(54)

Theorem: The category $\mathrm{ConcSh}(\widetilde{\Gamma}, J)$ is a quasitopos.

(See Baez & Hoffnung)

Moral: The base site $\underline{\mathrm{Diffeo}}$ (or, generally, any base site) provides "model spaces". The category of concrete sheaves (or generally of any sheaves, but less intuitively) is a "category of generalized spaces" whose structure is defined by maps into them from model spaces

---

Internal Constructions in Toposes

Idea: We can define "internal" structures of various kinds in any topos.

eg// · Group object, ring object etc...
· simplicial objects (eg. "simplicial sets") etc.

Roughly: The CATEGORY OF INTERNAL ___ in $\mathrm{Sh}(\mathcal{C})$ is the same as the CATEGORY OF ___-valued SHEAVES

First Example: Group object

In a category C with finite products (eg. $\mathrm{Sh}(\widetilde{\Gamma})$) a GROUP OBJECT in C is an object $G \in C$ equipped with:

$$m : G \times G \to G$$
$$e : 1 \to G$$
$$i : G \to G$$

which satisfy:

1)
$$\begin{tikzcd} G^3 \arrow[r, "id\times m"] \arrow[d, "m\times id"'] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"'] & G \end{tikzcd}$$
commutes (associativity)

2)
$$\begin{tikzcd} G \arrow[r, "e\times id"] \arrow[d, "id\times e"'] \arrow[dr, "id"] & G\times G \arrow[d] \\ G\times G \arrow[r] & G \end{tikzcd}$$
commutes (unit law)

3)
$$\begin{tikzcd} G \arrow[r, "(id,i)\circ\Delta"] \arrow[d, "(i,id)\circ\Delta"'] \arrow[dr, "1"] & G^2 \arrow[d, "m"] \\ G^2 \arrow[r, "m"'] & G \end{tikzcd}$$
commutes (inverse property)

Case: $C = Sh(\mathcal{T}, J)$ (or $PSh(\mathcal{T})$)

Then $G : \mathcal{T}^{op} \to Sets$ and m, e, i are natural trans.

At each $U \in \mathcal{T}$, we get $G(U)$, m_U, e_U, i_U all satisfy the axioms for groups.

~~Prop: Gp[C], the category of groups in C is [Gp][Sets] (i.e. Hom(Th(Gp), Sets)) where~~

So: A GROUP OBJECT in $Sh(\mathcal{T}, J)$ is the same as an OBJECT of $Sh((\mathcal{T}, J), [Grp])$ the category of GROUP-VALUED PRESHEAVES in $Hom(\mathcal{T}^{op}, Grp)$ satisfying the sheaf condition.

Abelian Categories

Def^{n}: A category $\mathcal{C}$ is PRE-ADDITIVE if $\forall x, y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is an abelian group & composition $\mathrm{Hom}_{\mathcal{C}}(y,z) \times \mathrm{Hom}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x,z)$ is bilinear.

eg// [Ab], the category of abelian groups i.e. abelian group objects in Set

[Group objects with the extra axiom that

$$\begin{tikzcd} G\times G \arrow[r, "m"] \arrow[d, "\text{twist}"'] & G \\ G\times G \arrow[ur, "m"'] & \end{tikzcd}$$

commutes]

Claim: Abelian group objects in $Sh(\mathcal{T}, J)$ form an additive category

Proof: $\mathrm{Hom}_{\mathcal{C}}(x,y)$ inherits the abelian group structure from y.

Def^{n}: An additive category $\mathcal{C}$ is one in which (also) → i.e. preadditive category

i) There is a zero object 0

ii) $\forall x_1, x_2 \in \mathcal{C}$, there is a product $x_1 \times x_2$ and a coproduct $x_1 \sqcup x_2$

iii) The morphism $r : x_1 \sqcup x_2 \to x_1 \times x_2$ given by

$$x_k \to x_1 \sqcup x_2 \xrightarrow{r} x_1 \times x_2 \to x_j = \begin{cases} \mathrm{id}_{x_k} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

is an isomorphism

(i.e. the product & coproduct are isomorphic, with compatible injections & projections so that both are just $X_1 \oplus X_2$, a "biproduct")

Note: In an additive category, any object X is an abelian group object with multiplication

$$X \times X \xrightarrow{r^{-1}} X \sqcup X \xrightarrow{\sigma} X$$

where σ is the "codiagonal map" [which comes from the universal property of the coproduct:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "id"'] & X \sqcup X \arrow[d, dashed, "\exists!"] & X \arrow[l, "i"'] \arrow[dl, "id"] \\ & X & \end{tikzcd}$$

]

exercise: Check this can be made into an abelian group object (and find the inverse map!)

