Cauchy problems for Lorentzian manifolds with special holonomy

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Outline

Project with H. Baum & A. Lischewski, Humboldt-Universität Berlin:

- Baum, Lischewski & L., Differential Geom. Appl. 45 (2016), arXiv:1411.3059
- Lischewski, arXiv:1503.04946
- Lischewski & L., in progress

1. Cauchy problems for Lorentzian manifolds and special holonomy
2. Evolution equations in the analytic setting
   - Einstein equations
   - Parallel null vector and spinor
3. The smooth case and quasilinear symmetric hyperbolic systems
   - Einstein equations
   - Parallel null vector and spinor
4. Riemannian manifolds satisfying the constraints

The Cauchy problem for a parallel null vector field

Given a Riemannian manifold $(\mathcal{M}, \mathrm{g})$, can we embed $(\mathcal{M}, \mathrm{g})$ as a Cauchy hypersurface into a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ of the form $\overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$ and

$$\boxed{\overline{\mathrm{g}} = -\lambda^2 \, dt^2 + \mathrm{g}_t,} \qquad (*)$$

such that $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ admits a parallel null vector field V, i.e., with

$$\overline{\nabla} V = 0 \quad \text{and} \quad \overline{\mathrm{g}}(V, V) = 0 \qquad ?$$

Here g_t = family of Riemannian metrics on $\mathcal{M}$ with $\mathrm{g}_0 = \mathrm{g}$, $\lambda = \lambda(t, x)$ *"lapse fct"*.

- Without requiring V being null finding $\overline{\mathrm{g}}$ with V parallel is trivial: The metric $\overline{\mathrm{g}} = -dt^2 + \mathrm{g}$ on $\mathbb{R} \times \mathcal{M}$ has a parallel time-like vector field ∂_t.
- If V is null and parallel, g has to satisfy the constraint equation

  $$\nabla U = u \mathrm{W},$$

  with $u = \mathrm{g}(U, U)$ and W the Weingarten operator [Helga's talk].
- For a Riemannian manifold $(\mathcal{M}, \mathrm{g})$ satisfying the constraints, find a (globally hyperbolic) Lorentzian manifold of the form $(*)$ with parallel null vector field.

Motivation 1: Special Lorentzian holonomy (more details in Helga's talk)

Let $(\mathcal{M}, \mathrm{g})$ be a semi-Riemannian manifold, and

$$\mathrm{Hol}(\mathcal{M}, \mathrm{g}) = \left\{ P_\gamma^\nabla \in \mathbf{O}(T_p\mathcal{M}, \mathrm{g}_p) \mid \gamma(0) = \gamma(1) = p \right\}$$

its holonomy group with Lie algebra $\mathfrak{hol}(\mathcal{M}, \mathrm{g})$.

- $(\mathcal{M}, \mathrm{g})$ has *"special holonomy"* $\iff$ $\mathfrak{hol} \subsetneq \mathfrak{so}(p, q)$ but the manifold is *indecomposable*, i.e., does *not* (locally) decompose as a product.
- Riemannian special holonomy: $\mathbf{U}(p)$ $\mathbf{SU}(p)$, $\mathbf{Sp}(q)$, $\mathbf{Sp}(q)\cdot\mathbf{Sp}(1)$, $\mathbf{G}_2$, $\mathbf{Spin}(7)$ [Berger, Bryant, ...] + isotropy groups of symmetric spaces.
- Lorentzian special holonomy: $\nexists$ irreducible subalgebras of $\mathfrak{so}(1, n+1)$! $\Longrightarrow$

$$\mathfrak{hol} \subset \mathfrak{stab}(\textit{null line}) = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n = \left\{ \left(\begin{array}{ccc} a & v^\top & 0 \\ 0 & A & -v \\ 0 & 0^\top & -a \end{array} \right) \middle| \begin{array}{l} a \in \mathbb{R} \\ v \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\}.$$

There is a classification of Lorentzian special holonomy algebras:

- indecomposable subalgebras of $\mathfrak{so}(1, n+1)$ [Berard-Bergery & Ikemakhen '93]
- $pr_{\mathfrak{so}(n)}(\mathfrak{hol})$ is a Riemannian holonomy algebra [L '03] $\rightsquigarrow$ Berger's list
- Construction of local metrics for all possible holonomy algebras [... Galaev '05]

Construction of Lorentzian manifolds with special holonomy

- Let $(\mathcal{M}, \mathrm{g}, \mu)$ be a Riemannian manifold with closed 1-form μ. Then

  $$\overline{\mathrm{g}} = \mu\, dv + \mathrm{g},$$

  is a Lorentzian metric on $\mathcal{M} \times \mathbb{R}$ with parallel null vector field ∂_v.
- Most constructions for prescribed holonomy are based on the local form of a Lorentzian manifold with parallel null vector field

  $$\overline{\mathrm{g}} = 2du(dv + f\, du + f_i\, dx^i) + h_{ij}\, dx^i dx^j,$$

  with f, f^i and h_{ij} functions of $x^1, \ldots x^{n-2}, u$. Then ∂_v is null and parallel.
- Need 'global' constructions for globally hyperbolic manifolds with complete Cauchy hypersurfaces and with special holonomy [Baum-Müller '08]
- A Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ is globally hyperbolic if it admits a Cauchy hypersurface $\mathcal{M}$, i.e., a spacelike hypersurface that is met by every maximal timelike curve exactly once. They are of the form $\overline{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$ with

  $$\overline{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}_t.$$

  [Geroch '70, ..., Bernal-Sánchez '03]

Motivation 2: Parallel spinors on Lorentzian manifolds

Let $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ be a Lorentzian spin manifold with spinor bundle $\overline{\mathbb{S}} \to \overline{\mathcal{M}}$. and $\psi \in \Gamma(\overline{\mathbb{S}})$ a parallel spinor, with induced causal and parallel Dirac current V_ψ.

- $\overline{\mathrm{g}}(V_\psi, V_\psi) = -1$: $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ locally is a product $-dt^2 + \mathrm{h}$ with h Riemannian, $\mathrm{Ric}^{\mathrm{h}} = 0$ and with a parallel spinor
  $\rightsquigarrow$ special holonomy Riemannian manifolds: $\mathbf{SU}(p)$, $\mathbf{Sp}(q)$, $\mathbf{G}_2$, $\mathbf{Spin}(7)$.
- $\overline{\mathrm{g}}(V_\psi, V_\psi) = 0$: $T\overline{\mathcal{M}}$ is filtered $\mathbb{R}V_\psi \subset V_\psi^\perp \subset T\overline{\mathcal{M}}$
  No induced product structure and *not Ricci-flat* but $\mathrm{Ric}^{\overline{\mathrm{g}}} = f\,(V_\psi^\flat)^2$.
- Constraints: Each spacelike hypersurface $(\mathcal{M}, \mathrm{g})$ admits a spinor field φ with

$$\nabla_X \varphi = \frac{i}{2}\mathrm{W}(X)\cdot\varphi, \quad \forall X \in T\mathcal{M}, \qquad U_\varphi \cdot \varphi = i\, u_\varphi\, \varphi, \tag{1}$$

  in which U_φ is defined by $\mathrm{g}(U_\varphi, X) = -i(X\cdot\varphi, \varphi)$, $u_\varphi = \sqrt{\mathrm{g}(U_\varphi, U_\varphi)} = \|\varphi\|^2$.
- A spinor with (1) is called *generalised imaginary Killing spinor (GIKS)*.
- $U_\varphi = pr_{T\mathcal{M}} V_\psi$ satisfies the constraint $\nabla_X U_\varphi = u_\varphi \mathrm{W}(X)$.

Solve the Cauchy problem for Lorentzian manifolds with parallel null vector field V and extend GIKS on $(\mathcal{M}, \mathrm{g})$ to parallel spinor on $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ by parallel transport.

