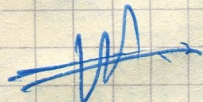


(54)

Theorem: The category  $\text{ConcSh}(\mathcal{C}, \mathcal{J})$  is a quasitopos. (See Baez & Hoffnung)



Moral: The base site  $\text{Diffco}$  (or, generally, any base site) provides "model spaces". The category of concrete sheaves (or generally of any sheaves, but less intuitively) is a "category of generalized spaces" whose structure is defined by maps into them from model spaces



### Internal Constructions in Toposes

Idea: We can define "internal" structures of various kinds in any topos.

- eg. Group object, ring object, etc...
- simplicial objects (eg. "simplicial sets") etc.

Roughly: The CATEGORY OF INTERNAL \_\_\_\_\_ in  $\text{Sh}(\mathcal{C})$  is the same as the CATEGORY OF \_\_\_\_\_-valued SHEAVES

### First Example: Group object

In a category  $\mathcal{C}$  with finite products (eg.  $\text{Sh}(\mathcal{C})$ ) a GROUP OBJECT in  $\mathcal{C}$  is an object  $G \in \mathcal{C}$  equipped with:

$$\begin{array}{c} m: G \times G \rightarrow G \\ \downarrow \quad \downarrow \\ e: 1 \rightarrow G \\ \downarrow \\ i: G \rightarrow G \end{array}$$

which satisfy:  $G^3 \xrightarrow{id \circ m} G^2$  commutes  
 (associativity)

1) 
$$\begin{array}{ccc} G^3 & \xrightarrow{id \circ m} & G^2 \\ m \circ id \downarrow & & \downarrow m \\ G^2 & \xrightarrow{m} & G \end{array}$$

2) 
$$\begin{array}{ccc} G & \xrightarrow{e \circ id} & G \times G & \text{commutes} \\ id \circ e \downarrow & \searrow id & \downarrow & \text{(unit law)} \\ G \times G & \xrightarrow{m} & G \end{array}$$

3) 
$$\begin{array}{ccc} G & \xrightarrow{(id, i) \circ \Delta} & G^2 & \text{commutes} \\ (i, id) \circ \Delta \downarrow & \searrow i & \downarrow m & \text{(inverse property)} \\ G^2 & \xrightarrow{m} & G \end{array}$$

(Case:  $C = Sh(\mathcal{T}, \mathcal{J})$  (or  $PSh(\mathcal{J})$ ))

Then  $G: \mathcal{T}^{op} \rightarrow \text{Sets}$  and  $m, e, i$  are natural trans.

At each  $U \in \mathcal{T}$ , we get  $G(U)$ ,  $m_U, e_U, i_U$  all satisfy the axioms for ~~groups~~ groups.

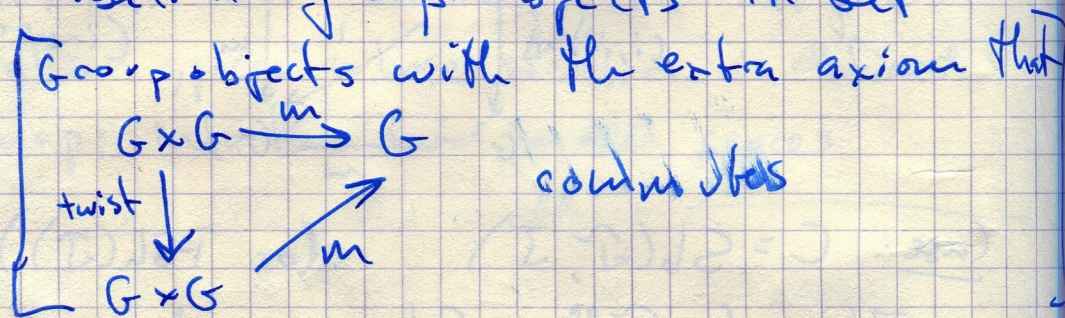
~~Prop. The category of  $\mathcal{A}$ -group is  $\mathcal{C}(\mathcal{J}, \text{Sets})$  (i.e.  $\text{Hom}(\mathcal{T}^{op}, \text{Sets})$ ) where~~

