(42) Overview: The + construction which worked to sheafify presheaves on spaces may not be enough for all sites...
But $a = ++$ (i.e. + applied twice) is!

(example: The presheaf B of BOUNDED FUNCTIONS (on Top) becomes $B^+ = C$, the sheaf of continuous f^s.)

Def^n: The + operation $+ : PSh(\mathbb{C}, J) \to PSh(\mathbb{C}, J)$ acts on a presheaf by

$$P \overset{+}{\longmapsto} P^+ := \varinjlim_{R \in J(C)} \mathrm{Match}(R, P) \quad (P^+(C))$$

where Match(R,P) is the set of matching families for the cover R of C.

The (co)limit $\varinjlim_{R \in J(C)}$ (i.e. $\underset{R \in J(C)}{\mathrm{colim}}$) is taken over the partial order of covers in $J(C)$

i.e.: $P^+(C)$ is an EQUIVALENCE CLASS of matching families (eq x_f's)

$$\overline{X} = \{ x_f \in P(D) \mid (D \xrightarrow{f} C) \in R \} \text{ s.t. } x_f \cdot k = x_{fk} \quad \forall\, E \xrightarrow{k} D \}$$

for $R \in J(C)$

where two such families are equivalent if they have a common refinement $T \subseteq R \cap S$, $T \in J(C)$

with $x_f = y_f \quad \forall f \in T$

- This is a presheaf - the "restriction maps" are well-defined on equivalence classes.
i.e. $\phi^+ : P^+ \to Q^+$ from $\phi : P \to Q$

(Intuition: Since P^+ is separated, it does not contain "too many" new elements to amalgamate (M&M Lemma III.5.5)

Note: In general, P^+ ~~may~~ need only have amalgamations for matching families from P, so it might not be a sheaf. Sometimes it is:

eg// $R^+ = C$

- i.e. a cts f's in $C(\mathcal{U})^{(m)}$ arises as a colimit of bounded functions on a cover

  such as: $f(x) = x$ on $\mathcal{U} = \mathbb{R}$

  from $R = \{(-n,n) \hookrightarrow \mathbb{R}\}$

  a matching family on R: $\{f(x) = x \text{ on } (-n,n)\}$

- But for matching families of cts f's we don't need to add any new functions as C is already a sheaf.

- In general, P^+ might not be a sheaf if matching families from P^+ have no amalgamation. But if they do, it is unique:

Defⁿ: A presheaf P is SEPARATED if any matching family has at most one amalgamation.

Prop: If $P \in PSh(\mathcal{C}, J)$, P^+ is separated.

Proof Idea: If $\bar{x}, \bar{y} \in P^+(\mathcal{U})$ and their restrictions to some cover match, we want $\bar{x} = \bar{y}$.

But if the restrictions agree as $P^+(U_i)$ elements there is some refinement where they agree exactly as $P(V_i)$-elements. So $\bar{x}, \bar{y}$ must be the same P^+-object ⊠

Claim: If P is separated, P^+ is a sheaf

(Cor: If $P \in PSh(\mathcal{C}, J)$, $(P^+)^+ \in Sh(\mathcal{C}, J)$)

(44) **Note:** The associated sheaf functor makes the inclusion $Sh(\mathbb{T},J) \overset{i}{\rightarrowtail} PSh(\mathbb{T},J)$ a REFLECTIVE one, i.e. has left adjoint:

$$\begin{tikzcd}
 & Sh(\mathbb{T},J) \arrow[d, shift left, "i"] \\
SepPSh \arrow[ur, "+"] & PSh(\mathbb{T},J) \arrow[l, "+"] \arrow[u, shift left, "a"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

[Think of it like an injection & projection for a sub-vector-space]

$$V_1 \underset{\pi}{\overset{i}{\rightleftarrows}} V$$

Note that many presheaves may have the same associated sheaf

eg// Let $B_k(X)$ be the (pre-)sheaf on $TOP(X)_{smooth}$ (X a mfld) with $B_k(U) = \{C^\infty$ f's $U \xrightarrow{f} \mathbb{R}$, with $\partial_k f$ bounded$\}$

→ All of these have $C^\infty(U)$ as the associated sheaf...

Theorem Any Grothendieck Topos (= category $Sh(\mathbb{T},J)$ for some site) is an elementary topos (a.k.a. Joyal-Tierney topos)

Proof Sketch:

1) Limits & colimits exist in $PSh(\mathbb{T})$ since presheaf categories are toposes.
   - Just need to ~~check~~ make them satisfy sheaf conditions

i.e.: If $\{F_i\}$ are in $Sh(\mathcal{J},J)$

then define a sheaf

$$\varinjlim F_i := a\left(\varinjlim i(F_i)\right)$$

(limit in $Sh(\mathcal{J},J)$) (limit in $PSh(\mathcal{J})$)

Since limits/colimits are found "pointwise"... in $PSh(\mathcal{J})$, $(\varinjlim i(F_i))(\mathcal{U}) := \left(\varinjlim (i(F_i)(\mathcal{U}))\right)$

-- this makes sense. Apply $+^2 = a$ to get associated sheaf.