Motivation 3: (Generalised) Killing spinors on Riemannian manifolds

$(\mathcal{M}, g)$ Riemannian mfd., φ a Killing spinor with Killing number λ, i.e.

$$\nabla_X \varphi = \lambda X \cdot \varphi, \qquad \lambda \in \mathbb{R} \cup i\,\mathbb{R}.$$

- Killing spinor $\Rightarrow (\mathcal{M}, g)$ Einstein with $\mathrm{scal} = 4n(n-1)\lambda^2$.
- $\{\text{Killing spinors}\} \simeq \{\text{parallel spinors on the cone } (\mathbb{R}^+ \times \mathcal{M}, \hat{g} = 2\lambda^2 dr^2 + r^2 g)\}$
- Parallel spinors are fixed under spin rep of $\mathrm{Hol} \rightsquigarrow$ use holonomy classification in order to classify mfd's with Killing spinors.
- $\lambda \in \mathbb{R}$ [Bär '93] : Riemannian cones are flat or irreducible [Gallot '79], Berger's list $\Rightarrow (\mathcal{M}, g) = S^n$, (3-)Sasaski, 6-dim nearly Kähler, nearly parallel $\mathbf{G}_2$
- $\lambda \in i\mathbb{R}$: $(\mathcal{M}, g) = H^n$ or $(\mathcal{M} = \mathrm{R} \times \mathcal{F}, g = ds^2 + e^{4i\lambda s} h)$ and $(\mathcal{F}, h)$ admits a parallel spinor [Baum '89]. This can be obtained using the time-like cone and a generalisation of Gallot's result [Alekseevski, Cortés, Galaev, L '08].

Use same approach — with the cone replaced by the solution to a more general Cauchy problem — and the classification of Lorentzian holonomy, to locally classify Riemannian manifolds with generalised imaginary Killing spinor.

Example: Cauchy problem for $\mathrm{Ric}(\overline{\mathrm{g}}) = 0$

Let $\overline{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}_t$ on $\mathcal{I} \times \mathcal{M}$ and $T = \frac{1}{\lambda}\partial_t$ be the timelike unit normal.

- $\mathrm{W} := -\overline{\nabla} T|_{T\mathcal{M}}$ the Weingarten operator, $\mathrm{W} = -\frac{1}{2\lambda}\dot{\mathrm{g}}$, where *dot* $= \partial_t$.
- Fundamental curvature equations, $\overline{\mathrm{R}} =$ curvature tensor of $\overline{\mathrm{g}}$:

$$
\begin{array}{rcll}
\overline{\mathrm{R}}|_{T\mathcal{M}} & = & \mathrm{R} + \mathrm{W} \wedge \mathrm{W} & \text{Gauß} \\
\overline{\mathrm{R}}(\cdot,\cdot,\cdot,T)|_{T\mathcal{M}} & = & d^{\nabla}\mathrm{W} & \text{Codazzi} \\
\overline{\mathrm{R}}(T,\cdot,\cdot,T)|_{T\mathcal{M}} & = & \frac{1}{\lambda}\left(\dot{\mathrm{W}} + \nabla^2(\lambda)\right) + \mathrm{W}^2 & \text{Mainardi}
\end{array}
$$

- Ricci-tensor of $\overline{\mathrm{g}}$, $\overline{\mathrm{Ric}} = \mathrm{Ric}(\overline{\mathrm{g}})$:

$$
\begin{array}{rcl}
\overline{\mathrm{Ric}}(T,T) & = & \frac{1}{\lambda}\left(\mathrm{tr}(\dot{\mathrm{W}}) + \Delta(\lambda)\right) + \mathrm{tr}(\mathrm{W}^2) \\
\overline{\mathrm{Ric}}(T,.)|_{T\mathcal{M}} & = & d(\mathrm{tr}\mathrm{W}) + \mathrm{div}\mathrm{W} \\
\overline{\mathrm{Ric}}|_{T\mathcal{M}\times T\mathcal{M}} & = & -\frac{1}{\lambda}\left(\dot{\mathrm{W}} + \nabla^2(\lambda)\right) + \mathrm{Ric} + \mathrm{tr}(\mathrm{W})\mathrm{W} - 2\mathrm{W}^2
\end{array}
\tag{2}
$$

- Scalar curvature:

$$
\overline{\mathrm{scal}} = \mathrm{scal} + (\mathrm{tr}(\mathrm{W}_t))^2 - 3\|\mathrm{W}\|^2 - \frac{2}{\lambda}\left(\mathrm{tr}(\dot{\mathrm{W}}) + \Delta(\lambda)\right) \tag{3}
$$

- Set (2) and (3) to zero and replace $\mathrm{tr}(\dot{\mathrm{W}})$ in (2) by (3).

Constraint and evolution equations for $\mathrm{Ric}(\overline{g}) = 0$

- $\overline{\mathrm{Ric}} = 0 \iff \left\{ \begin{array}{rcl} \mathrm{scal} & = & \mathrm{tr}(W^2) - \mathrm{tr}(W)^2 \\ d\,\mathrm{tr}(W) & = & -\mathrm{div}(W) \end{array} \right\}$ (constraints) and

$$\dot{W} = \lambda\Big(\mathrm{Ric} + \mathrm{tr}(W)W - 2W^2\Big) - \nabla^2(\lambda) \quad \text{(evolution equations)}$$

- Constraints are preserved under evolution equations.
- The evolution equations are of the form:

$$\ddot{g} = F(g, \dot{g}, \partial_i g, \partial_i \dot{g}, \partial_i \partial_j g),$$

  with initial data $g|_{t=0} = g$, $\dot{g}|_{t=0} = -2\lambda W$.
- If λ and initial data are real analytic: apply Cauchy-Kowalevski to get unique solution:
  - for Lorentzian metrics [Darmois '27, Lichnerowicz '39], this can be generalised to the smooth setting: Choquet-Bruhat (50's, second part of the talk).
  - Riemannian: solution for analytic data [Koiso '81], but in general no solution for smooth, non-analytic [counterexamples by Bryant '10, also Ammann, Moroianu & Moroianu '13] .

Constraint and evolution equations for a parallel vector field

On $\overline{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$ we can write a vector field $V \in \Gamma(T\overline{\mathcal{M}})$ as

$$V = uT + U \qquad \text{with } U \in \Gamma(T^\perp) \text{ and } u \in C^\infty(\mathbb{R} \times \mathcal{M}).$$

V is null for $\overline{\mathrm{g}} = -\lambda^2\, dt^2 + \mathrm{g}_t \implies u = -\mathrm{g}(T, V) = \sqrt{\mathrm{g}(U, U)}$.