So: A GROUP OBJECT in  $Sh(\mathcal{T}, \mathcal{J})$  is the same as an OBJECT of  $Sh(\mathcal{T}, \mathcal{J}, [Grp])$  the category of GROUP-VALUED PRESHEAVES in  $\text{Hom}(\mathcal{T}^{op}, Grp)$  satisfying the sheaf condition.

56 Abelian Categories

Def<sup>2</sup>: A category  $\mathcal{C}$  is PRE-ADDITIVE if  
 $\forall x, y \in \mathcal{C}$ ,  $\text{Hom}_{\mathcal{C}}(x, y)$  is an abelian group & composition  $\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$  is bilinear.

eg 1 [Ab], the category of abelian groups  
 i.e. abelian group objects in Set



Claim: Abelian group objects in  $\text{Sh}(\mathcal{S}, \mathcal{J})$  form an ~~abelian~~ additive category

Proof:  $\text{Hom}_{\mathcal{C}}(x, y)$  inherits the abelian group structure from  $y$ .

$\rightarrow$  i.e. preadditive category

Def<sup>3</sup>: An ~~abelian~~ additive category  $\mathcal{C}$  is one in which (also)

- i) There is a zero object 0
- ii)  $\forall x_1, x_2 \in \mathcal{C}$ , there is a product  $x_1 \times x_2$  and a coproduct  $x_1 \sqcup x_2$
- iii) The morphism  $r: x_1 \sqcup x_2 \rightarrow x_1 \times x_2$  given by
 
$$x_k \rightarrow x_1 \sqcup x_2 \xrightarrow{r} x_1 \times x_2 \rightarrow x_j = \begin{cases} \text{id}_{x_k} & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$$
 is an isomorphism

(i.e. the product & coproduct are isomorphic, with compatible injections & projections so that both are just  $X_1 \oplus X_2$ , a "biproduct")

(57)

Note: In an additive category, any object  $X$  is an abelian group object with multiplication

$$X \times X \xrightarrow{\pi} X \sqcup X \xrightarrow{\sigma} X$$

where  $\sigma$  is the "codiagonal map"

which comes from the universal property of the coproduct:

$$\begin{array}{ccc} X & \xrightarrow{i} & X \sqcup X & \xleftarrow{j} & X \\ & \searrow \text{id} & \downarrow \text{id} & \swarrow \text{id} & \\ & & X & & \end{array}$$

Exercise: Check this can be made into an abelian group object (and find the inverse map!)

In particular:  $\text{Sh}(\mathcal{T}, \mathcal{J}, \text{Ab}) \simeq [\text{Abelian Group objects in } \text{Sh}(\mathcal{T}, \mathcal{J})]$   
 is an abelian category equivalence.

## Simplicial Objects & Complexes

Idea: The above is a special case of an internal construction in a topos (of sheaves). There are many others with the same flavour...

Def<sup>n</sup>: The THEORY OF ABELIAN GROUPS,  $\text{Th}(\text{AbGrp})$  is a category with all products, generated by an abelian group object.

(i.e. it has all objects & morphisms implied by the diagrams above & the property of having products, and no others).

58) Then:

Prop: An abelian group object in  $\mathcal{C}$  is equivalent to a functor  $A: \text{Th}(\text{AbGrp}) \rightarrow \mathcal{C}$  and a morphism of abelian group objects is a natural transformation  $f: A \Rightarrow A'$ .