2) Cartesian Closed

- colimit argument is same as limit argument above.
- Exponentials: Want that $i(G^F) \cong i(G)^{i(F)}$

(i.e. sheafification preserves mapping objects) M&M p135 (more details)

(Follows from Yoneda Lemma (that presheaves are determined by

$$P(\mathcal{U}) = \mathrm{Hom}(y(\mathcal{U}), P)$$

action on representables)

and showing P mapped ($P = y(\mathcal{U})$ e.g.)

$$P \longrightarrow i(G^F)$$

$$\hookrightarrow P \longrightarrow i(G)^{i(F)}$$

(more details) M&M p136

(46) which comes from the defining feature of the exponential object:

$\forall\ f: C \times B \longrightarrow A$, $\exists!\ \hat{f}: C \times B \longrightarrow A^B \times B$

with

$$\begin{tikzcd} A^B \times B \arrow[r, "ev"] & A \\ C \times B \arrow[u, "\exists!\hat{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

(can be defined "pointwise") (for sheaves).

3) Subobject Classifier

Lemma: The presheaf (lattice of subobjects)

$\Omega(\mathcal{U}) = \{\text{closed sieves on } \mathcal{U}\}$

is a sheaf.

"Closed" sieve: A sieve M is closed for J if: $\forall\ f: V \to \mathcal{U}$

M covers f (i.e. $f^* M \in J(V)$)

$\Longrightarrow f \in M$

(So: Ω is not "all sieves" as in $PSh(\mathcal{J})$ but only those CLOSED FOR J)

- Initial object is $\mathcal{U} \to t_{\mathcal{U}}$ = maximal sieve.

Prop: The sheaf Ω, together with the obvious injection true: $\mathbb{1} \to \Omega$ is a subobject classifier for $Sh(\hat{\mathcal{J}}, J)$

Geometric Morphisms

Idea: We can associate $Sh(X) = Sh(TOP(X), \text{covers})$ to a topological space X, via the associated locale, $TOP(X)$.

Recall: Locales come from top. spaces. From continuous functions, we get adjoint pairs of locale maps:

$$\left(X \xrightarrow{f} Y\right) \longmapsto \left(\begin{tikzcd} TOP(X) \arrow[r, shift left, "f_*"] \arrow[r, phantom, "\perp"] & TOP(Y) \arrow[l, shift left, "f^{-1}"] \end{tikzcd}\right)$$

where $f^{-1}: \underset{TOP(Y)}{\underset{\Upsilon}{V}} \longmapsto \underset{TOP(X)}{\underset{\Upsilon}{f^{-1}(u)}}$

and $f_*: \underset{TOP(X)}{\underset{\Upsilon}{\mathcal{U}}} \longmapsto \bigvee_{\alpha} \{V_\alpha \mid f^{-1}(V) \subseteq \mathcal{U}\}$

(union of open sets with preimage landing in $\mathcal{U}$)

Note: Any geom. morphism of G. toposes comes from such a morphism of sites

Similarly: A GEOMETRIC MORPHISM of toposes is an adjoint pair

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathcal{F} \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

of functors, such as arise for sheaves:

$$\begin{tikzcd} Sh(X) \arrow[r, shift right, "f_*"'] & Sh(Y) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

(48) Conceptually, this is because locales are "contained" into sheaf category:

$$\widetilde{\mathcal{J}} \xrightarrow{\;y\;} \widehat{\widetilde{\mathcal{J}}} = PSh(\widetilde{\mathcal{J}}) \underset{i}{\overset{a}{\rightleftarrows}} Sh(\widetilde{\mathcal{J}}, J)$$

in particular, the equivalent of "cts $f: X \xrightarrow{f} Y$" is "adjoint pair" $\mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \widetilde{\mathcal{F}}$.

$f^* \dashv f_*$
L-adj R-adj

For sheaf categories, these are

f_* DIRECT IMAGE

f^* INVERSE IMAGE

$$Sh(X) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} Sh(Y)$$

$$(f_*(F))(U) = F(f^{-1}(U))$$

eg// $\{pt\} \xrightarrow{y} Y$, $F \in Sh(\{pt\})$

$$(y_* F)(V) = \begin{cases} F(\{pt\}) & y \in V \\ F(\phi) & y \notin V \end{cases}$$

"Skyscraper sheaf"

$f^{-1}G = a(f^{+}G)$ where

$$(f^{+}G)(U) \simeq \varinjlim_{U \to f(V)} G(V) \quad \text{for } G \in PSh(Y)$$

these are adjoint since f^{-1} and f_* inverse as locale maps

(Note there can also be the other limit:

$\varprojlim_{U \to f(V)} G(V)$ → right adjoint.).

Containment of the "base" category of spaces (locale or Grothendieck site) into sheaves motivates: sheaves as generalized spaces...

(Next: Diffeological Spaces as e.g.)