$$\overline{\nabla}_X V = du(X)T - u\mathrm{W}(X) + \underbrace{\overline{\nabla}_X U}_{= \nabla_X U - \mathrm{W}(X,U)T} \qquad \text{for } X \in T\mathcal{M}$$

$$\overline{\nabla}_X V = 0 \iff \boxed{\nabla U - u\mathrm{W} = 0 \qquad \& \qquad du - \mathrm{W} = 0}$$

constraint equations $(t = 0)$

$$\overline{\nabla}_{\partial_t} V = \dot{u}T + u \underbrace{\overline{\nabla}_{\partial_t} T}_{= \nabla\lambda} + \underbrace{\overline{\nabla}_{\partial_t} U}_{= [\partial_t, U] - \lambda\mathrm{W}(U) + d\lambda(U)T}$$

$$\overline{\nabla}_{\partial_t} V = 0 \iff \boxed{\dot{U} + u\nabla\lambda + \tfrac{1}{2}\dot{\mathrm{g}}_t(U) = 0 \qquad \& \qquad \dot{u} + d\lambda(U) = 0}$$

evolution equations

Second order evolution equations for a parallel null vector field

- If $\overline{\nabla} V = 0$, then for $X, Y \in T\mathcal{M}$ it is

$$\begin{aligned} \overline{\mathrm{R}}(\partial_t, X, Y, V) = \overline{\mathrm{R}}(\partial_t, X, V, \partial_t) &= 0 && \rightsquigarrow \text{evolution for } \mathrm{g} \text{ via Codazzi-Mainardi} \\ \overline{\nabla}_{\partial_t}\overline{\nabla}_{\partial_t} V &= 0 && \rightsquigarrow \text{evolution for for } U = pr_{T\mathcal{M}}(V) \\ \overline{\nabla}_X V|_{\{0\}\times\mathcal{M}} &= 0 && \rightsquigarrow \text{constraint for } U \text{ (initial condition for } \dot{\mathrm{g}}) \\ \overline{\nabla}_{\partial_t} V|_{\{0\}\times\mathcal{M}} &= 0 && \rightsquigarrow \text{initial condition for } \dot{U} \end{aligned}$$

- We can show that this is in fact equivalent to $\overline{\nabla} V = 0$ and leads to equivalent 2nd order evolution equations in Cauchy-Kowalevski form:

$$\left(\ddot{\mathrm{g}}, \ddot{U}, \ddot{u}\right) = \mathcal{F}(\mathrm{g}, \dot{\mathrm{g}}, \partial_i \mathrm{g}, \partial_i \dot{\mathrm{g}}, \partial_i\partial_j \mathrm{g}, U, \dot{U}, \dots, \partial_i\partial_j u),$$

however the first component of $\mathcal{F}$ is not necessarily symmetric!

- Observation:
  In the analytic case, $(*)$ can be replaced by $\overline{\mathrm{R}}(X, V, V, Y) = 0$ for all $X, Y \in T\mathcal{M}$.

Let $(\mathcal{M}, \mathrm{g})$ be a Riemannian mfd, W a symmetric endomorphisms field, U a vector field, u function on $\mathcal{M}$, all real analytic, with constraints

$$\nabla_i U^j = -u \mathrm{W}_i{}^j, \;\; \mathrm{g}(U, U) = u^2 > 0.$$

Then for any analytic fct. $\lambda = \lambda(t, x)$ the Lorentzian metric $\overline{g} = -\lambda^2 dt^2 + \mathrm{g}_t$ has parallel null vector field $V = \frac{u_t}{\lambda}\partial_t - U_t \Longleftrightarrow$

$$\begin{aligned}
\ddot{\mathrm{g}}_{ij} &= \tfrac{1}{u} U^k \left(\lambda \nabla_{[k} \dot{\mathrm{g}}_{(j]i)} - \dot{\mathrm{g}}_{(i[j)} \nabla_{k]} \lambda\right) + \tfrac{1}{2} \dot{\mathrm{g}}_{ik} \dot{\mathrm{g}}^k{}_j + \tfrac{\dot{\lambda}}{\lambda} \dot{\mathrm{g}}_{ij} + 2\lambda \nabla_i \nabla_j \lambda + \tfrac{2\lambda^2}{u^2} U^k U^\ell R_{ik\ell j} \\
\ddot{U}_i &= \tfrac{1}{2u} U^k U^l \left(\dot{\mathrm{g}}_{l[k} \nabla_{i]} \lambda - \lambda \nabla_{[i} \dot{\mathrm{g}}_{k]l}\right) - U^k \left(\dot{\mathrm{g}}_{ki} - \tfrac{\dot{\lambda}}{\lambda 2} \dot{\mathrm{g}}_{ki} - \lambda \nabla_k \nabla_i \lambda - \nabla_k \lambda \nabla_i \lambda\right) \\
&\quad + u \mathrm{g}_{ki} \nabla^k \dot{\lambda} + \tfrac{u}{2} \dot{\mathrm{g}}_{ki} \nabla^k \lambda + 2\dot{u} \nabla_i \lambda \\
\ddot{u} &= U^k \left(\mathrm{g}_{kl} \nabla^l \dot{\lambda} + \tfrac{3}{2} \dot{\mathrm{g}}_{kl} \nabla^l \lambda\right) + 2\dot{U}^k \nabla_k \lambda - u \nabla_k \lambda \nabla^k \lambda
\end{aligned}$$

with initial conditions
$$\left\{ \begin{array}{rclcrcl}
\mathrm{g}_{ij}(0) &=& \mathrm{g}_{ij}, & & \dot{\mathrm{g}}_{ij}(0) &=& -2\lambda \mathrm{W}_{ij}, \\
U(0) &=& U, & & \dot{U}_i(0) &=& u \nabla_i \lambda + \lambda U^k \mathrm{W}_{ki} \\
u(0) &=& u & & \dot{u}(0) &=& U^k \nabla_k \lambda
\end{array} \right. .$$

Theorem 1 (Baum, Lischewski, L '14)

If $(\mathcal{M}, \mathrm{g}, U)$ are real analytic satisfying the constraint equations, λ real analytic, then $(\mathcal{M}, \mathrm{g})$ can be embedded as Cauchy hypersurface into a Lorentzian manifold $(\overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}, \overline{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}_t)$ with parallel null vector field. For given initial conditions as above, $\overline{\mathrm{g}}$ is unique and $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ is globally hyperbolic.

- Apply Cauchy-Kowalevski to the evolution equations in coordinate neighbourhoods in $\mathcal{M}$. By uniqueness these patch together to a unique global solution on $\mathcal{M}$ defining the Lorentzian metric $\overline{\mathrm{g}}$ on $\widetilde{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$.
- Every $p \in \mathcal{M}$ then admits a neighbourhood $\mathcal{U}_p$ in $\widetilde{\mathcal{M}}$ such that $\mathcal{M} \cap \mathcal{U}$ is a Cauchy hypersurface in $\mathcal{U}$
- Then $\overline{\mathcal{M}} = \cup_{p \in \mathcal{M}} \mathcal{U}_p$ contains M as a Cauchy hypersurface.

Example [Baum & Müller '08]

$\lambda \equiv 1$, W Codazzi tensor, i.e. $\nabla_{[i} \mathrm{W}_{j]}{}^k = 0$. Solution to the above system:

$$
\begin{aligned}
\mathrm{g}_{ij}(t) &= \mathrm{g}_{ij} - 2t\mathrm{W}_{ij} + t^2 \mathrm{W}_{ik}\mathrm{W}^k{}_j \\
U^i(t) &= \mathrm{A}^i{}_k(t) U^k, \text{ mit } \mathrm{A}^i{}_k \text{ inverse of } (\delta_i{}^j - t\mathrm{W}_i{}^j) \\
u(t) &= u
\end{aligned}
$$

Application to parallel null spinors

Corollary 1

Let $(\mathcal{M}, \mathrm{g}, \varphi)$ be an analytic Riemannian spin manifold with an analytic GIKS φ.
$\Longrightarrow$ *On the Lorentzian manifold in Theorem 1 there exists a parallel null spinor.*

Proof: Take the Lorentzian manifold obtained as solution with parallel null vector field. Translate the initial GIKS φ parallel along t-lines $\rightsquigarrow$ spinor ψ with $\overline{\nabla}_{\partial_t}\psi = 0$.

