Global hyperbolicity and the completeness of geometric flows[1]

Part I: The spacetime viewpoint (today)
Part II: Singularity formation(tomorrow)

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[1]Based on joint work with I. Bakas

6 Singularities

- Singularity theorem
- Necessary conditions
- More general singularities
- Bel-Robinson energy

Einstein vs. non-relativistic geometric flows

- Treat geometric flows from unified viewpoint
- Spot and reflect on their differences and similarities
- Build bridges between the two disciplines
- Develop methods, constructions and results in new context
- Sometimes new concepts are needed
- Spacetime viewpoint of geometric flows will prevail throughout this talk

Generic properties of time slices

- Spacetime $(\mathcal{V}, \bar{g})$, time function $t : \mathcal{V} \to \mathbb{R}$
- Level sets of $t : (\Sigma_t, g_t)$, metric: $\bar{g} = -N^2 dt^2 + g_t$, where $N \equiv (-\bar{g}^{\mu\nu}\,\partial_\mu t\,\partial_\nu t)^{-1/2}$ measures normal separation of the Σ_t's.

Main issue: For 'sequence', or flow, of not just one time slice (Σ, g) but of a 1-parameter family of Riem manifolds parameterized by time (Σ_t, g_t),

Control during evolution

physics, given by fields ψ
geometry, given by g_t, k_t
topology of time 3-slice
dynamics, given by e.g., $\mathrm{Ric} - \frac{1}{2}Rg = \kappa T$.

Determine

- allowed initial state(s) $(\Sigma_0, g_0), \psi_0$
- possible final state(s) $(\Sigma_\infty, g_\infty), \psi_\infty$.

Geometric flows

At least two ways to do this:

↪Simplest: Consider (Σ, g_t), Σ is a fixed background manifold.

↪ More general: Arrange for genuine changes in topology, (Σ_t, g_t).

Comments:

- First case is special case of second, by setting $\Sigma_t = \Sigma \times \{t\}$.
- Need to consider generalized flows, flows-with-surgery for second case:

↪ Unknown even basic causal structure results!

- In generalized case, $\partial_t g \to L_{\partial_t} g$.
- For any smooth flow (Σ_t, g_t), by the chain rule, any other expression that depends on this metric, e.g., $\mathrm{Riem}(t), \mathrm{Ric}(t), R(t), l(t), \mathrm{vol}_{\Sigma_t}(t)$, should have rates of change that depend linearly on $\dot{g}_t$. Computations straightforward.

Examples of geometric flows

1. Trivial flow and rescalings

Except the trivial flow: $g(t) = g(0)$, the simplest flow is the rescaling

$$g(t) = F(t)g(0),$$

with $F(t) > 0, F(0) = 1$, so that the flow-law is given by,

$$\dot{g}(t) = f(t)g(t), \quad f(t) = \dot{F}/F$$

Easy to get expressions for the rates of change of the various quantities (from the general variational formulae).

Examples of geometric flows

2. Ricci flow

Suppose law for (Σ, g_t) evolution is

$$\dot{g} = -2\,\mathrm{Ric}(t).$$

This is Hamilton's Ricci flow. To get global evolution for $g(t)$ like before now becomes almost equivalent to the proof of the Poincaré conjecture!

However, local existence for this parabolic eqn is not that difficult:

Theorem

Suppose: Σ compact and g_0 a smooth metric on Σ. Then there is a unique Ricci flow g_t on the time interval $[0, T)$, for some $T > 0$. (Hamilton using Nash-Moser iterations, de Turck.)

Examples of geometric flows

3. The Einstein flow

Considerably more complicated (!) than the Ricci flow is the Einstein flow for (Σ_t, g_t).

• Standard choice Consider (Σ, g_t). We have:

Evolution equations:

$$\begin{aligned} \partial_t g_{ij} &= 2Nk_{ij} \\ \partial_t k_{ij} &= \nabla_i \partial_j N - N\left(R_{ij} + 3Hk_{ij} - 2k_{il}k^l_j\right). \end{aligned}$$

Constraints:

$$\begin{aligned} R + (\mathrm{tr}\, k)^2 - |k|^2_g &= 0 \\ \nabla^i k_{ij} - 3\partial_j H &= 0. \end{aligned}$$

(The mean curvature is: $H = \frac{1}{3}\mathrm{tr}\, k_{ij} \equiv \frac{1}{3}\tau$.)