In particular: $Sh((\mathcal{T}, J), Ab) \simeq \left[\begin{array}{c}\text{Abelian Group objects} \\ \text{in } Sh(\mathcal{T}, J)\end{array}\right]$

~~M~~ is an abelian category equivalence.

Simplicial Objects & Complexes

Idea: The above is a special case of an internal construction in a topos (of sheaves). There are many others with the same flavour...

Def^{n}: The THEORY OF ABELIAN GROUPS, $Th(AbGP)$ is a category with all products, generated by an abelian group object.

(i.e. it has all objects & morphisms implied by the diagrams above & the property of having products, and no OTHERS).

Then:

Prop: An abelian group object in $\mathcal{C}$ is equivalent to a functor
$$A : Th(AbGrp) \longrightarrow \mathcal{C}$$
and a morphism of abelian group objects is a natural transformation
$$f : A \Rightarrow A'$$

In particular: $[AbGrp] \cong PSh(Th(AbGrp)^{op})$

There are other similar cases:

eg// 1) Simplicial objects:

These are presheaves on Δ: $\begin{cases} \text{obj}: \mathbb{Z}^+ (= \mathbb{N}) \\ \text{Mor}: \text{order-preserving maps } n \to m \end{cases}$

(so $SSet = PSh(\Delta^{op}, set)$).

Simplicial objects in other categories are also functors from Δ^{op}:

(d: face maps, s: boundary maps) $X : \Delta^{op} \longrightarrow \mathcal{C} \iff \left[X_0 \underset{d_1^0}{\overset{d_0^0}{\underset{s_i}{\rightleftarrows}}} X_1 \underset{d_2^1}{\overset{d_0^1}{\rightrightarrows}} X_2 \underset{d_3^2}{\overset{d_0^2}{\rightrightarrows}} \cdots \right]$

2) Complexes:

These are presheaves on the additive category ~~category~~ $(\mathbb{Z}^+, d.)$

i.e. sequences of objects in an (additive) category $\mathcal{A}$:

(∂: Boundary of on grp. of chains) $X_0 \overset{\partial^0}{\longleftarrow} X_1 \overset{\partial^1}{\longleftarrow} X_2 \overset{\partial^2}{\longleftarrow} \cdots$

(where the ∂^i satisfy $\partial^{i+1} \circ \partial^i = 0$.) $\Rightarrow C_\bullet^+(\mathcal{A})$

(Note: This definition requires an additive category so that a 0-morphism exists, namely $X \xrightarrow{0} Y = X \xrightarrow[!]{} 0 \xrightarrow[!]{} Y$)

(So only ADDITIVE FUNCTORS, which preserve the relation, should be considered: complexes are inherently an additive-category concept.)

<u>Motivation</u>: Simplicial objects are good for capturing HOMOTOPICAL structures, since any simplicial set/object has a corresponding picture as a "space":

$$\mathrm{SSet} \xrightarrow[|\cdot|]{} \mathrm{Top}$$

(Takes a collection of simplexes, $X_n = \{n\text{-simplexes}\}$ and glues them to give a space).

Simplicial SHEAVES:

$$\mathrm{Sh}((\bar{\mathcal{J}}, J), \mathrm{SSet})$$

should be seen as giving a "space of maps" into a generalized space, just as sheaves give a "set of maps" into it.

But as with abelian groups:

$$\mathrm{Sh}((\bar{\mathcal{J}}, J), \mathrm{SSet}) \cong \mathrm{PSh}(\Delta, \mathrm{Sh}(\bar{\mathcal{J}}, J))$$

i.e. both are functorcategories

$$\mathrm{Hom}_{cat}((\bar{\mathcal{J}} \times \Delta)^{op}, \mathrm{Set})$$

satisfying the sheaf condition for J.