- $\mathcal{E} := \left(T^*\mathcal{M} \otimes \overline{\mathbb{S}}\right) \oplus \left(\Lambda^2 T^*\mathcal{M} \otimes \overline{\mathbb{S}}\right) \longrightarrow \overline{\mathcal{M}}$,
- $\begin{pmatrix} A := \overline{\nabla}\psi \\ B := \overline{\mathrm{R}}(\cdot,\cdot)\psi \end{pmatrix} \in \Gamma(\mathcal{E})$
- Check that $\begin{pmatrix} A \\ B \end{pmatrix}$ satisfies a PDE $\overline{\nabla}_{\partial_t}\begin{pmatrix} A \\ B \end{pmatrix} = Q\begin{pmatrix} A \\ B \end{pmatrix}$, with Q linear on $\mathcal{E}$.
- $A = B = 0$ along initial hypersurface and hence for all t.

Smooth case: the Cauchy problem for $\mathrm{Ric}(\overline{\mathrm{g}}) = 0$

- In the smooth setting we relax the condition on the explicit form of $\overline{\mathrm{g}}$: Given a Riemannian manifold $(\mathcal{M}, \mathrm{g})$ and symmetric endomorphism W satisfying the constraints

$$\mathrm{R} = \mathrm{tr}(\mathrm{W}^2) - \mathrm{tr}(\mathrm{W})^2, \qquad d\,\mathrm{tr}(\mathrm{W}) = -\mathrm{div}(\mathrm{W}),$$

  find a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ with $\mathrm{Ric}(\overline{\mathrm{g}}) = 0$ and such that $(\mathcal{M}, \mathrm{g})$ injects into $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ with Weingarten operator W.

- A system of PDEs for vector valued functions $u : \mathbb{R}^{n+1} \to \mathbb{R}^N$ of $x^0 = t$ and $x = (x^i)_{i=1,\dots,n}$

$$A^0(x^\mu, u)\,\partial_0 u = \sum_{i=1}^{n} A^i(x^\mu, u)\,\partial_i u + B(x^\mu, u), \qquad (4)$$

  is a 1st order quasilinear symmetric hyperbolic system of PDEs if
  - the $A^\mu(x^\mu, u)$ are symmetric $N \times N$ matrices, and
  - $A^0(x^\mu, u)$ is *positive definite* with a uniform positive lower bound.

  Such systems have a unique smooth solution for given initial conditions.

Hyperbolic reduction [Friedrich-Rendall]:

- In coordinates the Ricci tensor looks like

$$\mathrm{Ric}(\overline{g})_{\mu\nu} = -\frac{1}{2}\overline{g}^{\alpha\beta}\partial_\alpha\partial_\beta\overline{g}_{\mu\nu} + \underbrace{\partial_{(\mu}\overline{\Gamma}_{\nu)}}_{= \overline{g}^{\alpha\beta}\partial_\mu\partial_\nu\overline{g}_{\alpha\beta} + \mathrm{LOTs}} + LOTs$$

  where $\Gamma_\mu = \Gamma^\alpha_{\mu\alpha}$.

- Not fixing time coordinate and the form of the metric $\rightsquigarrow$ full diffeomorphism invariance

$$\phi \in \mathrm{Diff}(\overline{\mathcal{M}}) \implies \mathrm{Ric}(\phi^*\overline{g}) = \phi^*\mathrm{Ric}(\overline{g}) = 0.$$

- Hyperbolic reduction to break the diffeomorphism invariance:
  Fix background metric

$$\hat{g} = -\lambda^2 dt^2 + g$$

  on $\mathbb{R} \times \mathcal{M}$ with $\lambda \in C^\infty(\mathbb{R} \times \overline{\mathcal{M}})$, define $C = \overline{\nabla} - \hat{\nabla}$ and

$$E_\mu = \overline{g}_{\mu\nu}\overline{g}^{\alpha\beta}C^\nu{}_{\alpha\beta},$$

  and replace $\mathrm{Ric}(\overline{g}) = 0$ by the equation

$$\widehat{\mathrm{Ric}}(\overline{g})_{\mu\nu} := \mathrm{Ric}(\overline{g})_{\mu\nu} + \overline{\nabla}_{(\mu}E_{\nu)} = 0.$$

Einstein equation as quasilinear symmetric hyperbolic system

- Locally, $\widehat{\mathrm{Ric}}(\overline{\mathrm{g}})_{\mu\nu}$ is of the form

$$\widehat{\mathrm{Ric}}(\overline{\mathrm{g}})_{\mu\nu} = \mathrm{Ric}(\overline{\mathrm{g}})_{\mu\nu} + \overline{\nabla}_{(\mu} E_{\nu)} = -\tfrac{1}{2}\overline{\mathrm{g}}^{\alpha\beta}\partial_\alpha\partial_\beta\overline{\mathrm{g}}_{\mu\nu} + LOTs.$$

- $\widehat{\mathrm{Ric}}(\overline{\mathrm{g}}) = 0$ is a 1^{st} order quasilinear symmetric hyperbolic system of PDEs,

$$A^0(x^\mu, G)\partial_0 G = \sum_{i=1}^{n} A^i(x^\mu, G)\partial_i G + B(t, x, G), \qquad (5)$$

for functions $G = \begin{pmatrix} \overline{\mathrm{g}} \\ \partial_\mu \overline{\mathrm{g}} \end{pmatrix}$ of $x^0 = t$ and $x = (x^i)_{i=1,\dots,n}$, i.e., and hence has a unique smooth solution for given smooth initial conditions.

- Then $\widehat{\mathrm{Ric}}(\overline{\mathrm{g}}) = 0$ implies that E satisfies a wave equation

$$\overline{\Delta} E_\mu = \overline{\nabla}^\alpha \overline{\nabla}_\alpha E_\mu = -\underbrace{2\overline{\nabla}^\alpha \overline{\mathrm{R}}_{\alpha\mu}}_{= \overline{\nabla}_\mu \mathrm{scal} \; = \; -\overline{\nabla}_\mu \overline{\nabla}^\alpha E_\alpha} - \overline{\nabla}^\alpha \overline{\nabla}_\mu E_\alpha = \overline{\mathrm{R}}_{\mu\ \alpha}^{\ \alpha\ \beta} E_\beta = \overline{\mathrm{R}}_\mu^{\ \alpha} E_\alpha. \qquad (6)$$

Hence $E \equiv 0$ if $E|_{\mathcal{M}} = 0$ and $\overline{\nabla} E|_{\mathcal{M}} = 0$.

Initial conditions

- original initial data: $\overline{g}_{ij}|_{\mathcal{M}} = g_{ij}$, $\partial_0 \overline{g}_{ij}|_{\mathcal{M}} = -2\lambda W_{ij}$,
- choice: $\overline{g}_{00}|_{\mathcal{M}} = -\lambda^2|_{\mathcal{M}}$, $\overline{g}_{0i}|_{\mathcal{M}} = 0$, i.e., $\overline{g}_{\mu\nu}|_{\mathcal{M}} = \widehat{g}_{\mu\nu}$.
- determine initial data for $\partial_t \overline{g}_{00}$ and $\partial_t \overline{g}_{0i}$ such that $E|_{\mathcal{M}} = 0$:

$$\partial_t \overline{g}_{00}|_{\mathcal{M}} = -2F_0|_{\mathcal{M}} - \mathrm{tr}(W), \qquad \partial_t \overline{g}_{0i}|_{\mathcal{M}} = -F_i|_{\mathcal{M}} + \tfrac{1}{2} g^{kl}(2\partial_k g_{il} - \partial_i g_{kl}),$$

where $F_\mu = \overline{g}_{\mu\nu} \overline{g}^{\alpha\beta} \widehat{\Gamma}^{\nu}_{\alpha\beta}$.

- With $E|_{\mathcal{M}} = 0$ we have $\overline{\nabla}_i E_\mu|_{\mathcal{M}} = 0$. The constraints give

$$0 = \mathrm{Ric}(\overline{g})_{0i}|_{\mathcal{M}} = \tfrac{1}{2} \overline{\nabla}_0 E_i|_{\mathcal{M}}, \qquad 0 = \mathrm{Ric}(\overline{g})_{00} = \overline{\nabla}_0 E_0|_{\mathcal{M}},$$

and hence $\overline{\nabla} E|_{\mathcal{M}} = 0$.

