x)$ of seives on x s.t.

1) the maximal seive $t_x = \{$all morphisms into $x\}$ is in $J(x)$
2) (STABILITY) If $S \in J(x)$, then $\forall h: y \to x$, $h^*(S) \in J(y)$
3) (TRANSITIVITY) If $S \in J(x)$, and R is ANY seive on x s.t. $h^*(R) \in J(y)$ $\forall h: y \to x$ then $R \in J(x)$ also.

A GROTHENDIECK SITE is a pair $(\mathcal{T}, J)$

eg// • Top w/ "open covers"
(i.e. $J(x) = \{$all collections $\{U_i \to x\}$ which form an open cover, together with all composites$\}$)

• (Manifolds, Schemes, etc. with appropriate maps which are topological open covers)

• Measure spaces w/ "almost everywhere covers"

Note: All of these really start with generators of the sieves in $J(x)$

Def^n: A BASIS for a Grothendieck topology on $\mathcal{T}$ (if $\mathcal{T}$ has pullbacks!) is ~~[illegible]~~

$K: x$ objects of $\mathcal{T} \to$ collections of morphisms to x

1) containing all isos
2) if $\{f_i : x_i \to x\}_I \in K(x)$, $g: y \to x$
$\Rightarrow \{\pi_i : x_i \times_x y \to y\}_I \in K(y)$