So: - Simplicial Sheaves describe "generalized spaces over $\bar{\mathcal{J}}$" with HOMOTOPY structure on the SPACE of maps

- Sheaves of Complexes of Abelian Groups: "generalized spaces over $\bar{\mathcal{J}}$" with abelian group structure & differential structure (d)

<u>Claim</u> (Dold-Kan Correspondence)

There is an equivalence of categories

$$\mathrm{PSh}(\Delta, \mathrm{AbGrp}) \cong C(\mathrm{AbGrp})$$

<u>Cor.</u> (Dold-Puppe): For an ABELIAN category A,

$$\mathrm{PSh}(\Delta, A) \cong C^{+}_{\cdot}(A)$$

(In particular, this is true for $A = \mathrm{Sh}((\bar{\mathcal{J}}, J), \mathrm{AbGrp})$)

(60.) (The Dold-Puppe theorem follows from Dold-Kan
by applying the equivalence at each $\mathcal{U} \in \widetilde{J}$)

Recall def^n: An ABELIAN category is an additive category in which:

1) Every morphism has kernels & cokernels:

$$\mathrm{Ker}(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}(f)$$

2) Every morphism is strict ($\mathrm{Coim}(f) \cong \mathrm{Im}(f)$):

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r] & X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[r] & \mathrm{Coker}(f) \\
 & \mathrm{Coim}(f) \arrow[r, dashed, "\sim"] & \mathrm{Im}(f) \arrow[u, hook] &
\end{tikzcd}$$

Proof (idea) The equivalence is given by the following:

i) $N : PSh(\Delta, Ab) \longrightarrow C^+_\bullet(Ab)$ acts on simp. abelian group by:

$(\{NA_n\}, \partial)$ NORMALIZED CHAIN COMPLEX

$$A_n \longmapsto \bigcap_{i=0}^{n-1} \mathrm{Ker}(d_i) \subset A_n$$

which become a complex with
where

$$\partial^n = (-1)^n d_n : A_n \longrightarrow A_{n-1} \quad \text{(restricted to } NA_n)$$

(This satisfies $\partial^2 = 0$ by the simplicial identities for d_i)

ii) $\Gamma : C^+_\bullet(Ab) \longrightarrow PSh(\Delta, Ab)$ acts on complex $V_\bullet$ by:

$$V_n \longmapsto \bigoplus_{[n] \twoheadrightarrow [k]} V_k$$

← (sum over surjections in Δ - i.e. ordered)

For any given order-preserving $\theta : [m] \to [n]$, the corresponding map of simplices $\Gamma(V)_n \to \Gamma(V)_m$ is:

$$\theta^* : \bigoplus_{[n] \to [k]} V_k \longrightarrow \bigoplus_{[m] \to [k]} V_r$$

given by, (on the summand for a given $\theta:[n]\to[k]$

$$V_k \xrightarrow{d^*} V_s \hookrightarrow \bigoplus_{[m]\twoheadrightarrow[r]} V_r$$

where $[m] \xrightarrow{t} [s] \xrightarrow{d} [k]$ $\quad (s = \mathrm{Im}(\sigma\circ\theta))$

surjection, inclusion

factorizes $\sigma\circ\theta : [m] \longrightarrow [k]$.

(For $A = Sh((\mathcal{T}, J), Ab)$, this same construction works "pointwise" at each $U \in \mathcal{T}$.)

• The fact that this is an equivalence of $Pshv(\Delta, Ab)$ with $C_\bullet^+(Ab)$ means $\Gamma\circ N \cong \mathrm{Id}$ and $N\circ\Gamma \cong \mathrm{Id}$ by natural isomorphisms.

eg: For instance, $\Gamma\circ N \xrightarrow{\simeq} \mathrm{Id}$ is given by

$$\bigoplus_{[n]\twoheadrightarrow[k]} (NA)_k \longrightarrow A_n$$, defined on the $\sigma:[n]\to[k]$ part:

$$\begin{tikzcd} NA_k \arrow[rr, Rightarrow] \arrow[dr, hook] & & A_n \\ & A_k \arrow[ur, "\sigma^*"'] & \end{tikzcd}$$

(ex: Check this is a natural isomorphism.) ∎

(Note: more details found in §III of Goerss & Jardine's "Simplicial Homotopy Theory")