- With $E|_{\mathcal{M}} = 0$, $\overline{\nabla} E|_{\mathcal{M}} = 0$ and wave equation (6), we get $E = 0$.

We obtain a local solutions to $\widehat{\mathrm{Ric}}(\overline{g}) = \mathrm{Ric}(\overline{g}) = 0$ that patch together to a globally hyperbolic solution.

The smooth case: parallel vector field [Lischewski & L, in prep.]

Given a smooth Riemannian manifold $(\mathcal{M}, \mathrm{g})$ with U and W satisfying the constraints

$$\nabla U = u\mathrm{W}, \qquad u = \sqrt{\mathrm{g}(U,U)},$$

can we find a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ with parallel, null vector field V such that $\mathrm{pr}_{T\mathcal{M}}(V) = U$ and W is the Weingarten operator of $\mathcal{M} \subset \overline{\mathcal{M}}$?

- Let V be a parallel vector field on $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ and $\mu = V^\flat = \overline{\mathrm{g}}(V, .) \in \Omega^1(\overline{\mathcal{M}})$. Then
  - $V \lrcorner (\overline{\nabla}^{(k)} \mathrm{Ric}(\overline{\mathrm{g}})) = 0$ for $k = 0, 1, \dots$ hence there is a bilinear form Q with

$$\mathrm{Ric}(\overline{\mathrm{g}}) = Q, \qquad V \lrcorner Q = 0, \qquad \overline{\nabla}_V Q = 0. \tag{7}$$

  - $\overline{\nabla}\mu = 0$ and in particular $(d + \delta^{\overline{\mathrm{g}}})\mu = 0$. Recall that $d + \delta^{\overline{\mathrm{g}}} = c \circ \overline{\nabla}$, where c is the Clifford multiplication on forms

$$c : T\overline{\mathcal{M}} \otimes \Omega^* \ni X \otimes \omega \mapsto X^\flat \wedge \omega - X \lrcorner \omega \in \Omega^*$$

- Given a initial manifold $(\mathcal{M}, \mathrm{g})$ and a function $\lambda \in C^\infty(\mathbb{R} \times \mathcal{M})$, fix a background metric $\hat{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}$ on $\mathbb{R} \times \mathcal{M}$ that defines $\widehat{\mathrm{Ric}}(\overline{\mathrm{g}})$ as for the Einstein equations.

- Consider the PDE system

$$\boxed{\begin{array}{rcl} \widehat{\mathrm{Ric}}(\overline{\mathrm{g}}) & = & Q \;:\; \mu^\sharp \lrcorner Q = 0 \\ \overline{\nabla}_{\mu^\sharp} Q & = & 0 \\ (d + \delta^{\overline{\mathrm{g}}})\mu & = & 0 \end{array}} \tag{8}$$

  for a metric $\overline{\mathrm{g}}$, a one-form μ, and a symmetric BLF Q. Note that
  - $\widehat{\mathrm{Ric}}(\overline{\mathrm{g}})$ does not contain derivatives of Q, i.e., 1st eq. in (8) is like Einstein equation with energy-momentum tensor Q.
  - $d + \delta = c \circ \overline{\nabla} : \Lambda^* \to \Lambda^*$ is of Dirac type.
- (8) reduces to a 1st order quasilinear symmetric hyperbolic system of the form

$$\begin{pmatrix} A_1^0 & 0 & 0 \\ 0 & A_2^0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \partial_0 G \\ \partial_0 \mu \\ \partial_0 Q \end{pmatrix} = \begin{pmatrix} A_1^i & 0 & 0 \\ 0 & A_2^i & 0 \\ 0 & 0 & a_3^i \mathbf{1} \end{pmatrix} \begin{pmatrix} \partial_i G \\ \partial_i \mu \\ \partial_i Q \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

  for $G = (\overline{\mathrm{g}}, \partial_t \overline{\mathrm{g}}, \partial_i \overline{\mathrm{g}})$, μ and Q, with $A^0_{1/2}$ symmetric positive definite, $A^i_{1/2}$ symmetric, that has a unique solution for given initial data along $\mathcal{M}$.
- $\Psi := \left(\overline{\nabla} V, E\right)$ is a solution to a wave eq. $\mathcal{P}\Psi = 0$, for normally hyperbolic $\mathcal{P}$.
- Again, the initial data are determined by

$$\overline{\mathrm{g}}|_{\mathcal{M}} = \widehat{\mathrm{g}}, \quad \mu|_{\mathcal{M}} = \widehat{\mathrm{g}}(\tfrac{u}{\lambda}\partial_t - U, .), \quad \partial_t \overline{\mathrm{g}}|_{T\mathcal{M}\times T\mathcal{M}} = -2\lambda \mathrm{W},$$

  and the requirement that $\overline{\nabla} V|_{\mathcal{M}} = 0$ and $E|_{\mathcal{M}} = 0$.

Cauchy problems for Lorentzian manifolds and special holonomy

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Part 2

Recall from part 1:

Given a Riemannian manifold $(\mathcal{M}, \mathrm{g})$, with a vector field U and a symmetric endomorphism field W satisfying the constraint equation

$$\nabla U = u\mathrm{W}, \qquad \text{with } u = \sqrt{\mathrm{g}(U, U)},$$

and a function $\lambda \in C^\infty(\mathbb{R} \times \mathcal{M})$, we wanted to construct a Lorentzian manifold $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ which

- contains $(\mathcal{M}, \mathrm{g} = \overline{\mathrm{g}}|_{\mathcal{M}})$ as Cauchy hypersurface, with Weingarten operator W,
- admits a parallel null vector field V such that $\mathrm{pr}_{T\mathcal{M}}(V) = U$,
- possibly of the form

$$\overline{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}_t,$$

for a family of Riemannian metrics.

Theorem 2 (Lischewski-L, in progr.)

Let $(\mathcal{M}, \mathrm{g})$ *be a smooth Riemannian manifold with a vector field* U *and a symmetric endomorphism field* W *satisfying the constraint equation* $\nabla U = u\mathrm{W}$ *and a function* $\lambda \in C^\infty(\mathbb{R} \times \mathcal{M})$.
Then there is $\overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$ *with a Lorentzian metric*

$$\overline{\mathrm{g}} = -\tilde{\lambda}^2 dt^2 + \mathrm{g}_t \qquad \text{with } \tilde{\lambda}|_{\mathcal{M}} = \lambda,\ \mathrm{g}_0 = \mathrm{g},$$

such that $(\mathcal{M}, \overline{\mathrm{g}})$ *admits a parallel null vector field* V *with* $\mathrm{pr}_{T\mathcal{M}}(V) = U$*, and such that* $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ *is globally hyperbolic with* $\mathcal{M}$ *as Cauchy hypersurface with Weingarten operator* W.
Moreover, let $\hat{\mathrm{g}} = -\lambda^2 dt^2 + \mathrm{g}$ *be the background metric on* $\mathbb{R} \times \mathcal{M}$ *defined by the initial data* λ *and* g*. Then* $\overline{\mathrm{g}}$ *is the unique metric satisfying the additional conditions*

- $\overline{\mathrm{g}}|_{\mathcal{M}} = -\hat{\mathrm{g}}|_{\mathcal{M}}$,
- $\mathrm{tr}_{\overline{\mathrm{g}}}(C) = 0$*, where* $C(X, Y) = \overline{\nabla}_X Y - \hat{\nabla}_X Y$ *is the difference tensor between the Levi-Civita connections of* $\overline{\mathrm{g}}$ *and* $\hat{\mathrm{g}}$.