Metric forms for non-Einsteinian GFs I

On the 4-manifold $\mathcal{V} = \mathcal{M}_t \times [t_0, \infty]$, $t_0 \in \mathbb{R}$, we are given the following data:

- a smooth Riemannian metric g_{ij} on the 3-manifold $\mathcal{M}_t = \mathcal{M} \times \{t\}$,
- a smooth function $N(t, x^i)$ defined on $\mathcal{V}$, and
- a vector field $N^i(t, x^j)$ tangent to the 3-manifold $\mathcal{M}$.

Metric forms for non-Einsteinian GFs II

A basic geometric assumption of the geometric flow (eg., Hořava-Lifshitz) kinematics is the existence of a 'book-keeping' line element form

$$g_{HL} := ds^2_{HL} = -N^2 dt^2 + g_{ij}(t)\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right). \quad (1)$$

It is usually assumed that N is a function of t only. Here t is NOT proper time, but absolute time.
The form (1) is invariant under the action of the restricted group of foliation-preserving diffeomorphisms $t \to \tilde{t}(t), x \to \tilde{x}(t, x)$, not of the full group of spacetime transformations.

Metric forms for Einstein flow I

On the relativistic spacetime $(\mathcal{V}, g_4)$, where $\mathcal{V} = \mathcal{M}_t \times [t_0, \infty]$, we may take the submanifolds $\mathcal{M}_t$ to be spacelike.
Then we can always construct a Cauchy-adapted frame (e_0, e_i) with e_i tangent to the space slice $\mathcal{M}_t$ and e_0 orthogonal to it. The dual coframe θ^α has $\theta^0 = dt$, $\theta^i = dx^i + N^i dt$, where the tangent vector N^i to the spacelike hypersurfaces.
This then leads to the standard general relativistic splitted $(3+1)$-form for the spacetime metric g_4 defining proper time (or proper distance, in the case of spacelike separation) between any two events on $(\mathcal{V}, g_4)$,

$$g_4 : ds^2_{GR} = -N^2 dt^2 + g_{ij}(t) \left(dx^i + N^i dt\right) \left(dx^j + N^j dt\right) . \qquad (2)$$

Metric forms for Einstein flow II

Here N is a positive function called the lapse, N^i is the shift, and we may use the same symbols for both the Hořava-Lifshitz data (g_{ij}, N, N^i) defining the form ds^2_{HL} in (1), and the spacetime metric ds^2_{GR} given by Eq. (2), although normally N is only a function of t in (1).

We emphasize that the spacetime interval (2) is invariant under the full group of spacetime diffeomorphisms, not only under the subgroup of foliation-preserving ones as in Eq. (1).

Remark: It is believed that the correct geometric framework for Hořava-Lifshitz (or other geometric flows where there is a preferred time coordinate) is the so-called Newton-Cartan geometry. In this

Metric forms for Einstein flow III

case, Eq. (1) is still a Lorentz metric, but because of the restricted invariance of the associated action of the theory under only foliation-preserving diffeomorphisms (not the full group of spacetime transformation), we imagine that the metric given by Eq. (1) may loose its nondegeneracy in some places on the manifold. This implies a possible violation of the law of transformation of the $Detg_{ij}$ and its possible vanishing for non-foliation preserving chart changes. These changes, however, are irrelevant for a theory invariant only under the restricted group of transformations.

Difficulties with non-Einsteinian flows

Uses of ' space-time' vs. ' spacetime' if need to distinguish!

- no null structure on $(\mathcal{V}, g_{HL})$
- no standard notion of causality or chronology
- no usual trichotomy of timelike, null, spacelike, for vectors at any point p on $\mathcal{V}$
- no invariant definition of a notion of length for a given curve $C : I \subset \mathbb{R} \to \mathcal{V}$ on the manifold $\mathcal{V}$
- no notion of geodesic on $(\mathcal{V}, g_{HL})$
- hence no obvious way to talk about the usual route through geodesic (in-)completeness to spacetime singularities, maximal curves, etc.