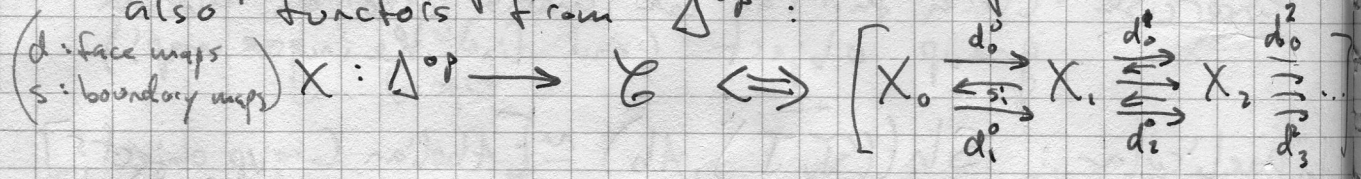
In particular:  $[\text{AbGrp}] \cong \text{PSh}(\text{Th}(\text{AbGrp})^{\text{op}})$

There are other similar cases:

eg/ 1) Simplicial objects:

These are presheaves on  $\Delta$ .  $\begin{cases} \text{obj} = \mathbb{Z}^+(\in \mathbb{N}) \\ \text{Mor} = \text{order-preserving maps } n \rightarrow m \end{cases}$   
 (so  $\text{SSet} = \text{PSh}(\Delta^{\text{op}}, \text{Set})$ ).

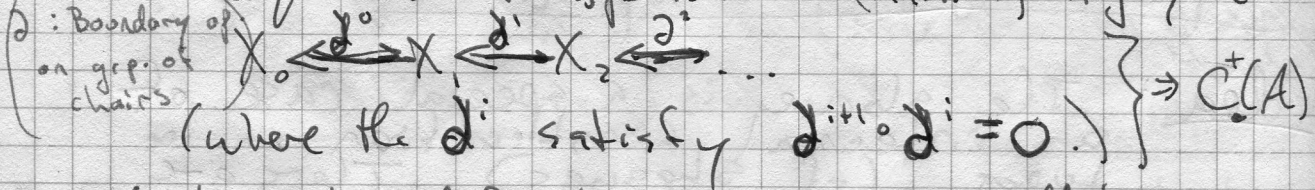
Simplicial objects in other categories are also functors from  $\Delta^{\text{op}}$ :



2) Complexes:

These are presheaves on the ~~category~~ <sup>additive category</sup>  $(\mathbb{Z}^+, d)$

i.e. sequences of objects in an (additive) category  $\mathcal{A}$ :



(Note: This definition requires an additive category so that a 0-morphism exists, namely  $X \xrightarrow{0} Y = X \xrightarrow{!} 0 \xrightarrow{!} Y$ )

(So only ADDITIVE FUNCTORS, which preserve the relation, should be considered: complexes are inherently an additive-category concept.)

Motivation: Simplicial objects are good for capturing HOMOTOPICAL structures, since any simplicial set/object has a corresponding picture as a "space": (59)

$$S\text{Set} \xrightarrow{1:1} \text{Top}$$

(Takes a collection of simplexes,  $X_n = \{n\text{-simplexes}\}$  and glues them to give a space).

Simplicial SHEAVES:

$$\text{Sh}(\mathbb{J}, J), S\text{Set}$$

should be seen as giving a "space of maps" into a generalized space, just as sheaves give a "set of maps" into it.

But as with abelian groups:

$$\text{Sh}(\mathbb{J}, J), S\text{Set} \cong \text{PSh}(\Delta, \text{Sh}(\mathbb{J}, J))$$

i.e. both are functor categories

$$\text{Hom}_{\text{cat}}(\mathbb{J} \times \Delta^{\text{op}}, \text{Set})$$

satisfying the sheaf condition for  $J$ .