Proof, step 1: the quasilinear symmetric hyperbolic system

Fix the background metric $\hat{g} = \lambda^2 dt^2 + g$ on $\mathbb{R} \times \mathcal{M}$. For a Lorentzian metric $\overline{g}$ define the difference difference tensor C between the Levi-Civita connections of $\overline{g}$ and $\hat{g}$, the 1-form E and the modified Ricci tensor by

$$E = \overline{g}(\mathrm{tr}_{\overline{g}}(C), .), \qquad \widehat{\mathrm{Ric}}(\overline{g})_{\alpha\beta} \ = \ \mathrm{Ric}(\overline{g})_{\alpha\beta} + \overline{\nabla}_{(\alpha} E_{\beta)}.$$

Consider the PDE system

$$\boxed{\widehat{\mathrm{Ric}}(\overline{g}) = Q \circ \mathrm{pr}_{T\mathcal{M}}^{\mu^\sharp} \qquad \overline{\nabla}_{\mu^\sharp} Q = 0, \qquad (d + \delta^{\overline{g}})\mu = 0} \tag{9}$$

for a metric $\overline{g}$, a one-form μ, and a symmetric BLF Q.

In local coordinates $x^0 = t$, $x = (x^i)_{i=1,\dots,n}$, this is equivalent to a 1^{st} order quasilinear symmetric hyperbolic system of the form

$$\begin{pmatrix} A_1^0 & 0 & 0 \\ 0 & A_2^0 & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \partial_0 G \\ \partial_0 \mu \\ \partial_0 Q \end{pmatrix} = \begin{pmatrix} A_1^i & 0 & 0 \\ 0 & A_2^i & 0 \\ 0 & 0 & a_3^i \, \mathbf{1} \end{pmatrix} \begin{pmatrix} \partial_i G \\ \partial_i \mu \\ \partial_i Q \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

for $G = (\overline{g}, \partial_\alpha \overline{g})$, μ and Q, with $A^\alpha_{1/2} = A^\alpha_{1/2}(x^\beta, G, \mu, Q)$ and $A^0_{1/2}$ symmetric positive definite, $A^i_{1/2}$ symmetric.

Proof, step 2: the wave equation

For $\overline{g}$, Q and $\mu = V^\flat$ be a solution of the system (9) we define the quantities

$$\begin{aligned} \Phi &= \left(\overline{\nabla} V, E, \overline{\nabla}_V E, \overline{\nabla} E(V)\right). \\ \Psi &= \mathrm{div}^{\overline{g}}\left(Q - \tfrac{1}{2}\mathrm{tr}_{\overline{g}}(Q)\overline{g}\right). \end{aligned}$$

Then show that Φ and Ψ satisfy the following PDEs

$$\Delta\Phi = L_1(\Phi, \overline{\nabla}\Phi, \Psi), \qquad \overline{\nabla}_V \Psi = L_2(\Phi, \overline{\nabla}\Phi), \tag{10}$$

where $\Delta = \overline{\nabla}^2$ is defined by the Bochner Laplacian on the appropriate bundle and L_1 and L_2 are linear.

Again, in local coordinates the system (10) is a equivalent to a 1^{st} order linear symmetric hyperbolic system for Φ, $\partial\Phi$ and Ψ, and hence has a unique solution for given initial values.

If we can show that Φ, $\overline{\nabla}\Phi$ and Ψ vanish along $\mathcal{M}$, then they vanish for all t.

Proof, step 3: Initial conditions

(i) the original initial conditions:

$$\overline{\mathrm{g}}|_{\mathcal{M}} = \hat{\mathrm{g}}|_{\mathcal{M}}, \qquad \partial_t \overline{\mathrm{g}}|_{T\mathcal{M}\times T\mathcal{M}} = 2\lambda \mathrm{W}, \qquad \mu|_{\mathcal{M}} = u\hat{\mathrm{g}}(\tfrac{u}{\lambda}\partial_t - U, .)$$

(ii) initial conditions for Q:

$$\begin{aligned} U \lrcorner\, Q|_{\mathcal{M}} &= d\,\mathrm{tr}(\mathrm{W}) = -\mathrm{div}(\mathrm{W}) \\ Q &= \mathrm{Ric} - \mathrm{W}^2 + \mathrm{tr}(\mathrm{W})\mathrm{W} - \mathrm{R}(N, ., .N) + \mathrm{W}(., N)\mathrm{W}(., N) - \mathrm{W}(N, N)\mathrm{W} \end{aligned}$$

where $N = \frac{1}{u}U$ and the second equation holds for $U^\perp \times U^\perp$ along $\mathcal{M}$.

(iii) initial data for $\partial_t \overline{\mathrm{g}}_{00}$ and $\partial_t \overline{\mathrm{g}}_{0i}$:

$$\begin{aligned} \partial_t \overline{\mathrm{g}}_{00}|_{\mathcal{M}} &= -2\lambda|^2_{\mathcal{M}} \left(F_0|_{\mathcal{M}} - \lambda|_{\mathcal{M}} \mathrm{tr}(\mathrm{W})\right), \\ \partial_t \overline{\mathrm{g}}_{0i}|_{\mathcal{M}} &= \lambda|^2_{\mathcal{M}} \Big(-F_i|_{\mathcal{M}} + \tfrac{1}{2}\mathrm{g}^{kl}(2\partial_k \mathrm{g}_{il} - \partial_i \mathrm{g}_{kl}) + \partial_i(\log \lambda|_{\mathcal{M}})\Big) \end{aligned}$$

where $F_\mu = \overline{\mathrm{g}}_{\mu\nu}\overline{\mathrm{g}}^{\alpha\beta}\widehat{\Gamma}^\nu_{\alpha\beta}$.

The initial conditions (ii) and (iii) imply that Φ, $\overline{\nabla}\Phi$ and Ψ from the previous slide vanish along $\mathcal{M}$.

Proof, step 4: the global metric and its form

From local to global:

- For each $p \in \mathcal{M}$ there is a globally hyperbolic neighbourhood $\mathcal{U}_p$ and solutions $\overline{\mathrm{g}}$, V, Q with $E = 0$ and $\overline{\nabla} V = 0$.
- On overlaps, these solutions coincide and thus give rise to a globally hyperbolic solution on $\overline{\mathcal{M}} = \cup_{p \in \mathcal{M}} \mathcal{U}_p$ containing $\mathcal{M}$ as Cauchy hypersurface.

The form of the metric as $\overline{\mathrm{g}} = -\widetilde{\lambda}^2 dt^2 + \mathrm{g}_t$:
Consider the vector field $F = \frac{1}{dt(\overline{\nabla} t)} \overline{\nabla} t$, i.e., with $dt(F) = 1$.

- The leafs of $F^\perp$ are given as $\mathcal{M}_t = \{t\} \times \mathcal{M} \subset \overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$.
- The flow ϕ of F satisfies $\phi_s(\mathcal{M}_t) = \mathcal{M}_{t+s}$ because

$$\frac{d}{ds}(t(\phi_s(p))) = dt|_{\phi_s(p)}(F) = 1, \quad \text{and hence } t(\phi_s(p)) = s + t(p).$$

- Then the metric $\Phi^* \overline{\mathrm{g}}$ for $\Phi(t, p) = \phi_t(p) \in \overline{\mathcal{M}}$ satisfies

$$\Phi^* \overline{\mathrm{g}}(\partial_t, X) = \overline{\mathrm{g}}(F, d\Phi(X)) = 0 \ \forall X \in T\mathcal{M}_t,$$

  with $\widetilde{\lambda}^2 = \Phi^* \overline{\mathrm{g}}(\partial_t, \partial_t) = \overline{\mathrm{g}}(F, F) = \frac{1}{dt(\overline{\nabla} t)}$.