Difficulties with non-Einsteinian flows

- How to talk sensibly about dynamics and asymptotic properties of spacetime fields near singularities in terms of standard space-time notions?
- How are we to somehow import a notion of geodesic (in-)completeness into these frameworks?
- How to compare such non-Einsteinian flows to the more usual ones that allow a spacetime interpretation?

Basic point: Due to having less symmetry, we cannot simply import the spacetime properties of (2) into (1) (this was a basic issue that initiated the joint work with IB, cf., 'Mixmaster in HL gravity').

Existence of Lorentzian metrics

Assume that $\mathcal{V}$ is a connected, C^∞ and Hausdorff manifold.

- given the Hořava-Lifshitz functions N, N^i, we can form the following nowhere vanishing vector field,

$$X = (N, N^i), \tag{3}$$

  defined on $\mathcal{V}$.
- Then the existence of X on $\mathcal{V}$ is equivalent to the condition that it admits a time-orientable Lorentz metric
- such a manifold is necessarily paracompact
- Then using partitions of unity, it is not difficult to show that there are an infinite number of such metrics defined on $\mathcal{V}$

Hence, we may also assume that on $\mathcal{V}$ there is a time-oriented Lorentz metric g_4 such that $(\mathcal{V}, g_4)$ is an Einstein spacetime.

Time shifts of infinite length I

No process of obtaining geodesics of max length in non-Einsteinian flows. However, we can show:

Theorem

If an inextendible geodesic has infinite $l_{GR}(C)$ length[a] (that is has no future endpoint), then as a curve it will also have infinite $l_{HL}(C)$ length.

[a] there are conditions for this to happen - see below

Here,

Time shifts of infinite length II

- the spacetime length of any curve C connecting the points $p, q \in \mathcal{N}$ (suitable global hyperbolic region with q in the future of p in the spacetime metric (2)) is given by (a dot denotes differentiation with respect to T),

$$l_{GR}(C) = \int_{T_0}^{T_1} \left(1 - g_{ij}\dot{X}^i\dot{X}^j\right)^{1/2} dT. \tag{4}$$

- introduce Minkowski normal coordinates (t, x^i) in the region $\mathcal{N}$ (that is ∂_t is timelike and future-pointing and the null cone $T_p\mathcal{V}$ is the set $t^2 - \sum(x^i)^2 = 0$).

Time shifts of infinite length III

- introduce Gaussian normal coordinates T, X, Y, Z on $\mathcal{V}$, where $T = (t^2 - \sum(x^i)^2)^{1/2}$, $X^1 = x^1/t, X^2 = x^2/t, X^3 = x^3/t$.
- *synchronous system*: the surfaces $T = \text{const.}$ are spacelike while the curves $X^i = \text{const.}$ are timelike geodesics orthogonal to these.
- The metric (2) then takes the standard form,

$$ds^2_{GR} = dT^2 - g_{ij}dX^i dX^j. \tag{5}$$

- The length functional attains its max for the curves $X^i = \text{const.}$ (the 3-metric g_{ij} is positive-definite). That is, the geodesic connecting the two points p, q has maximum length.

Time shifts of infinite length IV

- Then consider an inextendible geodesic, that is a curve $C : (T_0, \infty) \to \mathcal{V}$ with no future endpoint, of infinite length $l_{GR}(C)$ as given in (4) (there are conditions for this to happen, cf. below).
- Then for this curve, we may form the functional $l_{HL}(C)$ given precisely by the same form as in (4). In these coordinates (T, X^i) (no matter how they are constructed!) the 'length' $l_{HL}(C)$ is again infinite, but of course this value has no invariant 4-meaning (as in Eq. (4) where T measures proper time).

Time shifts of infinite length V

- We then have the freedom (in the projectable Hořava-Lifshitz framework) to multiply this infinite value of the length $l_{HL}(C)$ by a smooth function $N(T)$, that is to perform arbitrary time shifts. Therefore the particular value ∞ for the length $l_{HL}(C)$ *is* an invariant for the restricted (projectable) version of the symmetry group. Hence, we conclude that if a geodesic has infinite $l_{GR}(C)$ length, it will also have its $l_{HL}(C)$ length infinite (as a curve).