So: - Simplicial Sheaves describe "generalized spaces over  $\mathbb{J}$ " with homotopy structure on the SPACE of maps

- Sheaves of Complexes of Abelian Groups: "generalized spaces over  $\mathbb{J}$ " with abelian group structure & differential structure ( $d$ )

Claim (Dold-Kan Correspondence)

There is an equivalence of categories

$$\text{PSh}(\Delta, \text{AbGrp}) \cong C(\text{AbGrp})$$

Cor. (Dold-Puppe): For an ABELIAN category  $A$ ,

$$\text{PSh}(\Delta, A) \cong C^*(A)$$

(In particular, this is true for  $A = \text{Sh}(\mathbb{J}, J), \text{AbGrp}$ )

⑥ The Dold-Puppe theorem follows from Dold-Kan by applying the equivalence at each  $U \in \mathcal{J}$

Recall def: An ABELIAN category is an additive category in which:

1) Every morphism has kernels & cokernels:

$$\ker(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \operatorname{coker}(f)$$

2) Every morphism is strict ( $\operatorname{coim}(f) \cong \operatorname{im}(f)$ ):

$$\ker(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \operatorname{coker}(f)$$

$$\begin{array}{ccc} \downarrow & & \uparrow \\ \operatorname{coim}(f) & \xrightarrow{\cong} & \operatorname{im}(f) \end{array}$$

Proof (sketch) The equivalence is given by the following:

i)  $N: \operatorname{Psh}(\Delta, \operatorname{Ab}) \rightarrow C^+(\operatorname{Ab})$  acts on simp. abelian group by

$$A_n \longmapsto \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n$$

( $\{N_n\}, \partial$ )  
NORMALIZED  
CHAIN COMPLEX

which become a complex with  ~~$A_n \xrightarrow{\partial_n} A_{n-1}$~~   
where  $\partial^n = (-1)^n d_n: A_n \rightarrow A_{n-1}$  (restricted to  $N_n$ )

(This satisfies  $\partial^2 = 0$  by the simplicial identities for  $d_i$ )

ii)  $P: C^+(\operatorname{Ab}) \rightarrow \operatorname{Psh}(\Delta, \operatorname{Ab})$  acts on complex  $V_\bullet$  by

$$V_n \longmapsto \bigoplus_{[n] \rightarrow [k]} V_k \quad \leftarrow \text{(sum over surjections in } \Delta \text{ - i.e. ordered)}$$

For any given order-preserving  $\theta: [m] \rightarrow [n]$ , the corresponding map of simplices  $P(V)_n \rightarrow P(V)_m$  is

$$\theta^*: \bigoplus_{[n] \rightarrow [k]} V_k \longrightarrow \bigoplus_{[m] \rightarrow [k]} V_k$$

given by, (on the summand for a given  $\sigma: [n] \rightarrow [k]$ ) (61)

$$V_k \xrightarrow{d^*} V_s \hookrightarrow \bigoplus_{[m] \rightarrow [r]} V_r$$

where  $[m] \xrightarrow{t} [s] \xrightarrow{d} [k]$  ( $s = \text{Im}(\sigma \circ \theta)$ )

surjection inclusion

factorizes  $\sigma \circ \theta: [m] \rightarrow [k]$ .

(For  $A = \text{Sh}(\mathcal{F}, \mathcal{I})$ ,  $A_b$ ), this same construction works "pointwise" at each  $\mathcal{U} \in \mathcal{I}$ .

The fact that this is an equivalence of  $\text{Psh}(A, A_b)$  with  $C_*(A_b)$  means  $\Gamma \circ N \cong \text{Id}$  and  $N \circ \Gamma \cong \text{Id}$  by natural isomorphisms.

eg: For instance,  $\Gamma \circ N \cong \text{Id}$  is given by

$$\bigoplus_{[n] \rightarrow [k]} (NA)_{[k]} \rightarrow A_n, \text{ defined on the } \sigma: [n] \rightarrow [k] \text{ part:}$$

$$\begin{array}{ccc} NA_{[k]} & \xrightarrow{\quad} & A_n \\ & \searrow & \nearrow \sigma^* \\ & A_k & \end{array}$$

(ex: check this is a natural isomorphism.)

(Note: more details found in § III of Goerss & Jardine's "Simplicial Homotopy Theory")