Extension of the spinor

Extend generalised imaginary Killing spinor (GIKS) along $(\mathcal{M}, \mathrm{g})$ to parallel spinor on $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ by parallel transport along the flow of $V \implies$

Corollary 2 (Lischewski '15)

Let $(\mathcal{M}, \mathrm{g}, \varphi)$ be a smooth Riemannian mfd with smooth GIKS φ and $\lambda \in C^\infty(\mathbb{R} \times \mathcal{M})$. Then the above Lorentzian manifold $\overline{\mathcal{M}} \subset \mathbb{R} \times \mathcal{M}$ with Lorentzian metric $\overline{\mathrm{g}}$ admits a a parallel spinor ϕ such that $\phi|_{\mathcal{M}} = \varphi$.

The corollary was obtained independently by Lischewski by studying the system

$$\mathrm{Ric}(\overline{\mathrm{g}}) = f\,(V_\phi^\flat)^2, \qquad \not{D}^{\overline{\mathrm{g}}}\phi = 0, \qquad df(V_\phi) = 0$$

for a spinor ϕ, a metric $\overline{\mathrm{g}}$ and a function f. The wave operator in step 2 of the proof is made of multiple copies of $\not{D}^2$ and $\overline{\nabla}^2$.

Riemannian manifolds satisfying the constraints

Let $(\mathcal{M}, \mathrm{g})$ be a Riemannian manifold with a vector field U such that $\nabla_{[i}U_{j]} = 0$ and $u^2 := \mathrm{g}(U, U) \neq 0$.

- $dU^\flat = 0$ and $U^\perp$ is integrable.
- Locally, $U = \mathrm{grad}(f)$, and the leaves of $U^\perp$ are the level sets of f.
- For $Z = \frac{1}{u^2}U$ we have $\mathcal{L}_Z U^\flat = dU^\flat(Z, .) = 0$. Hence, the flow ϕ of Z is a diffeomorphism between the leaves of $U^\perp$ and there is a diffeomorphism

$$\Phi : \mathcal{I} \times \mathcal{U} \ni (s, x) \to \phi_s(x) \in \mathcal{W} \subset \mathcal{M}, \qquad \mathcal{I} \text{ an interval},$$

- The diffeomorphism Φ satisfies $d\Phi|_{(s,x)}(\partial_s) = Z|_{\phi_s(x)}$ and

$$\begin{aligned} \Phi^*\mathrm{g}(\partial_s, \partial_s) &= \mathrm{g}(Z, Z) = \frac{1}{\mathrm{g}(U,U)}, \\ \Phi^*\mathrm{g}(\partial_s, X) &= \mathrm{g}(Z, d\Phi(X)) = 0, \qquad \text{for } X \in U^\perp \end{aligned}$$

Riemannian manifolds satisfying the constraints, ctd.

If Z is complete, its flow is defined on $\mathbb{R}$. On the universal cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, $U = \mathrm{grad}(f)$ globally, and hence there is a diffeomorphism

$$\Phi : \mathbb{R} \times \underset{\substack{\uparrow \\ \text{level sets of } f \,=\, \text{integral mfds of } U^\perp.}}{\mathcal{U}} \ni (s,x) \quad \rightarrow \quad \phi_s(x) \in \widetilde{\mathcal{M}}$$

Theorem 3

$(\mathcal{M}, \mathrm{g}, U)$ satisfies the constraint $\nabla_{[i}U_{j]} = 0 \iff$ it is locally isometric to

$$\left(\mathcal{I} \times \mathcal{F}, \mathrm{g} = \tfrac{1}{u^2} ds^2 + \mathrm{h}_s\right), \qquad \mathrm{h}_s \text{ family of Riemannian metrics on } \mathcal{F}.$$

This isometry maps U to $u^2\partial_s$. Moreover: If the vector field $\frac{1}{u^2}U$ is complete, then $\widetilde{\mathcal{M}}$ is globally isometric to $\mathbb{R} \times \mathcal{F}$.
Conversely, if $\mathcal{F}$ is compact and $u \in C^\infty(\mathcal{F} \times \mathbb{R})$ bounded, then for any family of Riemannian metrics h_s

$$\left(\mathcal{M} = \mathbb{R} \times \mathcal{F}, \mathrm{g} = \tfrac{1}{u^2} ds^2 + \mathrm{h}_s\right)$$

is complete.

Lorentzian holonomy reductions and the screen bundle

Let $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ be a Lorentzian mfd. of dim $(n+2)$ with parallel null vector field V.

- $\mathfrak{hol}_p(\overline{\mathcal{M}}, \overline{\mathrm{g}}) \subset \mathfrak{stab}(V_p) = \mathfrak{so}(n) \ltimes \mathbb{R}^n = \left\{ \left(\begin{array}{ccc} 0 & v^\top & 0 \\ 0 & A & -v \\ 0 & 0^\top & 0 \end{array} \right) \middle| \begin{array}{l} v \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\}$
- Screen bundle over $\mathcal{M}$:

$$\mathbb{S} = V^\perp / \mathbb{R} \cdot V, \qquad \mathrm{h}^{\mathbb{S}}([X],[Y]) = \overline{\mathrm{g}}(X,Y), \qquad \nabla^{\mathbb{S}}_X [Y] = \left[\overline{\nabla}_X Y \right],$$

  is a vector bundle with positive def. metric and compatible connection $\nabla^{\mathbb{S}}$.
- $\mathfrak{hol}(\nabla^{\mathbb{S}}) = \mathrm{pr}_{\mathfrak{so}(n)} \mathfrak{hol}(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ is a Riemannian holonomy algebra, i.e., equal to (a product of) $\mathfrak{so}(k)$, $\mathfrak{u}(k)$, $\mathfrak{sp}(1) \oplus \mathfrak{sp}(k)$, $\mathfrak{su}(k)$, $\mathfrak{sp}(k)$, $\mathfrak{g}_2$ or $\mathfrak{spin}(7)$, or the isotropy of a Riemannian symmetric space.
- Fixing a time-like unit vf $T \in \Gamma(\overline{\mathcal{M}})$ gives a canonical identification of $\mathbb{S}$ with a tangent subbundle

$$V^\perp \cap T^\perp = \mathbb{S} \subset T\overline{\mathcal{M}}$$

- If $(\overline{\mathcal{M}}, \overline{\mathrm{g}}, V)$ arises as solution to the Cauchy problem for V from $(\mathcal{M}, \mathrm{g}, U)$ we identify $\mathbb{S}|_{\mathcal{M}}$ with $U^\perp \subset T\mathcal{M}$ and get

$$\nabla^{\mathbb{S}}_X \sigma|_{\mathcal{M}} = \nabla^{\perp}_X \sigma,$$

$\sigma \in \Gamma(\mathbb{S}|_{\mathcal{M}}) = \Gamma(U^\perp)$, $X \in T\mathcal{M}$ and $\nabla^\perp = \mathrm{pr}_{U^\perp} \circ \nabla^{\mathrm{g}}$ the induced connection.

- Locally $(\mathcal{M}, \mathrm{g}) = (\mathcal{I} \times \mathcal{F}, \frac{1}{u^2} ds^2 + \mathrm{h}_s)$ and we can interpret $\sigma \in \Gamma(U^\perp)$ as family $\{\sigma_s\}_{s \in \mathcal{I}}$
- We have the following vector bundles of the same rank

$$\begin{array}{ccccc} (T\mathcal{F}, \nabla^{\mathrm{h}_s}) & & (U^\perp, \nabla^\perp) & & (\mathbb{S}, \nabla^{\mathbb{S}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F} & \subset & \mathcal{M} & \subset & \overline{\mathcal{M}} \end{array}$$

- Lorentzian holonomy reductions from $\mathfrak{so}(1, n+1)$ to $\mathfrak{g} \ltimes \mathbb{R}^n$ with $\mathfrak{g} \subset \mathfrak{so}(n)$ are given by a parallel null vector field V and a parallel sections of $\otimes^{a,b}\mathbb{S} \to \overline{\mathcal{M}}$ such as a complex structure, a stable 3-form, etc.