Therefore: We can now import techniques from GR about completeness to decide on corresponding criteria in non-Einsteinian frameworks.

Sufficient conditions for global hyperbolicity

Y. Choquet-Bruhat & S.C. 2002, 2004:

Theorem

If $(\mathcal{V}, g)$ is regularly sliced (i.e., $(N, N^i, \mathbf{g}_t)$ uniformly bounded, such that (Σ_0, g_0) is a complete Riemannian manifold), then $(\mathcal{V}, g)$ is globally hyperbolic.

In other words,

- *Regular slicing* implies *global hyperbolicity*.

Global hyperbolicity for geometric flows

- For non-Einsteinian geometric flows, we give the following

Definition

A spacetime is called globally hyperbolic if it is regularly sliced.

Sufficient conditions for completeness I

Choquet-Bruhat and S.C. (2002): Use spatial norm of shape tensor $|K|_{\mathbf{g}_t}$:

Theorem

If:

- $(\mathcal{V}, g)$ *is regularly sliced*
- *for each finite* t_1, $|\nabla N|_{\mathbf{g}_t}$ *and* $|K|_{\mathbf{g}_t}$ *integrable on* $[t_1, +\infty)$,

then $(\mathcal{V}, g)$ *is future causally g-complete.*

Sufficient conditions for completeness II

⇝ Under same conditions, we obtain completeness criteria for any geometric flow. Namely, provided that

- Space-time is globally hyperbolic (the Hořava-Lifshitz data N, N^i, g_{ij} are all uniformly bounded), and
- the norms $(\nabla_i N)^2$ (this is trivially zero in the projectable case) and $K_{ij}K^{ij}$ (or equivalently, $K_{ij}K^{ij} - (1/3)K^2$) are also bounded,

then (1) will be complete.

Sufficient conditions for singularity

Hawking-Penrose (1970): Use mean curvature vector field H of spacelike hypersurface Σ of $(\mathcal{V}, g)$. It has shape tensor $K = \mathrm{nor}\,\bar{\nabla}$. For unit future pointing $U \perp \Sigma$, consider convergence $\theta = \langle U, H \rangle = \frac{1}{n-1}\,\mathrm{trace}\, K$. Then

Theorem If:

- $\mathrm{Ric}(X, X) \geq 0$ for all causal vector fields X of of $(\mathcal{V}, g)$
- $\theta \geq C > 0$, everywhere on Cauchy surface Σ,

then no future-directed causal curve from Σ can have length greater than $1/C$.

Necessary conditions for singularity formation I

From the completeness theorem, it then follows that:

Theorem

Any singularities in Hořava-Lifshitz gravity will be accompanied either by a loss of global hyperbolicity and/or by a necessary blow up in $K_{ij}K^{ij}$ (or equivalently, $K_{ij}K^{ij} - (1/3)K^2$) (assuming boundedness of all other data on $\mathcal{V}$).

Necessary conditions for singularity formation II

For *potentially infinite metrics*, we choose the lapse and shift as,

$$-N^2(t,x^i) = R(g_{ij})(t,x^i) + \frac{\xi}{2t} < 0, \quad N^i = 0, \tag{6}$$

with R being the scalar curvature of the 3-metric $g_{ij}(t,x^i)$, and ξ a suitable real constant.
This is apparently a restriction of the scalar curvature of $\mathcal{M}$ in the sense that $R < -\xi/2t$, the existence of a uniform bound for the scalar curvature of $\mathcal{M}$.

Necessary conditions for singularity formation III

Then, the length of any curve $C:(t_0,t_1)\to\mathcal{V}$ is given by (we now reinsert the lapse and shift),

$$l_{GR}(C)=\int_{t_0}^{t_1}\left(-N^2+g_{ij}\dot{C}^i\dot{C}^j\right)^{1/2}dt, \tag{7}$$

and takes a particularly interesting form:

$$l_{GR}(C)=\int_{t_0}^{t_1}\left(R+\frac{\xi}{2t}+\left|\frac{dC}{dt}\right|_{g(t)}\right)^{1/2}dt, \tag{8}$$

Necessary conditions for singularity formation IV

and this is well-defined provided the g_{ij}-length of C is bounded from below. Then we write the integrand as

$$\left(\frac{\xi}{2t}\right)^{1/2}(1+x)^{1/2}, \quad x = \frac{2t}{\xi}\left(R + \left|\frac{dC}{dt}\right|_{g(t)}\right), \tag{9}$$

Necessary conditions for singularity formation V

and expand $(1+x)^{1/2}$ keeping only the highest non-trivial term. We find,

$$l_{GR}(C) = \frac{\xi^{-1/2}}{\sqrt{2}} \int_{t_0}^{t_1} \sqrt{t} \left(R + \left| \frac{dC}{dt} \right|_{g(t)} \right) dt + O(\xi^{-3/2}), \quad (10)$$

Necessary conditions for singularity formation VI

so that the spacetime length is nothing but the Perelman length function for the spacetime curve C,

$$l_{GR}(C) = \frac{\xi^{-1/2}}{\sqrt{2}} l_{per}(C) \tag{11}$$

with

$$l_{per}(C) = \int_{t_0}^{t_1} \sqrt{t} \left(R + \left| \frac{dC}{dt} \right|_{g(t)} \right) dt, \tag{12}$$

Necessary conditions for singularity formation VII

(the so called reduced length is $l_{per}(C)/2(\sqrt{t_2} - \sqrt{t_1})$).

This shows that there may be a connection between the singularities met in various geometric flows such as the Ricci flow and the 'physical' spacetime singularities of gravitational theories defined as geodesic incompleteness.

Necessary conditions for singularity formation VIII

In the simplest case of *uniformly timelike* curves, that is when the integrand in (8) is bounded away from zero by a positive constant,

$$-N^2 + g_{ij}\dot{C}^i\dot{C}^j \geq M^2, \quad M \text{ constant}, \tag{13}$$

the length of such a curve on the interval (t_0, ∞) is infinite. Using (11), we see that complete solutions in Hořava-Lifshitz gravity defined here, correspond exactly to the various 'singularity models' if we regard it as the geometric flow.

Necessary conditions for existence of more elaborate singularities I

- Above we have also given criteria where the curves are not uniformly timelike, and so to prove completeness becomes more delicate.
- If the length is finite (so that (1) is not complete), then the integral in (11) is finite, and so condition (13) must be violated.

Necessary conditions for existence of more elaborate singularities II

- This then leads to the integrand in the Perelman integral satisfying certain conditions leading to other singularities of the geometric flow.
- One way to proceed is through the use the the Bel-Robinson energies

Necessary conditions for existence of more elaborate singularities I

Completeness and (bounds of) the Bel-Robinson energy

Let η_{ijk} volume element of space metric $\mathbf{g}_t$. Define the electric and magnetic tensors

$$E_{ij} = R^{0}_{i0j},\ D_{ij} = \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hklm},\ H_{ij} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j},\ B_{ji} = \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j},$$

The Bel-Robinson energy of the Bianchi field $(\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B})$ *at time* t is

$$\mathcal{B}(t) = \frac{1}{2}\int_{\Sigma_t} \left(|\mathbf{E}|^2_{\mathbf{g}_t} + |\mathbf{D}|^2_{\mathbf{g}_t} + |\mathbf{B}|^2_{\mathbf{g}_t} + |\mathbf{H}|^2_{\mathbf{g}_t}\right) d\mu_{\mathbf{g}_t}.$$

Necessary conditions for existence of more elaborate singularities II

For a RW universe: $\mathbf{B} = \mathbf{H} = 0$,
$|E|^2_{\mathbf{g}_t} = 3(\ddot{a}/a)^2, |D|^2_{\mathbf{g}_t} = 3\left((\dot{a}/a)^2 + k/a^2\right)^2.$

$$\mathcal{B}(t) \sim k_u^2(t) + k_\sigma^2(t), \quad k_u,\ k_\sigma,$$ the principal sectional curvatures.

Theorem A spatially closed, expanding at time t_*, FRW universe that satisfies $\gamma < \mathcal{B}(t) < \Gamma$ is causally g-complete.
Further, there is a minimum radius, $a_{\min} > \Delta^{-1/2}$, and these universes are eternally accelerating $(\ddot{a} > 0)$.

Open problem: Role of surgery in these new types of singularities?