The relation to Lorentzian holonomy reductions

Theorem 4

Let $(\mathcal{M}, \mathrm{g}, U)$ be a Riemannian mfd. satisfying the constraints and $(\overline{\mathcal{M}}, \overline{\mathrm{g}}, V)$ the Lorenzian mfd. arising as solution of the Cauchy problem. Then there is a 1-1 correspondence between

$$\left\{ \begin{array}{c} \hat{\eta} \in \Gamma(\otimes^{a,b}\mathbb{S}) : \\ \nabla^{\mathbb{S}}\hat{\eta} = 0 \end{array} \right\} \overset{(*)}{\leftrightarrow} \left\{ \begin{array}{c} \eta \in \Gamma(\otimes^{a,b}U^{\perp}) : \\ \nabla^{\perp}\eta = 0 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \eta_s \in \Gamma(\otimes^{a,b}T\mathcal{F}) : \\ \boxed{\begin{array}{rclr} \nabla^{\mathrm{h}_s}\eta_s & = & 0 & (i) \\ \dot{\eta}_s & = & \frac{1}{2}\dot{\mathrm{h}}_s^{\sharp} \cdot \eta_s & (ii) \end{array}} \end{array} \right\}$$

Hence, $\mathfrak{hol}(\nabla^{\mathbb{S}}) = \mathrm{pr}_{\mathfrak{so}(n)}\mathfrak{hol}(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ lies in the stabiliser of a tensor on $\mathbb{S}$ if and only if on $\mathcal{F}$ there is an induced s-dependent family of h_s-parallel tensors η_s with (ii).

Proof of $(*, \leftarrow)$: Extend $\eta \in \Gamma(\otimes^{a,b}U^{\perp} \to \mathcal{M})$ to $\hat{\eta} \in \Gamma(\otimes^{a,b}\mathbb{S} \to \overline{\mathcal{M}})$ by parallel transport along the flow of V. Then $A := \nabla^{\mathbb{S}}\eta \in \Gamma(T^{\perp} \otimes \otimes^{a,b}\mathbb{S})$ satisfies $\nabla^{\mathbb{S}}A = 0$, which is a linear symmetric hyperbolic system for A with initial condition $A|_{\mathcal{M}} = 0$. Hence, not only $\nabla^{\mathbb{S}}_V\hat{\eta} = 0$ but also $\nabla^{\mathbb{S}}_X\hat{\eta} = 0$ for $X \in T\mathcal{M}$.

Flows of special Riemannian structures

Let h_s be a family of Riemannian metrics admitting a family η_s of parallel tensors defining a holonomy reduction. What about condition (ii) $\dot{\eta}_s = \frac{1}{2}\dot{\mathrm{h}}_s^\sharp \cdot \eta_s$?

- h_s is family of Kähler metrics: there is a complex structure $\eta_s = J_s$ and Kähler form ω_s with

$$\dot{J}_s = \tfrac{1}{2}\dot{\mathrm{h}}_s^\sharp \cdot J_s, \qquad \dot{\omega}_s = \tfrac{1}{2}\dot{\mathrm{h}}_s^\sharp \cdot \omega_s,$$

  so (ii) is automatically satisfied for J_s.

- h_s is family of Ricci-flat Kähler metrics with $\mathrm{div}^{\mathrm{h}_s}(\dot{\mathrm{h}}_s) = 0$.
- $\mathrm{h}_s = \mathrm{h}_s^1 + \mathrm{h}_s^2$ is a family of product metrics $\iff$ $\exists\ \nabla^{\mathrm{h}_s}$-parallel decomposable p-forms $\mu_s^i = \mathrm{vol}^{\mathrm{h}_s^i}$. Volume forms evolve as $\dot{\mu}_s = \frac{1}{2}\dot{\mathrm{h}}_s^\sharp \cdot \mu_s$.
- h_s is a family of holonomy $\mathfrak{g}_2$ metrics defined by a family of stable 3-forms φ_s.
  Problem: Given a family h_s of holonomy $\mathfrak{g}_2$ metrics, does there exist a family of stable 3-forms φ_s defining h_s such that $\dot{\varphi}_s = \frac{1}{2}\dot{\mathrm{h}}_s^\sharp \cdot \varphi_s$ holds?

$$Sym^2(\mathbb{R}^7) \oplus \mathbb{R}^7 \ \ni\ (S, X) \longmapsto S \cdot \varphi + X \lrcorner (*\varphi) \ \in\ \Lambda^3\mathbb{R}^7$$

  If $\dot{\varphi} = S \cdot \varphi + X \lrcorner (*\varphi)$, then the associated metric satisfies $\dot{\mathrm{h}} = 2S$ ...

Consequence of Theorem 4

If $(\overline{\mathcal{M}}, \overline{\mathrm{g}})$ is a Lorentzian manifold obtained from the Cauchy problem. Then $\mathfrak{hol}(\nabla^{\mathbb{S}}) \subset \mathfrak{g}$ if and only if the associated family of Riemannian metrics h_s on $\mathcal{F}$ is given as follows:

$\mathfrak{g}$		h_s is family of
$\mathfrak{so}(p) \oplus \mathfrak{so}(q)$	$\Longleftrightarrow$	product metrics
$\mathfrak{u}(n/2)$	$\Longleftrightarrow$	Kähler metrics
$\mathfrak{su}(n/2)$	$\Longleftrightarrow$	Ricci-flat Kähler metrics with $\mathrm{div}^{\mathrm{h}_s}(\dot{\mathrm{h}}_s) = 0$
$\mathfrak{sp}(n/4)$	$\Longleftrightarrow$	hyper-Kähler metrics
$\mathfrak{g}_2$ / $\mathfrak{spin}(7)$	$\Longleftrightarrow$	of $\mathfrak{g}_2$ / $\mathfrak{spin}(7)$-metrics (with (ii)?)

Classification of Riemannian manifolds with GIKSs

Theorem 5

Let $(\mathcal{M}, g)$ be a Riemannian spin manifold admitting an GIKS φ. Then

1. $(\mathcal{M}, g)$ *is locally isometric to*

$$(\mathcal{M}, g) = \left(\mathcal{I} \times \mathcal{F}_1 \times \dots \times \mathcal{F}_k, g = \tfrac{1}{u^2} ds^2 + h^1_s + \dots + h^k_s\right) \quad (11)$$

for Riemannian manifolds $(\mathcal{F}_i, h^i_s)$ of dimension n_i, $u = \|\varphi\|^2$. Each h^i_s is one of the following families of special holonomy Riemannian metrics:

- h^i_s *is Ricci-flat Kähler and* $\mathrm{div}^{h^i_s}(\dot{h}^i_s) = 0$,
- *hyper-Kähler,* $\mathbf{G}_2$ *(?),* $\mathbf{Spin}(7)$ *(?) or a flat metric.*

2. *If $(\mathcal{M}, g)$ is simply connected and the vector field $\frac{1}{u^2_\varphi} U_\varphi$ is complete, the isometry* (11) *is global with $\mathcal{I} = \mathbb{R}$.*

3. *Conversely, every Riemannian manifold $(\mathcal{M}, g)$ of the form* (11) *with $\mathcal{I} \in \{S^1, \mathbb{R}\}$, where u is any positive function and $(\mathcal{F}_i, h^i_s)$ are families of special holonomy metrics (subject to the above flow equations ...) is spin and admits an GIKS.*